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# On evolution algebras and their derivations



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## Abstract

This work investigates the derivations of $n$-dimensional complex evolution algebras, based on the rank of the structural matrix. The spaces of the derivations of evolution algebras under three different conditions that make the rank of the structural matrix equals to $n-2$ are investigated.

**Keywords:** Evolution algebras, rank of the structural matrix, derivation.

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## 1. Introduction

In Tian [13], a new kind of algebra which is called evolution algebras was introduced as a type of genetic algebra that makes possible to deal algebraically with the self-reproduction of alleles in non-Mendelian genetics. According to the author and [9], evolution algebras also constitute a fundamental application between algebra, dynamic systems, Markov processes, Knot Theory, Graph Theory and Group Theory.

In between algebras and dynamical systems, evolution algebras are presented as a new field connected with both previously mentioned fields and they could be defined algebraically, their structure has a table of multiplication, which satisfies the conditions of commutative Banach algebra as non-associative Banach algebra in general; dynamically, they represent discrete dynamical systems [5]. Evolution algebras have the following elementary properties: Evolution algebras are not associative, in general; they are commutative, flexible, but not power-associative, in general; direct sums of evolution algebras are also evolution algebras; Kronecker products of evolution algebras are also evolution algebras [13]. Remarkably, evolution algebras are not considered as a well-known class of non-associative algebras, for instance, Lie, alternative and Jordan algebras, since they are not defined by identities. Therefore, the research on these algebras follows different paths [3, 4, 14].

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For any algebra, the space of all its derivations is Lie algebra with respect to the commutator multiplication. In the theory of non-associative algebras, particularly, in genetic algebras, the Lie algebra of derivations of a given algebra is one of the important tools for studying its structure [2, 6, 7, 11–13]. In this regard, the system of equations that describe the derivations of evolution algebras have been posed in [13]. Notably, there are several genetic interpretations of derivation of genetic algebra which have been given in [8, 10].

The space of derivations of evolution algebras for non-singular matrices and when rank of the structural matrix equals to $n-1$ were described in [1]. The purpose of the present study is to investigate the derivations of $n$-dimensional complex evolution algebras, based on certain conditions which cause the structural matrix to have rank equals to $n-2$. This paper is organized as follows. Section 2 highlights the definitions and facts about evolution algebras, derivations and related result. Section 3 investigates the derivation of evolution algebras based on certain conditions which cause the structural matrix to have rank equals to $n-2$.

## 2. Preliminaries

In this section, definitions, and related known result about evolution algebras and the derivation are recalled.

**Definition 2.1.** Let $E$ be a vector space over a field $\mathbb{K}$ with a basis $\{e_1, e_2, \ldots\}$ and a multiplication rule $\cdot$ such that

$$e_i \cdot e_j = \begin{cases} 0, & i \neq j, \\ \sum_m a_{ik} e_m, & i = j. \end{cases}$$

Then, $E$ is called evolution algebras.

Accurately, the basis $\{e_1, e_2, \ldots\}$ is called a natural basis and the matrix $A = (a_{ij})_{i,j=1}^{n}$ is denoted as the structural matrix of the finite-dimensional evolution algebras $E$. Accordingly, the evolution algebras are commutative, and this fact denotes that evolution algebras are flexible. The rank of the structural matrix of finite-dimensional evolution algebras does not depend on the choice of natural basis since $\mathrm{Rank} A = \dim(E \cdot E)$.

**Definition 2.2.** The derivation of algebra $E$ is a linear operator $d : A \to A$ such that

$$d(u \cdot v) = d(u) \cdot v + u \cdot d(v), \quad \text{for all } u, v \in A.$$

It is recognized that for any algebra, the space $\mathrm{Der}(E)$ of all derivations is a Lie algebra with the commutator multiplication. In [13], it was illustrated that the space of derivations of evolution algebras $E$ can be described as follows:

$$\mathrm{Der}(E) = \left\{ d \in \mathrm{End}(E) \mid a_{kj} d_{ij} + a_{ki} d_{ij} = 0, \text{ for } i \neq j; \, 2a_{ji} d_{ii} = \sum_{k=1}^{n} a_{ki} d_{jk} \right\},$$

where $d$ be a derivation of evolution algebras $E$ with natural basis $\{e_1, \ldots, e_n\}$ and $d(e_i) = \sum_{j=1}^{n} d_{ij} e_j$, $1 \leqslant i \leqslant n$.

**Theorem 2.1** ([10])**.** *Let $d : E \to E$ be a derivation of evolution algebra $E$ with non-singular evolution matrix in basis $\langle e_1, \ldots, e_n \rangle$. Then, the derivation $d$ is zero.*

**Theorem 2.2** ([10])**.** *Let $e_n e_n = \sum_{k=1}^{n-1} b_k (e_k e_k)$ and $b_p \neq 0$, $b_q \neq 0$ for some $1 \leqslant p \neq q \leqslant n$. Then, $d = 0$.*

**Theorem 2.3** ([10])**.** *Let $e_n e_n = b(e_1 e_1), b \neq 0$. Then, derivation $d$ is either zero or it is in one of the following forms up to basis permutation:*

(i) $D_1$ *where,* $d_{11} = \dfrac{\delta}{2^{n-s}-1}$, $1 \leqslant s \leqslant n-1$ *and* $\delta^2 = -b d_{1n}^2$;

(ii) $D_2$ *where,* $d_{22} = \dfrac{1-2^{m-k}}{2^{k-1}} d_{11}, d_{11} = \dfrac{\delta}{2^{m-k+1}-1}$, $1 \leqslant k < m \leqslant n-1$ *and* $\delta^2 = -bd_{1n}^2$;

(iii) $D_3$ *where,* $d_{11} = \delta$ *and* $\delta^2 = -bd_{1n}^2$.

*where* $D_i, i = \overline{1,3}$ *in the following forms:*

$$\begin{pmatrix} d_{11} & 0 & \dots & 0 & 0 & \dots & 0 & d_{1n} \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 2d_{11} & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 2^{n-s-1}d_{11} & 0 \\ -bd_{1n} & 0 & \dots & 0 & 0 & \dots & 0 & d_{11} \end{pmatrix}, \tag{$D_1$}$$

$$\begin{pmatrix} d_{11} & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & d_{1n} \\ 0 & d_{22} & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 2^{k-1}d_{22} & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 2d_{11} & \dots & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 2^{m-k}d_{11} & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ -bd_{1n} & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & d_{11} \end{pmatrix}, \tag{$D_2$}$$

$$\begin{pmatrix} d_{11} & 0 & \dots & 0 & 0 & \dots & 0 & 0 & d_{1n} \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & \dots & 0 & \frac{d_{11}}{2^{n-s-2}} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & \frac{d_{11}}{2} & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & d_{11} & 0 \\ -bd_{1n} & 0 & \dots & 0 & 0 & \dots & 0 & 0 & d_{11} \end{pmatrix}. \tag{$D_3$}$$

**Theorem 2.4** ([10])**.** *Let evolution algebras has a matrix* $A = (a_{ij})_{1 \leqslant i,j \leqslant n}$ *in the natural basis* $e_1, \dots, e_n$ *such that* $e_n e_n = 0$ *and* $\mathrm{Rank} A = n-1$*. Then, derivation* $d$ *of this evolution algebra is either zero or it is in one of the following forms up to basis permutation:*

$$\begin{pmatrix} 0 & \dots & 0 & d_{1n} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & d_{n-1n} \\ 0 & \dots & 0 & 0 \end{pmatrix}, \tag{$D_4$}$$

*where* $\sum\limits_{k=1}^{n-1} a_{ik} d_{kn} = 0, 1 \leqslant i \leqslant n-1$,

$$\begin{pmatrix} 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & \frac{d_{nn}}{2^{n-k-1}} & \dots & 0 & d_{k+1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \dots & \frac{d_{nn}}{2} & d_{n-1n} \\ 0 & \dots & 0 & 0 & \dots & 0 & d_{nn} \end{pmatrix}, \tag{$D_5$}$$

*where* $d_{i+1n} = \frac{a_{in}}{a_{ii+1}}\left(\frac{1}{2^{n-i-1}} - 1\right) d_{nn}$, $a_{ii+1} \neq 0$, $k+1 \leqslant i \leqslant n-2, 1 \leqslant k \leqslant n-1$ *and* $d_{k+1n} \in \mathbb{C}$.

It is noticeable that the aforementioned theorems describe the derivations of $n$-dimensional complex evolution algebras depending on non-singular structural matrix and the rank of the structural matrix that equals to $n-1$ which were investigated in [1]. In the next section, three different cases, which make the rank of the structural matrix equals to $n-2$ are going to be described.

## 3. Derivations of evolution algebras

In this section, some auxiliary results should be illustrated about the derivation of evolution algebras when the rank of the structural matrix equals to $n-2$. Accordingly, we will be ready to begin investigating the derivation of evolution algebras.

According to the definition of derivation of evolution algebras, it is easy to see that $de_ie_j + e_ide_j = 0$ and $de_i^2 = 2de_i^2$ for all $1 \leqslant i \neq j \leqslant n$. Let $de_k = \sum_{i=1}^{n} d_{ki}e_i$. Then, the following is obtained:

$$d_{ij}e_j^2 + d_{ji}e_i^2 = 0, \tag{3.1}$$

$$de_i^2 = 2d_{ii}e_i^2 \text{ for all } 1 \leqslant i \neq j \leqslant n. \tag{3.2}$$

Plainly, $e_1^2, \dots, e_{n-2}^2$ should be linearly independent since $\text{Rank}A = n-2$. As consequence of performing a suitable basis permutation, $e_i^2$ can be rewritten as follows:

$$e_i^2 = \sum_{k=1}^{n-2} b_k e_k^2, \quad b_1, \dots, b_{n-2} \in \mathbb{C}, i \in \{n-1, n\}. \tag{3.3}$$

Now, (3.2) indicates that $2d_{ii}$ is an eigenvalue of $d$ for all $1 \leqslant i \leqslant n-2$. Therefore, $\text{spec}(d) \supseteq \{2d_{11}, 2d_{22}, \dots, 2d_{n-2n-2}\}$ is verified. In addition, (3.1) fulfills that $d_{ij} = d_{ji} = 0$ for all $1 \leqslant i \neq j \leqslant n-2$. Let us replace $i$ by $n$ and $n-1$ in (3.1) separately, whereupon the following is established:

$$d_{mj}e_j^2 + d_{jm}e_m^2 = 0, \quad m \in \{n-1, n\}, \tag{3.4}$$

$$(d_{n-1j} + d_{jn-1}b_j)e_j^2 + \sum_{k=1, k\neq j}^{n-2} d_{jn-1}b_k e_k^2 = 0, \tag{3.5}$$

$$(d_{nj} + d_{jn}b_j)e_j^2 + \sum_{k=1, k\neq j}^{n-2} d_{jn}b_k e_k^2 = 0. \tag{3.6}$$

Consequently, (3.5) and (3.6) result that $d_{jn}b_k = 0$, $d_{nj} + d_{jn}b_j = 0$, $d_{jn-1}b_k = 0$ and $d_{n-1j} + d_{jn-1}b_j = 0$ for all $1 \leqslant k \neq j \leqslant n-2$.

It can be observed from (3.3) that there are different possible values for $b_k$. Therefore, (3.3) can be rewritten by different formulations. In this work, we are going to consider three different values for $b_k$, which are used as one of a main conditions in the following theorems. In this regard, we introduce the following example.

**Example 3.1.** Let $E$ be an evolution algebras with the following structural matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & \dots & a_{15} & a_{16} \\ a_{21} & a_{22} & \dots & \dots & a_{25} & a_{26} \\ a_{31} & a_{32} & \dots & \dots & a_{25} & a_{26} \\ a_{41} & a_{42} & \dots & \dots & a_{45} & a_{46} \\ a_{51} & a_{52} & \dots & \dots & a_{55} & a_{56} \\ a_{61} & a_{62} & \dots & \dots & a_{65} & a_{66} \end{pmatrix},$$

that has rank $n-2$.

As discussed, $e_5^2$ and $e_6^2$ can be rewritten as arbitrary linear combination. For instance, the following is obtainable:

(i) Let $e_6^2 = b_1e_1^2 + b_3e_3^2$ and $e_5^2 = b_2e_2^2 + b_4e_4^2$ such that $b_i \neq 0$ for all $i \in \{1,2,3,4\}$, which means that $e_6^2 = b_1(a_{11}e_1 + \cdots + a_{16}e_6) + b_3(a_{31}e_1 + \cdots + a_{36}e_6)$ and $e_5^2 = b_2(a_{21}e_1 + \cdots + a_{26}e_6) + b_4(a_{41}e_1 + \cdots + a_{46}e_6)$. This satisfies the rank of the structural matrix $A$ and illustrates the main condition of Theorem 3.1.

(ii) Let $e_6^2 = b_1e_1^2$ and $e_5^2 = b_2e_2^2$ such that $b_i \neq 0$ for all $i \in \{1,2\}$, which means that $e_6^2 = b_1(a_{11}e_1 + \cdots + a_{16}e_6)$ and $e_5^2 = b_2(a_{21}e_1 + \cdots + a_{26}e_6)$. This satisfies the rank of the structural matrix $A$ and illustrates the main condition of Theorem (3.2).

(iii) Let $e_5^2 = e_6^2 = 0$, which means that $a_{61} = \cdots = a_{66} = 0$ and $a_{51} = \cdots = a_{56} = 0$. This satisfies the rank of the structural matrix $A$ and illustrates the main condition of Theorem 3.3.

**Theorem 3.1.** *Let $E$ be an evolution algebras with structural matrix $A$ that has rank $n-2$ such that $e_n^2 \neq e_{n-1}^2$ and*

$$\mathbf{e}_i^2 = \begin{cases} \sum_{k=1}^{n-2} b_k e_k^2, & b_p \neq 0,\, b_q \neq 0,\; 1 \leqslant p \neq q \leqslant n,\; i = n; \\ \sum_{m=1}^{n-2} b_m e_m^2, & b_x \neq 0,\, b_y \neq 0,\; 1 \leqslant x \neq y \leqslant n,\; i = n-1. \end{cases}$$

*Then, the derivation $d$ is zero.*

*Proof.* Based on (3.5), (3.6) and meaning of assumption, $d_{jn}b_p = 0$, $d_{jn}b_q = 0$, $d_{jn-1}b_x = 0$ and $d_{jn-1}b_y = 0$ for all $1 \leqslant j \neq p \leqslant n-2$ can simply be obtained. Now, $d_{n-1j} + d_{jn-1}b_j = 0$ and $d_{nj} + d_{jn}b_j = 0$ follow that $d_{jn} = 0$, $d_{nj} = 0$, $d_{n-1j} = 0$ and $d_{jn-1} = 0$ for all $1 \leqslant j \leqslant n-2$, whereby indicates that $d = \mathrm{diag}\{d_{11}, d_{22}, \ldots, d_{nn}\}$. On the other hand, (3.2) gives that $2d_{nn}$ and $2d_{n-1n-1}$ are also eigenvalues of $d$. Hence, $\mathrm{spec}(d) = \{d_{11}, d_{22}, \ldots, d_{nn}\} = \{2d_{11}, 2d_{22}, \ldots, 2d_{nn}\}$, which can only be satisfied if $d = 0$. □

**Theorem 3.2.** *Let $E$ be an evolution algebras with structural matrix $A$ that has rank $n-2$ such that $e_n^2 = b_1e_1^2, b_1 \neq 0$, $e_{n-1}^2 = b_2e_2^2, b_2 \neq 0$ and $(d_{1n} \neq 0$, or $d_{2n-1} \neq 0)$. Then, the derivation $d$ can be expressed in one of the following forms:*

(i) $d = 0$, *if* $d_{1n} = d_{2n-1} = 0$;

(ii) $D_1$, *where* $d_{22} = \frac{\delta_1}{2^{m-(s-1)}-1}$, $d_{11} = \frac{\delta_2}{2^{n-(q+1)}-1}$, $\delta_2^2 = -b_1d_{1n}^2$ *and* $\delta_1^2 = -b_2d_{2n-1}^2$;

(iii) $D_2$, *where* $d_{22} = \frac{\delta_1}{2^{k-k_1+2}-1}$, $d_{33} = \frac{(1-2^{k-k_1+1})}{2^{k_1-2}}d_{22}$, $d_{11} = \frac{\delta_2}{2^{m-m_1+2}-1}$, $d_{k+1k+1} = \frac{(1-2^{m-m_1+1})}{2^{m_1-2}}d_{11}$, $\delta_2^2 = -b_1d_{1n}^2$ *and* $\delta_1^2 = -b_2d_{2n-1}^2$;

(iv) $D_3$, *where* $d_{11} = \delta_2$, $d_{22} = \delta_1$, $\delta_2^2 = -b_1d_{1n}^2$ *and* $\delta_1^2 = -b_2d_{2n-1}^2$.

*For more information see Appendix.*

*Proof.* Let $e_n^2 = b_1(e_1^2), b_1 \neq 0$ and $e_{n-1}^2 = b_2(e_2^2), b_2 \neq 0$. Now, (3.1) indicates the following: $d_{jn} = 0$ for all $j \in \{2, \ldots n-1\}$, $d_{n1} = -b_1d_{1n}$, $d_{jn-1} = 0$ for all $j \in \{2, \ldots n-2\} \cup \{n\}$ and $d_{n-12} = -b_2d_{2n-1}$. Insertion $n-1$ and $n$ instead of $i$ in (3.2) independently, the following can be established:

$$\begin{aligned} 2b_1d_{11}e_1^2 &= b_1de_1^2 = d(b_1e_1^2) = de_n^2 = 2d_{nn}e_n^2 = 2d_{nn}b_1e_1^2, \\ 2b_2d_{22}e_2^2 &= b_2d(e_2^2) = db_2e_2^2 = de_{n-1}^2 = 2d_{n-1n-1}e_{n-1}^2 = 2d_{n-1n-1}b_2e_2^2. \end{aligned}$$

This means that $d_{11} = d_{nn}$ and $d_{22} = d_{n-1n-1}$. Again, by using (3.2), the following is deduced:

$$a_{i1}(d_{11}e_1 + d_{1n}e_n) + a_{i2}(d_{22}e_2 + d_{2n-1}e_{n-1}) + \sum_{j=3}^{n-2} a_{ij}d_{jj}e_j + a_{in-1}(-b_2d_{2n-1}e_2 + d_{22}e_{n-1})$$

$$+ a_{in}(-b_1 d_{1n} e_1 + d_{11} e_n) = de_i^2 = 2d_{ii} e_i^2 = 2d_{ii} \sum_{j=1}^{n} a_{ij} e_j.$$

Which implies the following:

$$a_{i1}(2d_{ii} - d_{11}) = -a_{in} d_{1n} b_1, \tag{3.7}$$
$$a_{in}(2d_{ii} - d_{11}) = a_{i1} d_{1n}, \tag{3.8}$$
$$a_{i2}(2d_{ii} - d_{22}) = -a_{in-1} d_{2n-1} b_2, \tag{3.9}$$
$$a_{in-1}(2d_{ii} - d_{22}) = a_{i2} d_{2n-1}, \tag{3.10}$$
$$a_{ij}(2d_{ii} - d_{jj}) = 0, \tag{3.11}$$

for all $1 \leqslant i \leqslant n-2$ and $3 \leqslant j \leqslant n-2$.

Let $d_{1n} = d_{2n-1} = 0$. Then, $d = \{d_{11}, d_{22} \dots, d_{n-2n-2}\}$ whereby indicates that $d = 0$. Assume that both of $d_{1n}$ and $d_{2n-1}$ are not equal to zero. Thus, the set of all eigenvalues of $d$ is given by $\text{spec}(d) = \{d_{33}, \dots, d_{n-2n-2}, \alpha_1, \alpha_2, \beta_1, \beta_2\}$, where $\alpha_1 = d_{22} + \delta_1$, $\beta_1 = d_{22} - \delta_1$, $\delta_1^2 = -b_2 d_{2n-1}^2$, $\alpha_2 = d_{11} + \delta_2$, $\beta_2 = d_{11} - \delta_2$, $\delta_2^2 = -b_1 d_{1n}^2$. Let $\lambda \in \text{spec}(d)$ be such that $|\lambda| = \max\{|\alpha_1|, |\alpha_2|, |\beta_2|, |\beta_2|, |d_{33}|, \dots, |d_{n-2n-2}|\}$. If $\lambda \in \{d_{33}, \dots, d_{n-2n-2}\}$, then $2\lambda$ is also an eigenvalue, which contradicts module maximality of $\lambda$. Therefore, $\lambda = \alpha_1$, $\lambda = \beta_1$, $\lambda = \alpha_2$ or $\lambda = \beta_2$ are obtained. Evidently, (3.7), (3.8) and (3.9), (3.10) follow that $a_{i1} = 0$ ($a_{i2} = 0$) if and only if $a_{in} = 0$ ($a_{in-1} = 0$). Let $a_{i1} \neq 0$ and ($a_{i2} \neq 0$). Then, multiplying (3.7) in (3.8) and (3.9) in (3.10) implies that $(2d_{ii} - d_{11})^2 = -b_1 d_{1n}^2$ and $(2d_{ii} - d_{22})^2 = -b_2 d_{1n-1}^2$ or $2d_{ii} = d_{22} \pm \delta_1$ and $2d_{ii} = d_{11} \pm \delta_2$, respectively. Hence, for these $i$ we have

$$d_{ii} = \frac{1}{2}\alpha_1, \quad d_{ii} = \frac{1}{2}\beta_1, \quad d_{ii} = \frac{1}{2}\alpha_2 \text{ or } d_{ii} = \frac{1}{2}\beta_2. \tag{3.12}$$

Now, several cases are going to be examined.

Case 1. Let $\alpha_1\alpha_2\beta_1\beta_2 \neq 0$ such that $\alpha_1 + \beta_1 \neq 0$ and $\alpha_2 + \beta_2 \neq 0$. Clearly, $\alpha_1 + \beta_1 = 2d_{22}$, $\alpha_2 + \beta_2 = 2d_{11}$ and $2d_{11}, 2d_{22} \in \text{spec}(d)$ such that $\alpha_1 + \beta_1, \alpha_2 + \beta_2 \notin \{\alpha_1, \beta_1 \alpha_2, \beta_2\}$. This means that there exist $i_1$ and $j_1$ such that $d_{j_1 j_1} = \alpha_1 + \beta_1$ and $d_{i_1 i_1} = \alpha_2 + \beta_2$. Then, $2d_{i_1 i_1}, 2d_{j_1 j_1} \in \text{spec}(d)$, which implies that $2d_{j_1 j_1} = d_{j_2 j_2}$, $2d_{i_1 i_1} = d_{i_2 i_2}$ for some $i_2, j_2$ or $2d_{j_1 j_1} \in \{\alpha_1, \beta_1\}$ and $2d_{i_1 i_1} \in \{\alpha_2, \beta_2\}$. If $2d_{j_1 j_1} = d_{j_2 j_2}$ and $2d_{i_1 i_1} = d_{i_2 i_2}$, then the same process can be repeated until the following is obtained:

$$2^{k_1} d_{j_1 j_1} = \cdots = 2d_{j_{k_1} j_{k_1}} \in \{\alpha_1, \beta_1\} \text{ for some } 1 \leqslant k_1 \leqslant n-4,$$

$$2^{k_2} d_{i_1 i_1} = \cdots = 2d_{j_{k_2} i_{k_2}} \in \{\alpha_2, \beta_2\} \text{ for some } 1 \leqslant k_2 \leqslant n-4.$$

Therefore, $2^{k_1}(\alpha_1 + \beta_1) \in \{\alpha_1, \beta_1\}$ and $2^{k_2}(\alpha_2 + \beta_2) \in \{\alpha_2, \beta_2\}$ for some $1 \leqslant k_1, k_2 \leqslant n-4$ are obtained. For this, it can be possible to assume that $2^{k_1}(\alpha_1 + \beta_1) = \alpha_1$ and $2^{k_2}(\alpha_2 + \beta_2) = \alpha_2$. Thereby, the following can be concluded:

$$d_{22} = \frac{\alpha_1}{2^{k_1+1}}, \ d_{j_1 j_1} = \frac{\alpha_1}{2^{k_1}}, \dots, d_{j_{k_1} j_{k_1}} = \frac{\alpha_1}{2}, \quad \beta_1 = -(1 - \frac{1}{2^{k_1}})\alpha_1,$$
$$d_{11} = \frac{\alpha_2}{2^{k_2+1}}, \ d_{i_1 i_1} = \frac{\alpha_2}{2^{k_2}}, \dots, d_{i_{k_2} i_{k_2}} = \frac{\alpha_2}{2}, \quad \beta_2 = -(1 - \frac{1}{2^{k_2}})\alpha_2,$$

$|\beta_1| < |\alpha_1|$, $|\beta_2| < |\alpha_2|$ and $2^s \beta_m \neq 2^r \alpha_m$ for any $r, s \in Z, m = 1, 2$.

Let $d_1 < \cdots < d_m$ and $d_{m+1} < \cdots < d_p$ be the possible non-zero values of $|d_{33}|, \dots, |d_{n-2n-2}|$. Evidently, it is recognized that

$$\{\frac{|\alpha_1|}{2^{k_1}}, \dots, \frac{|\alpha_1|}{2}, \frac{|\alpha_2|}{2^{k_2}}, \dots, \frac{|\alpha_2|}{2}\} \subseteq \{d_1, \dots, d_p\}.$$

It is observable that $\{2d_{33}, \dots, 2d_{n-2n-2}\} \subseteq \text{spec}(d)\}$. Therefore, we deduce $2d_1, \dots, 2d_p \in \{d_1, \dots, d_p, |\alpha_1|, |\beta_1|, |\alpha_2|, |\beta_2|\}$. Accordingly, $2d_p \leqslant |\alpha_2|$ ($2d_m \leqslant |\alpha_1|$) and $d_{j_{k_1} j_{k_1}} = \frac{\alpha_1}{2}$ ($d_{i_{k_2} i_{k_2}} = \frac{\alpha_2}{2}$) are verified, which means that $d_p = \frac{|\alpha_2|}{2}$, and $d_m = \frac{|\alpha_1|}{2}$. Now, there is only possible one eigenvalue $d_{j_{k_1}j_{k_1}} = \frac{\alpha_1}{2}$ ($d_{i_{k_2}i_{k_2}} = \frac{\alpha_2}{2}$) with module $d_m$ ($d_p$). Actually, if we have $|d_{ii}| = d_m, d_{ii} \neq \frac{\alpha_1}{2}$ ($|d_{ii}| = \frac{\alpha_2}{2}$, $d_{ii} \neq \frac{\alpha_2}{2}$) for some $i$, then $\alpha_1 \neq 2d_{ii} \in \mathrm{spec}(d)$ ($\alpha_2 \neq 2d_{ii} \in \mathrm{spec}(d)$) and $|2d_{ii}| = |\alpha_1|$ ($|2d_{ii}| = |\alpha_2|$). Thus, there exists $j$ such that $d_{jj} = 2d_{ii}$ and $2d_{jj}$ must be in $\mathrm{spec}(d)$ and $|2d_{jj}| = 2|\alpha_1| > |\alpha_1|$ $|2d_{jj}| = 2|\alpha_2|$, which is a contradiction. Since there is only one eigenvalue $d_{j_{k_1}j_{k_1}} = \frac{\alpha_1}{2}$ ($d_{i_{k_2}i_{k_2}} = \frac{\alpha_2}{2}$) with module $\frac{1}{2}|\alpha_1|$ ($\frac{1}{2}|\alpha_2|$), one obtains that there is only one eigenvalue $d_{p-1}$ ($d_{m-1}$) of module $\frac{1}{4}\alpha_2$ ($\frac{1}{4}\alpha_1$) and etc. By applying the same process to $|\beta_1|$ and $|\beta_2|$, we obtain that there exist at most only one eigenvalue with module $\frac{1}{2}\beta_1$ ($\frac{1}{2}\beta_2$) and etc. Hence,

$$\{d_{33}, \dots, d_{n-2n-2}\} \setminus \{0\} = \bigcup_{i=1}^{s} \{\frac{1}{2^i}\alpha_1\} \bigcup_{j=1}^{r} \{\frac{1}{2^j}\alpha_2\}$$

or

$$\{d_{33}, \dots, d_{n-2n-2}\} \setminus \{0\} = \bigcup_{i=1}^{s_1} \{\frac{1}{2^i}\alpha_1\} \bigcup_{j=1}^{s_2} \{\frac{1}{2^j}\alpha_1\} \bigcup_{i_1=1}^{r_1} \{\frac{1}{2^{i_1}}\beta_1\} \bigcup_{j_1=1}^{r_2} \{\frac{1}{2^{j_1}}\beta_2\}.$$

Case 1.1. Let $\{d_{33}, \dots, d_{n-2n-2}\} \setminus \{0\} = \bigcup_{i=1}^{s} \{\frac{1}{2^i}\alpha_1\} \bigcup_{j=1}^{r} \{\frac{1}{2^j}\alpha_2\}$. Then, from (3.12) for these $i$ such that $a_{in} \neq 0$ and $a_{in-1} \neq 0$ we obtain $2d_{ii} = \alpha_1$ and $2d_{ii} = \alpha_2$ for different value of $i$. Accordingly, (3.8) and (3.10) make $a_{i1} = \frac{\alpha_2 - d_{11}}{d_{1n}} a_{in}$ and $a_{i2} = \frac{\alpha_1 - d_{22}}{d_{1n-1}} a_{in-1}$. Thereby, it indicates that $1^{st}$, $2^{nd}$ and $n^{th}$, $(n-1)^{th}$ columns of the structural matrix A are collinear, respectively. Thus, the $\mathrm{Rank}A = n-2$ is satisfied, i.e., all the other columns of the structural matrix A should be non-zero and linearly independent.

Assume that we have $s-2$ and $r-2$ zeros belong to $\{d_{33}, \dots, d_{mm}\}$ and $\{d_{m+1m+1}, \dots, d_{n-2n-2}\}$, respectively. Consequently, $\{d_{ii}\}_{n=3}^{s} \subseteq \{0\}$ and $\{d_{ii}\}_{n=m+1}^{q} \subseteq \{0\}$, $d_{ss} < |d_{s+1s+1}| \leqslant |d_{s+2s+2}| \leqslant \cdots \leqslant |d_{mm}|$ and $d_{qq} < |d_{q+1q+1}| \leqslant |d_{q+2q+2}| \leqslant \cdots \leqslant |d_{n-2n-2}|$ where $q = (s-2)-(m+1)+1$ are obtained. For this, (3.11) indicates that $a_{ij} = 0$ if the following is satisfied: $3 \leqslant i \leqslant s$ and $s+1 \leqslant j \leqslant m, s+1 \leqslant i \leqslant m$ and $3 \leqslant j \leqslant s, m+1 \leqslant i \leqslant q$ and $q+1 \leqslant j \leqslant n-2, q+1 \leqslant i \leqslant n-2$ and $m+1 \leqslant j \leqslant q, 3 \leqslant i \leqslant s$ and $q+1 \leqslant j \leqslant n-2, s+1 \leqslant i \leqslant m$ and $m+1 \leqslant j \leqslant q$.

Given that the $d_{s+1s+1} \neq 2d_{ii}$, $d_{q+1q+1} \neq 2d_{ii}$ for all $3 \leqslant i \leqslant n-2$. Therefore, (3.11) implies that $a_{iq+1} = 0$ and $a_{is+1} = 0$ for all $3 \leqslant i \leqslant n-2$. This means that $d_{q+1q+1} = 2d_{22}$ and $d_{s+1s+1} = 2d_{11}$. Otherwise, we obtain $(q+1)^{th}$ and $(s+1)^{th}$ columns are equal to zero, which contradicts $\mathrm{Rank}A = n-2$.

Now, $\{d_{s+1s+1}, \dots, d_{mm}\}$ and $\{d_{q+1q+1}, \dots, d_{n-2n-2}\}$ do not contain equal elements are going to be shown. Suppose that there exist equal elements namely $d_{q+2q+2}$ and $d_{s+2s+2}$ are equal to $d_{q+1q+1}$ and $d_{s+1s+1}$, respectively. Firstly, let us start when $d_{s+1s+1} = d_{s+2s+2}$. Certainly, $d_{s+2s+2} \neq d_{ii}$ for all $3 \leqslant i \leqslant n-2$, and hence $a_{is+2} = 0$ for all $3 \leqslant i \leqslant n-2$ is deduced. Therefore, the column $(s+2)^{th}$ should be either zero or collinear to the column $(s+1)^{th}$ of the structural matrix A, which is a contradiction. Suppose that $d_{jj}$ is equal to $d_{j+1j+1}$ for some $s+2 < j \leqslant m-1$. Then, $2d_{ii} - d_{jj} = 0$ if and only if $i = j-1$ and therefore (3.11) implies that $a_{ij} = 0$ for all $i \neq j-1$. In the same manner, $2d_{ii} - d_{j+1j+1} = 0$ if and only if $i = j-1$ and therefore (3.11) yields that $a_{ij+1} = 0$ for all $i \neq j-1$. This makes the columns $j^{th}$ and $(j+1)^{th}$ are collinear or at least one of them is zero, which contradicts $\mathrm{Rank}A = n-2$. By using the same process on $\{d_{q+1q+1}, \dots, d_{n-2n-2}\}$, one can show that all elements have different values. Hence, there are no equal elements among $\{d_{s+1s+1}, \dots, d_{mm}\}$ and $\{d_{q+1q+1}, \dots, d_{n-2n-2}\}$. Furthermore, we immediately conclude that $2^{m-(q-1)}d_{22} = 2^{m-(q-1)-1}d_{q+1q+1} = \cdots = 2d_{mm} = \alpha_1$ and $2^{n-(s+1)}d_{11} = 2^{n-(s+1)-1}d_{s+1s+1} = \cdots = 2d_{n-2n-2} = \alpha_2$. After simple calculation, we obtain that $d_{22} = \frac{\delta_1}{2^{m-(q-1)}-1}$, $d_{11} = \frac{\delta_2}{2^{n-(s+1)}-1}$. Finally, the structural matrix A should be in $A_1$ form (see Appendix).

The derivation of evolution algebras should be in the form $D_1$ with $d_{22} = \frac{\delta_1}{2^{m-(q-1)}-1}$, and $d_{11} = \frac{\delta_2}{2^{n-(s+1)}-1}$.

Case 1.2. Let $\{d_{33},\dots,d_{n-2n-2}\}\setminus\{0\}=\bigcup_{i=1}^{s_1}\{\frac{1}{2^i}\alpha_1\}\bigcup_{j=1}^{s_2}\{\frac{1}{2^j}\alpha_2\}\bigcup_{i_1=1}^{r_1}\{\frac{1}{2^{i_1}}\beta_1\}\bigcup_{j_1=1}^{r_2}\{\frac{1}{2^{j_1}}\beta_2\}$. To begin, assume that

$$\{d_{33},\dots,d_{k_1k_1}\}=\bigcup_{i_1=1}^{r_1}\{\frac{1}{2^{i_1}}\beta_1\},\ \{d_{k_1+1k_1+1},\dots,d_{kk}\}=\bigcup_{i=1}^{s_1}\{\frac{1}{2^i}\alpha_1\},$$
$$\{d_{k+1k+1},\dots,d_{m_1m_1}\}=\bigcup_{j_1=1}^{r_2}\{\frac{1}{2^{j_1}}\beta_2\},\ \{d_{m_1+1m_1+1},\dots,d_{mm}\}=\bigcup_{j=1}^{s_2}\{\frac{1}{2^j}\alpha_2\}$$

and $d_{m+1m+1}=\dots=d_{n-2n-2}=0$ such that $|d_{33}|\leqslant\dots\leqslant|d_{k_1k_1}|$, $|d_{k_1+1k_1+1}|\leqslant\dots\leqslant|d_{kk}|$, $|d_{k+1k+1}|\leqslant\dots\leqslant|d_{m_1m_1}|$, $|d_{m_1+1m_1+1}|\leqslant\dots\leqslant|d_{mm}|$.

Accordingly, $2d_{ii}-d_{jj}\neq 0$ for all $1\leqslant i\leqslant n$ such that $j\in\{3,k+1\}$ is deduced. According to (3.11) and the following facts: $a_{in}=b_1a_{i1}$ and $a_{in-1}=b_2a_{i2}$, we conclude that $a_{i3}$ and $a_{ik+1}$ have to be zero for all $1\leqslant i\leqslant n$. Moreover, the remainder of columns of the structural matrix A should be non-zero and linearly independent, i.e., $\mathrm{Rank}A=n-2$ is satisfied. Similarly, as in Case 1.1, we get that $d_{44}\neq d_{33}$ and so on. Hence,

$$d_{k_1k_1}=2d_{k_1-1k_1-1}=\dots=2^{k_1-3}d_{33}, \tag{3.13}$$
$$d_{m_1m_1}=2d_{m_1-1m_1-1}=\dots=2^{m_1-3}d_{k+1}. \tag{3.14}$$

This means $a_{j-1j}\neq 0$ and $a_{ij}=0\,(i\neq j-1)$ for all $4\leqslant j\leqslant k_1$ and $(k+1)+1\leqslant j\leqslant m_1$.

Now, it is recognized that $d_{k_1+1k_1+1}$ and $d_{m_1+1m_1+1}$ are in form $\frac{1}{2^{s_1}}\alpha_1$ and $\frac{1}{2^{s_2}}\alpha_2$, respectively. On the other hand, $2d_{ii}-d_{jj}\neq 0$ for all $3\leqslant i\leqslant n-2$ such that $j\in\{k_1+1,m_1+1\}$. Thereby, $d_{k_1+1k_1+1}=2d_{22}$ and $d_{m_1+1m_1+1}=2d_{11}$ are inferred. Otherwise, the columns $(k_1+1)^{\text{th}}$ and $(m_1+1)^{\text{th}}$ become zero, which contradicts $\mathrm{Rank}A=n-2$. By using the same way used in Case 1.1, the following can be concluded:

$$d_{kk}=2d_{k-1k-1}=\dots=2^{k-k_1}d_{k_1+1k_1+1}=2^{k-k_1+1}d_{22}, \tag{3.15}$$

$$d_{mm}=2d_{m-1m-1}=\dots=2^{m-m_1}d_{m_1+1m_1+1}=2^{m-m_1+1}d_{11}. \tag{3.16}$$

This means $a_{j-1j}\neq 0$ and $a_{ij}=0\,(i\neq j-1)$ for all $k_1+1\leqslant j\leqslant k$ and $(m_1+1)+1\leqslant j\leqslant m$.

Furthermore, from (3.11) and (3.12), we obtain that $a_{ij}=0$ for all $1\leqslant i\leqslant m$ and $m+1\leqslant j\leqslant n-2$. In addition, we have $a_{nj}=b_1a_{1j}$ and $a_{n-1j}=b_2a_{2j}$. This leads to the inference that $a_{nj}=a_{n-1j}=0$ for $m+1\leqslant j\leqslant n$. Hence, $d_{k_1k_1}=\frac{1}{2}\beta_1$, $d_{kk}=\frac{1}{2}\alpha_1$, $d_{m_1m_1}=\frac{1}{2}\beta_2$ and $d_{mm}=\frac{1}{2}\alpha_2$. From (3.8), (3.10) and (3.12), the following can be found: $a_{k_1n-1}=\frac{d_{2n-1}}{\beta_1-d_{22}}a_{k_12}\neq 0$, $a_{kn-1}=\frac{d_{2n-1}}{\alpha_1-d_{22}}a_{k2}\neq 0$, $a_{m_1n}=\frac{d_{1n}}{\beta_2-d_{11}}a_{m_11}\neq 0$, $a_{mn}=\frac{d_{1n}}{\alpha_2-d_{11}}a_{m1}\neq 0$.

Now, (3.13), (3.14), (3.15), (3.16) and simple calculation imply the following: $d_{22}=\frac{\delta_1}{2^{k-k_1+2}-1}$, $d_{33}=\frac{(1-2^{k-k_1+1})}{2^{k_1-3}}d_{22}$, $d_{11}=\frac{\delta_2}{2^{m-m_1+2}-1}$, $d_{k+1k+1}=\frac{(1-2^{m-m_1+1})}{2^{m_1-3}}d_{11}$. Therefore, the structural matrix A should be in $A_2$ form (see Appendix).

Case 2. Let $\alpha_1\beta_1\alpha_2\beta_2\neq 0$ such that $\alpha_1=-\beta_1$ and $\alpha_2=-\beta_2$, i.e., $d_{11}=d_{22}=0$. Noticeably, it is impossible of presence zero element among $\{d_{33},\dots,d_{n-2n-2}\}$. To achieve this, suppose that we have zero element among $\{d_{33},\dots,d_{n-2n-2}\}$. Then, (3.7), (3.8), (3.9) and (3.10) show that the $1^{\text{st}}$, $2^{\text{nd}}$, $(n-1)^{\text{th}}$ and $n^{\text{th}}$ columns should be zero, which is a contradiction. Therefore, $0\notin\{d_{ii}\}_{n=3}^{n-2}$ is obtained. Now, consider $d_1<\dots<d_p$ and $d_{p+1}<\dots<d_m$ are the possible non-zero values of $|d_{33}|,\dots,|d_{n-2n-2}|$. As conclusion of $\{2d_{33},\dots,2d_{n-2n-2}\}\subseteq\mathrm{spec}(d)$, we conclude that $2d_1,\dots,2d_m,\in\{d_1,\dots,d_m,|\alpha_1|,|\alpha_2|\}$. Moreover, accordance with all these values which are not equal to zero, the following is derived:

$$|\alpha_1|=2d_p, d_p=2d_{p-1},\dots,d_2=2d_1,\quad |\alpha_2|=2d_m, d_m=2d_{m-1},\dots,d_{p+2}=2d_{p+1}.$$

By performing the same way which used in the Case 1.1, we deduce that $\pm\frac{1}{2}\alpha_1$ and $\pm\frac{1}{2}\alpha_2$ are all possible eigenvalues with modules $d_p$ and $d_m$, respectively, and thus for all the other eigenvalues. Hence,

$$\{d_{33},\dots,d_{n-2n-2}\}\setminus\{0\}\subseteq\bigcup_{i_1=1}^{s_1}\{\frac{1}{2^{i_1}}\alpha_1\}\bigcup_{i_2=1}^{s_2}\{-\frac{1}{2^{i_2}}\alpha_1\}\bigcup_{j_1=1}^{r_1}\{\frac{1}{2^{j_1}}\alpha_2\}\bigcup_{j_2=1}^{r_2}\{-\frac{1}{2^{j_2}}\alpha_2\}.$$

Let $\frac{1}{2}\alpha_1,\frac{1}{2}\alpha_2\notin\{d_{33},\dots,d_{n-2n-2}\}$ and $-\frac{1}{2}\alpha_2,-\frac{1}{2}\alpha_2\notin\{d_{33},\dots,d_{n-2n-2}\}$. Then, (3.8), (3.10) and (3.12) indicate that $1^{(st)}$, $2^{(nd)}$, $(n-1)^{(th)}$ and $(n)^{(th)}$ are zero columns, which contradicts $\mathrm{Rank}A=n-2$. Hence, there exists $k_1,k_2$ such that $d_{k_1k_1},\in\{\frac{1}{2}\alpha_1,-\frac{1}{2}\alpha_1\}$ and $d_{k_2k_2}\in\{\frac{1}{2}\alpha_2,-\frac{1}{2}\alpha_2\}$ for all $3\leqslant k_1,k_2\leqslant n-2$. Now, suppose that $\{d_{33},\dots,d_{pp}\}\setminus\{0\}\supseteq\bigcup_{i_1=1}^{s_1}\{\frac{1}{2^i}\alpha_1\}$, $\{d_{p+1p+1},\dots,d_{mm}\}\setminus\{0\}\supseteq\bigcup_{i_2=1}^{s_2}\{\frac{1}{2^i}\alpha_2\}$ and $-\frac{1}{2}\alpha_1,-\frac{1}{2}\alpha_2\notin\{d_{33},\dots,d_{n-2n-2}\}$. Based on (3.8), (3.10) and (3.12), we immediately conclude that the $1^{st}$, $2^{nd}$ and $(n-1)^{th}$, $(n)^{th}$ columns of the structural matrix $A$ are linearly dependent. Thus, the first and second columns of the structural matrix $A$ can be expressed as follows: $a_{i1}=\frac{\alpha_2}{d_{1n}}a_{in}$, $a_{i2}=\frac{\alpha_1}{d_{2n-1}}a_{in-1}$ for all $1\leqslant i\leqslant n$. Accordingly, $\mathrm{Rank}A=n-2$ is satisfied , i.e., the remainder columns of the structural matrix $A$ are linearly independent and non-zero. However, assumption shows that there exist $d_{pp}=\frac{1}{2^{s_1}}\alpha_1$, $d_{mm}=\frac{1}{2^{s_2}}\alpha_2$. On the basis of this fact and (3.11), the $p^{th}$ and $m^{th}$ columns must be zero columns, which is a contradiction to $\mathrm{Rank}A=n-2$. By applying the same process on $\{d_{33},\dots,d_{pp}\}\setminus\{0\}\supseteq\bigcup_{i_1=1}^{s_1}\{-\frac{1}{2^i}\alpha_1\}$, $\{d_{p+1p+1},\dots,d_{mm}\}\setminus\{0\}\bigcup_{i_2=1}^{s_2}\{-\frac{1}{2^i}\alpha_2\}$ and $\frac{1}{2}\alpha_1,\frac{1}{2}\alpha_2\notin\{d_{33},\dots,d_{n-2n-2}\}$, we deduce a contradiction to $\mathrm{Rank}A=n-2$.

Suppose that $\{d_{33},\dots,d_{n-2n-2}\}\setminus\{0\}=\bigcup_{i_1=1}^{s_1}\{\frac{1}{2^{i_1}}\alpha_1\}\bigcup_{i_2=1}^{s_2}\{\frac{-1}{2^{i_2}}\alpha_1\}\bigcup_{j_1=1}^{r_1}\{\frac{1}{2^{j_1}}\alpha_2\}\bigcup_{j_2=1}^{r_2}\{\frac{-1}{2^{j_2}}\alpha_2\}$. Then, there exists $d_{pp}=\frac{1}{2^{s_1}}\alpha_1$, $d_{p_1p_1}=-\frac{1}{2^{s_2}}\alpha_1$, $d_{q_1q_1}=-\frac{1}{2^{r_1}}\alpha_2$ and $d_{q_1q_1}=\frac{-1}{2^{r_2}}\alpha_2$. So, (3.11) indicates $p^{th}$, $p_1^{th}$, $q^{th}$ and $q_1^{th}$ columns must be zero, which contradicts $\mathrm{Rank}A=n-2$. Hence, this case is impossible.

Case 3. Let $\alpha_1\alpha_2\neq 0$ such that $\beta_1=\beta_2=0$ i.e. $d_{11}=\delta_2$ and $d_{22}=\delta_1$. Let $d_1<\cdots<d_m$ and $d_{m+1}<\cdots<d_p$ be the possible non-zero values of $|d_{33}|,\dots,|d_{n-2n-2}|$. Plainly, $2d_1,\dots,2d_p\in\{d_1,\dots,d_p,|\alpha_1|,|\alpha_2|\}$. Based on these values which are non-zero, we obtain that

$$|\alpha_1|=2d_m, d_m=2d_{m-1},\dots,d_2=2d_1,\quad |\alpha_2|=2d_p, d_p=2d_{p-1},\dots,d_{m+2}=2d_{m+1}.$$

To avoid repetitions, we refer that applying the same argument used in proof Case 1.1 in order to prove that there is only one eigenvalue $\frac{\alpha_1}{2}$ with module $d_m$ can be reusable again. Hence,

$$\mathrm{spec}(d)=\{\frac{1}{2^p}\alpha_2,\dots,\frac{1}{2}\alpha_2,\alpha_2,\frac{1}{2^m}\alpha_1,\dots,\frac{1}{2}\alpha_1,\alpha_1\},\quad \mathrm{spec}(d)=\{0,\frac{1}{2^p}\alpha_2,\dots,\frac{1}{2}\alpha_2,\alpha_2,\frac{1}{2^m}\alpha_1,\dots,\frac{1}{2}\alpha_1,\alpha_1\}.$$

By performing a suitable basis permutation, we assume that $|d_{33}|\leqslant\cdots\leqslant|d_{mm}|$, $|d_{m+1m+1}|\leqslant\cdots\leqslant|d_{n-2n-2}|$. Now, assume that there exist $s-2$ and $r-2$ zeros among $\{d_{33},\dots,d_{mm}\}$, $\{d_{m+1m+1},\dots,d_{n-2}\}$, respectively. Then, $\{d_{ii}\}_{n=3}^{s}\subseteq\{0\}$ and $\{d_{ii}\}_{n=m+1}^{q}\subseteq\{0\}$, $d_{ss}<|d_{s+1s+1}|\leqslant|d_{s+2s+2}|\leqslant\cdots\leqslant|d_{mm}|$ and $d_{qq}<|d_{q+1q+1}|\leqslant|d_{q+2q+2}|\leqslant\cdots\leqslant|d_{n-2n-2}|$ where $q=(r-2)-(m+1)+1$ are deduced. Consequently, (3.11) makes $a_{ij}=0$ if the following is satisfied:
$1\leqslant i\leqslant s$ and $s+1\leqslant j\leqslant m$, $s+1\leqslant i\leqslant m$ and $1\leqslant j\leqslant s, m+1\leqslant i\leqslant q$ and $q+1\leqslant j\leqslant n-2$, $q+1\leqslant i\leqslant n-2$ and $m+1\leqslant j\leqslant q$, $1\leqslant i\leqslant s$ and $q+1\leqslant j\leqslant n-2$, $s+1\leqslant i\leqslant m$ and $m+1\leqslant j\leqslant q$.

Now, accordance with $d_{q+1q+1}\neq 2d_{ii}$ and $d_{s+1s+1}\neq 2d_{ii}$ for all $s+1\leqslant i\leqslant m$, $q+1\leqslant i\leqslant n-2$ and (3.11), one can see that $a_{is+1}=0$ $a_{iq+1}=0$ for all $s+1\leqslant i\leqslant n-2$, $q+1\leqslant i\leqslant n-2$, respectively. Thus, the rank of the structural matrix $A$ is satisfied, i.e., the other columns should be non-zero. By following the same process that used in Case 1.1, one shows that $\{d_{s+1s+1},\dots,d_{mm}\}$ and $\{d_{q+1q+1},\dots,d_{n-2n-2}\}$ do not contain equal elements. Therefore, $\{d_{s+1s+1}, \dots, d_{mm}\}$ and $\{d_{q+1q+1}, \dots, d_{n-2n-2}\}$ have distinct elements. In addition, (3.11) follows that $a_{i1} = a_{in} = 0$, $a_{i2} = a_{in-1} = 0$ for all $s+1 \leqslant i \leqslant m$, $s+1 \leqslant i \leqslant n-2$, respectively. Hence, the structural matrix A should be in in $A_3$ form (see Appendix).

Note that in a symmetrical case such that $\alpha_1 = 0, \beta_1 \neq 0$ and $\alpha_2 = 0, \beta_2 \neq 0$. One can obtain in a similar way that d is in the form $D_3$ with $d_{22} = -\delta_1$ and $d_{11} = -\delta_2$. □

**Theorem 3.3.** *Let E be an evolution algebras with structural matrix A that has rank $n-2$ such that $e_n^2 = e_{n-1}^2 = 0$. Then, the derivation d equals to zero or in one of the following forms:*

(i)

$$\begin{pmatrix} 0 & \dots & 0 & d_{1n-1} & d_{1n} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & d_{n-2n-1} & d_{n-2n} \\ 0 & \dots & 0 & d_{n-1n-1} & d_{n-1n} \\ 0 & \dots & 0 & d_{nn-1} & d_{nn} \end{pmatrix}, \tag{$D_4$}$$

*where* $\sum_{k=1}^{n} a_{ik} d_{kn-1} = 0$, $\sum_{k=1}^{n} a_{ik} d_{kn} = 0$ *for all* $1 \leqslant i \leqslant n-2$;

(ii)

$$\begin{pmatrix} \frac{\alpha}{2^{k-1}} & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & d_{1n-1} & d_{1n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & \dots & \alpha & 0 & \dots & 0 & 0 & \dots & 0 & d_{kn-1} & d_{kn} \\ 0 & \dots & 0 & \frac{\beta}{2^{m-(k+1)}} & \dots & 0 & 0 & \dots & 0 & d_{k+1n-1} & d_{k+1n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \dots & \beta & 0 & \dots & 0 & d_{mn-1} & d_{mn} \\ 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & d_{n-1n-1} & d_{n-1n} \\ 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & d_{nn-1} & d_{nn} \end{pmatrix}, \tag{$D_5$}$$

*where*

$$\begin{aligned} d_{i+1n} &= \frac{1}{a_{ii+1}} \left( a_{in} \left( \frac{\alpha}{2^{k-i}} - d_{nn} \right) - a_{in-1} d_{n-1n} \right), \quad 1 \leqslant i \leqslant k-1, \\ d_{i+1n-1} &= \frac{1}{a_{ii+1}} \left( a_{in-1} \left( \frac{\beta}{2^{m-i}} - d_{n-1n-1} \right) - a_{in} d_{nn-1} \right), \quad k+1 \leqslant i \leqslant m-1, d_{in-1}, d_{in} \in \mathbb{C} \end{aligned}$$

*for all* $1 \leqslant i \leqslant m$;

(iii)

$$\begin{pmatrix} 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & \dots & 0 & \frac{\alpha}{2^{(n-2)-(k+1)}} & \dots & 0 & d_{k+1n-1} & d_{k+1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \dots & \frac{\alpha}{2} & d_{n-2n-1} & d_{n-2n} \\ 0 & \dots & 0 & 0 & \dots & 0 & d_{n-1n-1} & d_{n-1n} \\ 0 & \dots & 0 & 0 & \dots & 0 & d_{nn-1} & d_{nn} \end{pmatrix}, \tag{$D_6$}$$

*where*

$$d_{i+1n-1} = \frac{a_{in-1}(2d_{ii} - d_{n-1n-1}) + a_{in}d_{nn-1}}{a_{ii+1}}, \quad a_{ii+1} \neq 0,$$
$$d_{i+1n} = \frac{a_{in}(2d_{ii} - d_{nn}) + a_{in-1}d_{n-1n}}{a_{ii+1}}, \quad a_{ii+1} \neq 0, \quad k+1 \leqslant i \leqslant n-2, \quad 1 \leqslant k \leqslant n-1,$$

*and* $d_{k+1n-1}, d_{k+1n} \in \mathbb{C}$.

*Proof.* Let $e_n^2 = e_{n-1}^2 = 0$. Then, (3.1) implies that $d_{nj} = 0$ and $d_{n-1j} = 0$ for all $1 \leqslant j \leqslant n-2$. Now, a derivation of this case based on the possible eigenvalues are going to be described. To begin, the set of all eigenvalues of $d$ is given by $\mathrm{spec}(d) = \{d_{11}, \dots, d_{n-2n-2}, \alpha, \beta\}$ where

$$\alpha = \frac{1}{2}(\sqrt{(d_{n-1n-1} - d_{nn})^2 + 4d_{n-1n-1}d_{nn}} + d_{n-1n-1} + d_{nn}),$$
$$\beta = \frac{-1}{2}(\sqrt{(d_{n-1n-1} - d_{nn})^2 + 4d_{n-1n-1}d_{nn}} + d_{n-1n-1} + d_{nn}).$$

Case 1. Let $\lambda \in \{d_{11}, \dots, d_{n-2n-2}\}$. Since $\mathrm{spec}(d) \supseteq \{2d_{11}, 2d_{22}, \dots, 2d_{n-2n-2}\}$, we obtain that $2\lambda \in \mathrm{spec}(d)$. This implies that $\lambda = 0$, which means that $d_{11} = \cdots = d_{n-2n-2} = \alpha = \beta = 0$ and $d(e_i) = d_{in-1}e_{n-1} + d_{in}e_n$ for all $1 \leqslant i \leqslant n-2$. Now, (3.2) yields $\sum_{j=1}^{n} a_{ij}d_{jn}e_n = 0$ and $\sum_{j=1}^{n} a_{ij}d_{jn-1}e_{n-1} = 0$ for all $1 \leqslant i \leqslant n-2$. Therefore, the two vectors $(d_{1n}, \dots, d_{n-2n}, d_{n-1n}, d_{nn})$, $(d_{1n-1}, \dots, d_{n-2n-1}, d_{n-1n-1}, d_{nn-1})$ are roots of homogeneous linear system of equations $Ax = 0$ and $Ay = 0$, respectively. Obviously, if the first $n-2$ columns are linearly independent, then $d = 0$. To obtain non-zero deviation, consider the first $n-2$ columns of the structural matrix $A$ are linearly dependent and denote the new matrix by $(A_4)$. Hence, $d$ is represented by $D_4$.

Case 2. Let $\lambda \notin \{d_{11}, \dots, d_{n-2n-2}\}$. Then, $\lambda = \alpha \neq 0$ or $\lambda = \beta \neq 0$. Let $d_1 < \cdots < d_{p_1}, d_{p_1+1} < \cdots < d_p$ be the possible non-zero values of $|d_{11}|, \dots, |d_{n-2n-2}|$. Clearly, $\{2d_{11}, \dots, 2d_{n-2n-2}\} \subseteq \mathrm{spec}(d)$, which indicates that $2d_1, \dots, 2d_{p_1}, 2d_{p_1+1}, \dots, 2d_p \in \{d_1, \dots, 2d_{p_1}, d_{p_1+1}, \dots, d_p, |\alpha|, |\beta|\}$. Accordingly, we conclude that $|\alpha| = 2d_{p_1}, d_{p_1} = 2d_{p_1-1}, \dots, d_2 = 2d_1$, $|\beta| = 2d_p, d_p = 2d_{p-1}, \dots, d_{p_1+2} = 2d_{p_1+1}$ provided that all eigenvalues are in the form $\alpha$ or $\beta$, and $|\alpha| = 2d_p, d_p = 2d_{p-1}, \dots, d_2 = 2d_1$ when all eigenvalues are in the form $\alpha$. In the same manner which used in Case 1.1, one can show that there exists one eigenvalue $\frac{1}{2}\alpha$ $(\frac{1}{2}\beta)$ with module $d_{p_1}(d_p)$ and so on. Hence,

$$\mathrm{spec}(d) = \{\frac{\alpha}{2^{p_1}}, \dots, \frac{\alpha}{2}, \alpha\} \text{ or } \mathrm{spec}(d) = \{0, \frac{\alpha}{2^{p_1}}, \dots, \frac{\alpha}{2}, \alpha\},$$
$$\mathrm{spec}(d) = \{\frac{\alpha}{2^{p_1}}, \dots, \frac{\alpha}{2}, \alpha, \frac{\beta}{2^{p}}, \dots, \frac{\beta}{2}, \beta\} \text{ or } \mathrm{spec}(d) = \{0, \frac{\alpha}{2^{p_1}}, \dots, \frac{\alpha}{2}, \alpha, \frac{\beta}{2^{p}}, \dots, \frac{\beta}{2}, \beta\}.$$

By performing an appropriate basis permutation, we assume that $|d_{11}| \leqslant \cdots \leqslant |d_{kk}| < |\alpha|$, $|d_{k+1k+1}| \leqslant \cdots \leqslant |d_{n-2n-2}| < |\beta|$. Now, (3.2) follows that $\sum_{j=1}^{n-2} a_{ij}d_{jj}e_j + \sum_{j=1}^{n} a_{ij}d_{jn-1}e_{n-1} + \sum_{j=1}^{n} a_{ij}d_{jn}e_n = d(e_i^2) = 2d_{ii}(e_i^2) = 2d_{ii}\sum_{j=1}^{n} a_{ij}e_j$. This means the following:

$$\sum_{j=1}^{n} a_{ij}d_{jn} = 2d_{ii}a_{in}, \quad \sum_{j=1}^{n} a_{ij}d_{jn-1} = 2d_{ii}a_{in-1}, \tag{3.17}$$
$$a_{ij}(2d_{ii} - d_{jj}) = 0, \quad \textit{for all } 1 \leqslant i, j \leqslant n-2. \tag{3.18}$$

Case 2.1. Let

$$\{d_{11}, \dots, d_{kk}\} = \bigcup_{i=0}^{s}\{\frac{1}{2^i}\alpha\}, \quad \{d_{k+1k+1}, \dots, d_{mm}\} = \bigcup_{j=0}^{r}\{\frac{1}{2^j}\beta\},$$

and $d_{m+1m+1} = \cdots = d_{n-2n-2} = 0$. Accordingly, (3.18) follows that $a_{ij} = 0$ if $1 \leqslant i \leqslant k$ and $k+1 \leqslant j \leqslant m$, $1 \leqslant i \leqslant k$ and $m+1 \leqslant j \leqslant n-2$, $k+1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant k$, $k+1 \leqslant i \leqslant m$ and $m+1 \leqslant j \leqslant n-2$. The consequence of using $d_{ii} \neq 2d_{jj}$ for all $k+1 \leqslant i \leqslant n-2$, $j \in \{1, k+1\}$ and (3.18) is that $a_{ik+1} = a_{i1} = 0$ for all $k+1 \leqslant i \leqslant n-2$. Therefore, the $1^{(st)}$ and $(k+1)^{th}$ columns of the structural matrix $A$ are zero. Thus, Rank$A$ is satisfied, i.e., the other columns must be non-zero and linearly independent. The used method for proving existence of different elements in Case 1.1 can be repeated to show that $d_{11}, \ldots, d_{n-2n-2}$ are different elements. Therefore,

$$d_{kk} = 2d_{k-1k-1}, \ldots, d_{22} = 2^{k-1}d_{11}, \quad d_{mm} = 2d_{m-1m-1}, \ldots, d_{k+2k+2} = 2^{m-(k+1)}d_{k+1k+1}.$$

From (3.18), we obtain $a_{ij} = 0$ for all $1 \leqslant i \leqslant m$, $j \neq i-1$. Hence, $d_{kk} = \frac{\alpha}{2}$ and $d_{mm} = \frac{\beta}{2}$ and so $d_{ii} = \frac{\alpha}{2^{k-i}}$ and $d_{ii} = \frac{\beta}{2^{m-i}}$ for all $1 \leqslant i \leqslant k$, $k+1 \leqslant i \leqslant m$, respectively. Finally, the structural matrix $A$ should be in the following form:

$$\begin{pmatrix}
0 & a_{12} & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & a_{1n-1} & a_{1n} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & a_{k-1k} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & a_{k-1n-1} & a_{k-1n} \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & a_{kn-1} & a_{kn} \\
0 & 0 & \cdots & 0 & 0 & a_{k+1k+2} & \cdots & 0 & & \cdots & 0 & 0a_{k+1n-1} & a_{k+1n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & a_{m-1m} & 0 & \cdots & 0 & a_{m-1n-1} & a_{m-1n} \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & a_{m-1n-1} & a_{m-1n} \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & a_{m+1m+1} & \cdots & a_{m+1n-2} & a_{m+1n-1} & a_{m+n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & a_{n-2m+1} & \cdots & a_{n-2n-2} & a_{n-2n-1} & a_{n-2n} \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0
\end{pmatrix} \quad (A_5)$$

Now, consider the $M \times M$ sub matrix $A_M = (a_{ij})_{m+1 \leqslant i,j \leqslant n-2}$ of the structural matrix $A$ where $M = ((n-2)-(m+1)+1)$. It is easy to see that (3.18) implies

$$\begin{pmatrix} a_{m+1m+1} & \cdots & a_{m+1n-2} \\ \vdots & & \vdots \\ a_{n-2m+1} & \cdots & a_{n-2n-2} \end{pmatrix} \begin{pmatrix} d_{m+1n-1} \\ \vdots \\ d_{n-2n-1} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} a_{m+1m+1} & \cdots & a_{m+1n-2} \\ \vdots & & \vdots \\ a_{n-2m+1} & \cdots & a_{n-2n-2} \end{pmatrix} \begin{pmatrix} d_{m+1n} \\ \vdots \\ d_{n-2n} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix},$$

and $a_{ii+1}d_{i+1n-1} + a_{in-1}d_{n-1n-1} = 2d_{ii}a_{in-1}$, $a_{ii+1}d_{i+1n} + a_{in}d_{nn} = 2d_{ii}a_{in}$ for all $1 \leqslant i \leqslant k-1$, $k+1 \leqslant i \leqslant m-1$, respectively and $a_{jn}d_{nn-1} = a_{jn-1}(2d_{jj} - d_{n-1-n-1})$, $a_{jn-1}d_{n-1n} = a_{jn}(2d_{jj} - d_{nn})$ for all $j \in \{k, m\}$.

Now, we already know that these columns of the sub matrix $A_M$ are linearly independent, which signifying that $d_{m+1n} = \cdots = d_{n-2n} = 0$ and $d_{m+1n-1} = \cdots = d_{n-2n-1} = 0$. Based on $a_{ii+1}d_{i+1n-1} = (2d_{ii} - d_{n-1n-1})$ and $a_{ii+1}d_{i+1n} = (2d_{ii} - d_{nn})$, one readily derives the following:

$$d_{i+1n} = \frac{1}{a_{ii+1}}\left(a_{in}\left(\frac{\alpha}{2^{k-i}} - d_{nn}\right) - a_{in-1}d_{n-1n}\right), \quad 1 \leqslant i \leqslant k-1$$

and

$$d_{i+1n-1} = \frac{1}{a_{ii+1}}\left(a_{in-1}\left(\frac{\beta}{2^{m-i}} - d_{n-1n-1}\right) - a_{in}d_{nn-1}\right), \quad k+1 \leqslant i \leqslant m-1.$$

Hence, the derivation $d$ of this sub case is in the form $D_5$.

**Case 2.2** Let $\{d_{22}, \ldots, d_{n-2n-2}\} \setminus \{0\} = \bigcup_{i=1}^{s} \{\frac{1}{2^i}\alpha\}$. In the same manner that used in the previous sub case, we can show that the structural matrix $A$ should be in the following form:

$$\begin{pmatrix}
a_{11} & \dots & a_{1k-1} & 0 & 0 & 0 & \dots & 0 & a_{1n-1} & a_{1n} \\
\vdots & \ddots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\
a_{k1} & \dots & a_{kk-1} & 0 & 0 & 0 & \dots & 0 & a_{kn-1} & a_{kn} \\
0 & \dots & 0 & 0 & 0 & a_{k+1k+2} & \dots & 0 & a_{k+1n-1} & a_{k+1n} \\
\vdots & & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & \dots & 0 & 0 & 0 & 0 & \dots & a_{n-2n-2} & a_{n-2n-1} & a_{n-2n} \\
0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0
\end{pmatrix}. \tag{$A_6$}$$

Whereupon, the derivation $d$ of this sub case is in the form $D_6$. □

## Appendix

$$\begin{pmatrix}
d_{11} & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & d_{1n} \\
0 & d_{22} & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & d_{2n-1} & 0 \\
0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \dots & \vdots & \vdots & \dots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 \\
0 & 0 & 0 & \dots & 0 & 2d_{22} & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \dots & \vdots & \vdots & \ddots & \vdots & \vdots & \dots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \dots & 0 & 0 & \dots & X & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 \\
0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \dots & \vdots & \vdots & \dots & \vdots & \vdots & \ddots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 \\
0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 2d_{11} & \dots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \dots & \vdots & \vdots & \dots & \vdots & \vdots & \dots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & Y & 0 & 0 \\
0 & -b_2 d_{2n-1} & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & d_{22} & 0 \\
-b_1 d_{1n} & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & d_{11}
\end{pmatrix} \tag{$D_1$}$$

where $X = 2^{m-(s-1)-1} d_{22}$, $Y = 2^{n-(q+1)-1} d_{11}$.

$$\begin{pmatrix}
d_{11} & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & d_{1n} \\
0 & d_{22} & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & d_{2n-1} & 0 \\
0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \dots & \vdots & \vdots & \dots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 \\
0 & 0 & 0 & \dots & 0 & X & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \dots & \vdots & \vdots & \ddots & \vdots & \vdots & \dots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \dots & 0 & 0 & \dots & d_{22} & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 \\
0 & 0 & \vdots & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \dots & \vdots & \vdots & \dots & \vdots & \vdots & \ddots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 \\
0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & Y & \dots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \dots & \vdots & \vdots & \dots & \vdots & \vdots & \dots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & d_{11} & 0 & 0 \\
0 & -b_2 d_{1n-1} & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & d_{22} & 0 \\
-b_1 d_{1n} & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & d_{11}
\end{pmatrix} \tag{$D_3$}$$

where $X = \frac{d_{22}}{2^{m-(s+1)}}$, $Y = \frac{d_{11}}{2^{(n-2)-(r-2+1)}}$.

$$\begin{pmatrix}
d_{11} & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & d_{1n} \\
0 & d_{22} & 0 & \dots & 0 & 0 & \vdots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & d_{2n-1} & 0 \\
0 & 0 & d_{33} & \dots & 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \dots & 2^{k_1-1}d_{33} & 0 & 0 & \dots & \dots & \vdots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 \\
0 & 0 & 0 & \dots & 0 & 2d_{22} & \dots & 0 & 0 & \vdots & \vdots & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \dots & 0 & 0 & \dots & 2^{k-k_1+1}d_{22} & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 \\
0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 2d_{k+1k+1} & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 2^{m_1-1}d_{k+1k+1} & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 \\
0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 2d_{11} & \dots & 0 & 0 & \dots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 2^{m-m_1+1}d_{11} & 0 & \dots & 0 & 0 & 0 \\
0 & 0 & 0 & \dots & 0 & 0 & \vdots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \dots & 0 & 0 & \vdots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 \\
0 & -b_2d_{2n-1} & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & d_{22} & 0 \\
-b_1d_{1n} & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & d_{11}
\end{pmatrix} \quad (D_2)$$

$$
\begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & a_{1q+1} & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & \vdots & 0 & a_{2s+1} & \vdots & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & a_{33} & \cdots & a_{3s} & 0 & \vdots & \cdots & 0 & a_{3m+1} & \cdots & a_{3q} & 0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\
0 & 0 & a_{s3} & \cdots & a_{ss} & 0 & \vdots & \cdots & 0 & a_{sm+1} & \cdots & a_{sq} & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & \vdots & 0 & 0 & a_{s+1s+2} & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \vdots & 0 & 0 & 0 & \cdots & a_{m-1m} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & a_{m2} & 0 & \vdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & a_{mn-1} & 0 \\
0 & 0 & a_{m+13} & \vdots & a_{m+1s} & 0 & 0 & \cdots & 0 & a_{m+1m+1} & \cdots & a_{m+1q} & 0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & a_{q3} & \cdots & a_{qs} & 0 & 0 & \cdots & 0 & a_{qm+1} & \cdots & a_{qq} & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & \vdots & 0 & 0 & 0 & \cdots & 0 & 0 & \vdots & 0 & 0 & a_{q+1q+2} & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \vdots & 0 & 0 & 0 & \cdots & 0 & 0 & \vdots & 0 & 0 & 0 & \cdots & a_{n-3n-2} & 0 & 0 \\
a_{n-21} & 0 & 0 & \vdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & a_{n-2n} \\
0 & 0 & 0 & \vdots & 0 & a_{n-1s+1} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & \vdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & a_{nq+1} & 0 & \cdots & 0 & 0 & 0
\end{pmatrix}
\quad (A_1)
$$

$$\begin{pmatrix}
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & a_{1m_1+1} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & a_{2k_1+1} & 0 & \cdots & 0 & \vdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & a_{34} & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & a_{k_1-1k_1} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & a_{k_12} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & a_{k_1n-1} & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & a_{k_1+1k_1+2} & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & a_{k-1k} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & a_{k2} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & a_{kn-1} & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & a_{k+1k+2} & \cdots & 0 & \vdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & a_{m_1-1m_1} & \vdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\
a_{m_11} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & a_{m_1n} \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \vdots & 0 & 0 & 0 & \vdots & a_{m_1+1m_1+2} & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & a_{m-1m} & 0 & \cdots & 0 & 0 & 0 \\
a_{m1} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \vdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & a_{mn} \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \vdots & 0 & 0 & 0 & \vdots & 0 & \cdots & 0 & a_{m+1m+1} & \cdots & a_{m+1n-2} & 0 & 0 \\
\vdots & \vdots & \vdots & 0 & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \vdots & 0 & \cdots & 0 & d_{n-2m+1} & \cdots & d_{n-2n-2} & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & a_{n-1k_1+1} & 0 & \cdots & 0 & \vdots & 0 & 0 & 0 & \vdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & a_{nm_1+1} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0
\end{pmatrix} \quad (A_2)$$

$$
\begin{pmatrix}
a_{11} & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & a_{1n} \\
0 & a_{22} & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & a_{2n-1} & 0 \\
a_{31} & a_{32} & a_{33} & \dots & a_{3s} & 0 & 0 & \dots & 0 & a_{3m+1} & \dots & a_{3q} & 0 & 0 & \dots & 0 & a_{3n-1} & a_{3n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\
a_{s1} & a_{s1} & a_{s3} & \dots & a_{ss} & 0 & 0 & \dots & 0 & a_{sm+1} & \dots & a_{sq} & 0 & 0 & \dots & 0 & a_{sn-1} & a_{sn} \\
0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
0 & 0 & 0 & \dots & 0 & 0 & a_{s+1s+2} & \dots & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \vdots & 0 & 0 & 0 & \dots & a_{m-1m} & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
0 & a_{m2} & 0 & \vdots & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & a_{mn-1} & 0 \\
0 & 0 & a_{m+13} & \vdots & a_{m+1s} & 0 & 0 & \dots & 0 & a_{m+1m+1} & \dots & a_{m+1q} & 0 & 0 & \dots & 0 & a_{m+1n-1} & a_{m+1n} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\
0 & 0 & a_{q3} & \dots & a_{qs} & 0 & 0 & \dots & 0 & a_{qm+1} & \dots & a_{qq} & 0 & 0 & \dots & 0 & a_{qn-1} & a_{qn} \\
0 & 0 & 0 & \vdots & 0 & 0 & 0 & \dots & 0 & 0 & \vdots & 0 & 0 & a_{q+1q+2} & \dots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \vdots & 0 & 0 & 0 & \dots & 0 & 0 & \vdots & 0 & 0 & 0 & \dots & a_{n-3n-2} & 0 & 0 \\
a_{n-21} & 0 & 0 & \vdots & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & a_{n-2n} \\
0 & b_2 a_{2n-1} & 0 & \vdots & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
b_1 a_{1n} & 0 & 0 & \vdots & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0
\end{pmatrix}
\quad (A_3)
$$

## References


[1] L. M. Camacho, J. R. Gómez, B. A. Omirov, R. M. Turdibaev, *The derivations of some evolution algebras*, Linear Multilinear Algebra, **61** (2013), 309–322. 1, 2

[2] Y. C. Casado, P. Cadavid, M. L. R. Montoya, P. M. Rodriguez, *Rodriguez On the characterization of the space of derivations in evolution algebras*, arXiv, **2019** (2019), 18 pages. 1

[3] J. M. Casas, M. Ladra, B. A. Omirov, U. A. Rozikov, *On nilpotent index and dibaricity of evolution algebras*, Linear Algebra Appl., **439** (2013), 90–105. 1

[4] J. M. Casas, M. Ladra, B. A. Omirov, U. A. Rozikov, *On evolution algebras*, Algebra Colloq., **21** (2014), 331–342. 1

[5] A. Dzhumadil'daev, B. A. Omirov, U. A. Rozikov, *Constrained evolution algebras and dynamical systems of a bisexual population*, Linear Algebra Appl., **496** (2016), 351–380. 1

[6] R. Ganikhodzhaev, F. Mukhamedov, A. Pirnapasov, I. Qaralleh, *Genetic Volterra algebras and their derivations*, Comm. Algebra, **46** (2018), 1353–1366. 1

[7] J. C. Gutierrez Fernandez, C. I. Garcia, *Derivations of Lotka-Volterra algebras*, Sao Paulo J. Math. Sci., **13** (2019), 292–304. 1

[8] P. Holgate, *The interpretation of derivations in genetic algebras*, Linear Algebra Appl., **85** (1987), 75–79. 1

[9] A. K. Khudoyberdiyev, B. A. Omirov, I. Qaralleh, *Few remarks on evolution algebras*, J. Algebra Appl., **14** (2015), 16 pages. 1

[10] A. Micali, P. Revoy, *Sur les algébres gamétiques*, Proc. Edinburgh Math. Soc. (2), **29** (1986), 187–197. 1, 2.1, 2.2, 2.3, 2.4

[11] F. Mukhamedov, O. Khakimov, B. Omirov, I. Qaralleh, *Derivations and automorphisms of nilpotent evolution algebras with maximal nilindex*, J. Algebra Appl., **18** (2019), 23 pages. 1

[12] I. Qaralleh, F. Mukhamedov, *Volterra evolution algebras and their graphs*, Linear Multilinear Algebra, **2019 2019**, 17 pages.

[13] J. P. Tian, *Evolution algebras and their applications*, Springer, Berlin, (2008). 1, 2

[14] J. P. Tian, Y. M. Zou, *Finitely generated nil but not nilpotent evolution algebras*, J. Algebra Appl., **13** (2014), 10 pages. 1