

Extended cubic B-spline method for solving a system of non-linear second-order boundary value problems



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Abstract

In this paper, we develop the extended cubic B-spline method for solving a system of nonlinear second-order boundary value problems. A quasilinearization approach is used and the error estimate is obtained. The accuracy of the method depends on two free parameters. We apply the method on four examples and the results which obtained using extended cubic B-spline indicate that the method is efficient and accurate.

Keywords: Boundary value problems, quasilinearization, extended cubic B-spline.

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1. Introduction

In this study, we consider the following system of nonlinear second-order boundary value problems:

$$\begin{cases} u''(x) + a_1(x)u'(x) + a_2(x)u(x) + a_3(x)v''(x) + a_4(x)v'(x) + a_5(x)v(x) + G_1(x, u, v) = f_1(x), \\ v''(x) + b_1(x)v'(x) + b_2(x)v(x) + b_3(x)u''(x) + b_4(x)u'(x) + b_5(x)u(x) + G_2(x, u, v) = f_2(x), \\ u(0) = u(1) = 0, v(0) = v(1) = 0, \end{cases} \quad (1.1)$$

where $0 \leq x \leq 1$, $f_1(x)$ and $f_2(x)$ are continuous functions, G_1 and G_2 are nonlinear functions of u and v . Also $a_i(x)$ and $b_i(x)$, $i = 1, 2, 3, 4, 5$ are real-valued functions of x that are sufficiently smooth. System of nonlinear boundary value problems involving ordinary differential equations arise in real life applications and the existence of solutions to such system was discussed in [7, 8, 28].

Various approximate methods have been developed and these include He's homotopy perturbation method [25], Laplace homotopy analysis method [23], variational iteration method [21], homotopy perturbation-reproducing kernel method [14], reproducing kernel method [13], sinc-collocation method [11, 12], local radial basis function based differential quadrature [10], Chebyshev finite difference [26],

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continuous genetic algorithm [2], modified homotopy analysis method [5], three positive solution method [19], spline collocation approach [20], cubic B-spline scaling functions [9], Chebyshev wavelet finite difference [22], B-spline method [6], multidimensional triangular models [16], multistage optimal homotopy asymptotic method [3, 4], and hybrid cubic B-spline method [18]. The extended cubic B-spline was first proposed by Han and Liu in 2003 [27]. It involved one free parameter, λ . This parameter was found to increase the flexibility of the spline curve. The cubic B-spline was developed to obtain an extended cubic B-spline with degree 4, 5, and 6 in [29]. Extended cubic B-spline method (ECBSM) was developed to solve linear and singular two point boundary value problems involving ordinary differential equations in [1, 15]. The same approach was also used to solve a linear system of second order boundary value problems by the present authors [17]. Two free parameters were optimized to obtain accurate results.

In this paper, we extend the use of ECBSM to solve system of nonlinear second order boundary value problems involving ordinary differential equations. The organization of this paper is as follows. The quasilinearization technique to handle the nonlinearity is discussed in Section 2. The extended cubic B-spline method is developed and the solution of the system is discussed in Section 3. The error estimate is obtained in Section 4. Finally, Section 5 presents four numerical examples.

2. Quasilinearization

The quasilinearization method is utilized to reduce the given nonlinear problem (1.1) to a sequence of linear problems. The method used is adopted from [1, 15]. From (1.1), we have

$$\begin{cases} G_1(x, u, v, u_x, v_x) = L_1(x, u, v, u_x, v_x) + f_1(x), \\ G_2(x, u, v, u_x, v_x) = L_2(x, u, v, u_x, v_x) + f_2(x), \end{cases} \tag{2.1}$$

where

$$\begin{cases} L_1(x, u, v, u_x, v_x) = -u''(x) - a_1(x)u'(x) - a_2(x)u(x) - a_3(x)v''(x) - a_4(x)v'(x) - a_5(x)v(x), \\ L_2(x, u, v, u_x, v_x) = -v''(x) - b_1(x)v'(x) - b_2(x)v(x) - b_3(x)u''(x) - b_4(x)u'(x) - b_5(x)u(x). \end{cases}$$

Initial approximations for the functions $u(x)$ and $v(x)$ in $G_1(x, u, v, u_x, v_x)$ and $G_2(x, u, v, u_x, v_x)$, denoted by $u^{(0)}(x) = c_1$ and $v^{(0)}(x) = c_3$, are chosen. $G_1(x, u, v, u_x, v_x)$ and $G_2(x, u, v, u_x, v_x)$ are then expanded about these approximations, to obtain

$$\begin{cases} G_1(x, u^{(1)}, u_x^{(1)}, v^{(1)}, v_x^{(1)}) = G_1(x, u^{(0)}, u_x^{(0)}, v^{(0)}, v_x^{(0)}) + (u^{(1)} - u^{(0)}) \left(\frac{\partial G_1}{\partial u} \right)_{(x, u^{(0)})} \\ \quad + (u_x^{(1)} - u_x^{(0)}) \left(\frac{\partial G_1}{\partial u_x} \right)_{(x, u_x^{(0)})} + (v^{(1)} - v^{(0)}) \left(\frac{\partial G_1}{\partial v} \right)_{(x, v^{(0)})} + (v_x^{(1)} - v_x^{(0)}) \left(\frac{\partial G_1}{\partial v_x} \right)_{(x, v_x^{(0)})}, \\ G_2(x, u^{(1)}, u_x^{(1)}, v^{(1)}, v_x^{(1)}) = G_2(x, u^{(0)}, u_x^{(0)}, v^{(0)}, v_x^{(0)}) + (u^{(1)} - u^{(0)}) \left(\frac{\partial G_2}{\partial u} \right)_{(x, u^{(0)})} \\ \quad + (u_x^{(1)} - u_x^{(0)}) \left(\frac{\partial G_2}{\partial u_x} \right)_{(x, u_x^{(0)})} + (v^{(1)} - v^{(0)}) \left(\frac{\partial G_2}{\partial v} \right)_{(x, v^{(0)})} + (v_x^{(1)} - v_x^{(0)}) \left(\frac{\partial G_2}{\partial v_x} \right)_{(x, v_x^{(0)})}. \end{cases}$$

Generally for $s = 0, 1, 2, \dots$ ($s =$ iteration index), we can write

$$\begin{cases} G_1(x, u^{(s+1)}, u_x^{(s+1)}, v^{(s+1)}, v_x^{(s+1)}) = G_1(x, u^{(s)}, u_x^{(s)}, v^{(s)}, v_x^{(s)}) + (u^{(s+1)} - u^{(s)}) \left(\frac{\partial G_1}{\partial u} \right)_{(x, u^{(s)})} \\ \quad + (u_x^{(s+1)} - u_x^{(s)}) \left(\frac{\partial G_1}{\partial u_x} \right)_{(x, u_x^{(s)})} + (v^{(s+1)} - v^{(s)}) \left(\frac{\partial G_1}{\partial v} \right)_{(x, v^{(s)})} + (v_x^{(s+1)} - v_x^{(s)}) \left(\frac{\partial G_1}{\partial v_x} \right)_{(x, v_x^{(s)})}, \\ G_2(x, u^{(s+1)}, u_x^{(s+1)}, v^{(s+1)}, v_x^{(s+1)}) = G_2(x, u^{(s)}, u_x^{(s)}, v^{(s)}, v_x^{(s)}) + (u^{(s+1)} - u^{(s)}) \left(\frac{\partial G_2}{\partial u} \right)_{(x, u^{(s)})} \\ \quad + (u_x^{(s+1)} - u_x^{(s)}) \left(\frac{\partial G_2}{\partial u_x} \right)_{(x, u_x^{(s)})} + (v^{(s+1)} - v^{(s)}) \left(\frac{\partial G_2}{\partial v} \right)_{(x, v^{(s)})} + (v_x^{(s+1)} - v_x^{(s)}) \left(\frac{\partial G_2}{\partial v_x} \right)_{(x, v_x^{(s)})}. \end{cases} \tag{2.2}$$

Here, we use Taylor’s expansion. By substituting (2.1) into the expansion (2.2), the following sequence of linear equations are resulted:

$$\begin{cases} L_1(x, u^{(s+1)}, u_x^{(s+1)}, v^{(s+1)}, v_x^{(s+1)}) \cong L_1(x, u^{(s)}, u_x^{(s)}, v^{(s)}, v_x^{(s)}) + (u^{(s+1)} - u^{(s)}) \left(\frac{\partial G_1}{\partial u} \right)_{(x, u^{(s)})} \\ \quad + (u_x^{(s+1)} - u_x^{(s)}) \left(\frac{\partial G_1}{\partial u_x} \right)_{(x, u_x^{(s)})} + (v^{(s+1)} - v^{(s)}) \left(\frac{\partial G_1}{\partial v} \right)_{(x, v^{(s)})} + (v_x^{(s+1)} - v_x^{(s)}) \left(\frac{\partial G_1}{\partial v_x} \right)_{(x, v_x^{(s)})}, \\ L_2(x, u^{(s+1)}, u_x^{(s+1)}, v^{(s+1)}, v_x^{(s+1)}) \cong L_2(x, u^{(s)}, u_x^{(s)}, v^{(s)}, v_x^{(s)}) + (u^{(s+1)} - u^{(s)}) \left(\frac{\partial G_1}{\partial u} \right)_{(x, u^{(s)})} \\ \quad + (u_x^{(s+1)} - u_x^{(s)}) \left(\frac{\partial G_1}{\partial u_x} \right)_{(x, u_x^{(s)})} + (v^{(s+1)} - v^{(s)}) \left(\frac{\partial G_1}{\partial v} \right)_{(x, v^{(s)})} + (v_x^{(s+1)} - v_x^{(s)}) \left(\frac{\partial G_1}{\partial v_x} \right)_{(x, v_x^{(s)})}. \end{cases} \tag{2.3}$$

We can compute $\left(\frac{\partial G_1}{\partial u} \right)_{(x, u^{(s)})}$, $\left(\frac{\partial G_1}{\partial u_x} \right)_{(x, u_x^{(s)})}$, $\left(\frac{\partial G_1}{\partial v} \right)_{(x, v^{(s)})}$, and $\left(\frac{\partial G_1}{\partial v_x} \right)_{(x, v_x^{(s)})}$, respectively. The same technique can be used to compute the derivative of G_2 . Equation (2.3) can be written as follows.

$$\begin{cases} u_{xx}^{(s+1)}(x) + a_1^{(s)}(x)u_x^{(s+1)}(x) + a_2^{(s)}(x)u^{(s+1)}(x) + a_3^{(s)}(x)v_{xx}^{(s+1)}(x) \\ \quad + a_4^{(s)}(x)v_x^{(s+1)}(x) + a_5^{(s)}(x)v^{(s+1)}(x) = f_1^{(s)}(x), \\ v_{xx}^{(s+1)}(x) + b_1^{(s)}(x)v_x^{(s+1)}(x) + b_2^{(s)}(x)v^{(s+1)}(x) + b_3^{(s)}(x)u_{xx}^{(s+1)}(x) \\ \quad + b_4^{(s)}(x)u_x^{(s+1)}(x) + b_5^{(s)}(x)u^{(s+1)}(x) = f_2^{(s)}(x), \end{cases} \tag{2.4}$$

with boundary conditions

$$\begin{cases} u^{(s+1)}(0) = \alpha, & u^{(s+1)}(1) = \beta, \\ v^{(s+1)}(0) = \gamma, & v^{(s+1)}(1) = \delta. \end{cases} \tag{2.5}$$

To solve the sequence of linear equations BVPs given in (2.4) with boundary conditions (2.5), we apply the ECBS method. Consider a partition π of $[a, b]$ as equally-spaced knots x_i into n segments $[x_i, x_{i+1}]$, $i = 0, 1, \dots, n$, where $h = \frac{b-a}{n}$ such that $x_i = x_0 + ih$, $i \in \mathbb{Z}$. Suppose that $U(x, \lambda_1)$ and $V(x, \lambda_1)$ be ECBS solution to (2.4), satisfying the interpolating conditions. Then the ECBS approximations for $U(x, \lambda_1)$, $U'(x, \lambda_1)$, $U''(x, \lambda_1)$, $V(x, \lambda_2)$, $V'(x, \lambda_2)$, and $V''(x, \lambda_2)$ at i^{th} knots. Discretizing (2.4) at x_i , we obtain

$$\begin{cases} U_{xx}^{(s+1)}(x_i) + a_1^{(s)}(x_i)U_x^{(s+1)}(x_i) + a_2^{(s)}(x_i)U^{(s+1)}(x_i) + a_3^{(s)}(x_i)V_{xx}^{(s+1)}(x_i) \\ \quad + a_4^{(s)}(x_i)V_x^{(s+1)}(x_i) + a_5^{(s)}(x_i)V^{(s+1)}(x_i) = f_1^{(s)}(x_i), \\ V_{xx}^{(s+1)}(x_i) + b_1^{(s)}(x_i)V_x^{(s+1)}(x_i) + b_2^{(s)}(x_i)V^{(s+1)}(x_i) + b_3^{(s)}(x_i)U_{xx}^{(s+1)}(x_i) \\ \quad + b_4^{(s)}(x_i)U_x^{(s+1)}(x_i) + b_5^{(s)}(x_i)U^{(s+1)}(x_i) = f_2^{(s)}(x_i). \end{cases} \tag{2.6}$$

Substituting $U(x, \lambda_1)$, $U'(x, \lambda_1)$, $U''(x, \lambda_1)$, $V(x, \lambda_2)$, $V'(x, \lambda_2)$, and $V''(x, \lambda_2)$ into equation

$$\begin{cases} U(x, \lambda_1) = \sum_{i=-3}^{n-1} \alpha_i E_i^4(x, \lambda_1), \quad x \in [x_0, x_n], \quad \alpha_i \in \mathbb{R}, \\ V(x, \lambda_2) = \sum_{i=-3}^{n-1} \beta_i E_i^4(x, \lambda_2), \quad x \in [x_0, x_n], \quad \beta_i \in \mathbb{R}, \end{cases} \tag{2.7}$$

we get a system of $2(n + 1)$ equations in $2(n + 3)$ unknowns. Therefore, we still need two more equations, which are obtained from the boundary conditions (2.5) as

$$\begin{cases} U^{(s+1)}(0) = \alpha, & U^{(s+1)}(1) = \beta, \\ V^{(s+1)}(0) = \gamma, & V^{(s+1)}(1) = \delta. \end{cases} \tag{2.8}$$

The iterations can be stopped by setting the prescribed absolute error criterion. For any $\epsilon > 0$, $|u_i^{(s+1)}(x) - u_i^{(s)}(x)| \leq \epsilon$ and $|v_i^{(s+1)}(x) - v_i^{(s)}(x)| \leq \epsilon$, where ϵ must be chosen small enough, for all i that appears. In this study, the value of ϵ is taken to be 10^{-7} as used by [24].

3. Extended cubic B-spline method

After quasilinearization, we consider equations (2.6) and (2.8), system of linear second order two-point boundary value problems, where $a \leq x \leq b$, $f_1(x)$ and $f_2(x)$ are continuous functions, and $a_i(x)$ and $b_i(x)$ for $i = 1, 2, 3, 4, 5$ are real-valued functions of x that are sufficiently smooth.

In this section, the ECBSM is developed for solving a system of nonlinear second order boundary value problems (1.1). Consider a partition π of $[a, b]$ of equally-spaced knots x_i with n segments, $[x_i, x_{i+1}]$, for $i = 0, 1, \dots, n$, where $a = x_0 < x_1 < \dots < x_n = b$, such that

$$h = \frac{b - a}{n}, \quad x_0 = a, \quad x_i = x_0 + ih, \quad i \in \mathbb{Z}.$$

The approximate solution $U(x, \lambda_1)$ and $V(x, \lambda_2)$ to the exact solutions $u(x)$ and $v(x)$, respectively, are given by equation (2.7). The extended cubic B-spline basis is given by [1]

$$E_i^4(x, \lambda) = \frac{1}{24h^4} \begin{cases} 4h(1 - \lambda)(x - x_i)^3 + 3\lambda(x - x_i)^4, & x \in [x_i, x_{i+1}], \\ (4 - \lambda)h^4 + 12h^3(x - x_{i+1}) + 6h^2(2 + \lambda)(x - x_{i+1})^2 \\ - 12h(x - x_{i+1})^3 - 3\lambda(x - x_{i+1})^4, & x \in [x_{i+1}, x_{i+2}], \\ (4 - \lambda)h^4 + 12h^3(x_{i+3} - x) + 6h^2(2 + \lambda)(x_{i+3} - x)^2 \\ - 12h(x_{i+3} - x)^3 - 3\lambda(x_{i+3} - x)^4, & x \in [x_{i+2}, x_{i+3}], \\ 4h(1 - \lambda)(x_{i+4} - x)^3 + 3\lambda(x_{i+4} - x)^4, & x \in [x_{i+3}, x_{i+4}]. \end{cases}$$

The values of E_i , E_i' , and E_i'' at the nodal points are given in Table 1.

Table 1: Coefficient of E_i , E_i' , and E_i'' .

| x | x_i | x_{i+1} | x_{i+2} | x_{i+3} | x_{i+4} |
|---------|-------|--------------------------|--------------------------|--------------------------|-----------|
| E_i | 0 | $\frac{4-\lambda}{24}$ | $\frac{8+\lambda}{12}$ | $\frac{4-\lambda}{24}$ | 0 |
| E_i' | 0 | $\frac{-1}{2h}$ | 0 | $\frac{1}{2h}$ | 0 |
| E_i'' | 0 | $\frac{2+\lambda}{2h^2}$ | $\frac{-2-\lambda}{h^2}$ | $\frac{2+\lambda}{2h^2}$ | 0 |

Substituting (2.7) into (2.6) at $x = x_i$ yields equations (3.1).

$$\begin{aligned} & \sum_{i=-3}^{n-1} \alpha_i [E_i''(x_i, \lambda_1) + a_1(x_i)E_i'(x_i, \lambda_1) + a_2(x_i)E_i(x_i, \lambda_1)] \\ & + \sum_{i=-3}^{n-1} \beta_i [a_3(x_i)E_i''(x_i, \lambda_2) + a_4(x_i)E_i'(x_i, \lambda_2) + a_5(x_i)E_i(x_i, \lambda_2)] = f_1(x_i), \quad i = 0, 1, \dots, n, \\ & \sum_{i=-3}^{n-1} \beta_i [E_i''(x_i, \lambda_2) + b_1(x_i)E_i'(x_i, \lambda_2) + b_2(x_i)E_i(x_i, \lambda_2)] \\ & + \sum_{i=-3}^{n-1} \alpha_i [b_3(x_i)E_i''(x_i, \lambda_1) + b_4(x_i)E_i'(x_i, \lambda_1) + b_5(x_i)E_i(x_i, \lambda_1)] = f_2(x_i), \quad i = 0, 1, \dots, n, \\ & \sum_{i=-3}^{n-1} \alpha_i E_i(x, \lambda_1) = 0, \quad x = 0, 1, \\ & \sum_{i=-3}^{n-1} \beta_i E_i(x, \lambda_2) = 0, \quad x = 0, 1. \end{aligned} \tag{3.1}$$

These equations produce a system of linear equations consisting of $2(n + 3)$ equations with $2(n + 3)$

unknowns, $\alpha_{-3}, \alpha_{-2}, \dots, \alpha_{n-1}, \beta_{-3}, \beta_{-2}, \dots, \beta_{n-1}$. This system can be written as a matrix vector

$$QM = R, \tag{3.2}$$

where $M = [\alpha_{-3}, \alpha_{-2}, \dots, \alpha_{n-1}, \beta_{-3}, \beta_{-2}, \dots, \beta_{n-1}]^T$, $R = [0, f_1(x_0), \dots, f_1(x_n), 0, 0, f_2(x_0), \dots, f_2(x_n), 0]^T$, and Q is a $2(n+3) \times 2(n+3)$ matrix given by

$$Q = \begin{pmatrix} H_1 & | & H_2 \\ \dots & \dots & \dots \\ H_4 & | & H_3 \end{pmatrix}.$$

The four sub-matrices H_1, H_2, H_3 , and H_4 can be written as the following.

$$\begin{aligned}
 H_1 &= \begin{pmatrix} \frac{4-\lambda_1}{24} & \frac{8+\lambda_1}{12} & \frac{4-\lambda_1}{24} & 0 & \dots & 0 & 0 \\ \epsilon_1(x_0) & \rho_1(x_0) & \tau_1(x_0) & 0 & \dots & 0 & 0 \\ 0 & \epsilon_1(x_1) & \rho_1(x_1) & \tau_1(x_1) & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \epsilon_1(x_n) & \rho_1(x_n) & \tau_1(x_n) \\ \cdot & \cdot & \cdot & \cdot & \frac{4-\lambda_1}{24} & \frac{8+\lambda_1}{12} & \frac{4-\lambda_1}{24} \end{pmatrix}_{(n+3) \times (n+3)}, \\
 H_2 &= \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \epsilon_2(x_0) & \rho_2(x_0) & \tau_2(x_0) & 0 & \dots & 0 & 0 \\ 0 & \epsilon_2(x_1) & \rho_2(x_1) & \tau_2(x_1) & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \epsilon_2(x_n) & \rho_2(x_n) & \tau_2(x_n) \\ \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 \end{pmatrix}_{(n+3) \times (n+3)}, \\
 H_3 &= \begin{pmatrix} \frac{4-\lambda_2}{24} & \frac{8+\lambda_2}{12} & \frac{4-\lambda_2}{24} & 0 & \dots & 0 & 0 \\ \epsilon_3(x_0) & \rho_3(x_0) & \tau_3(x_0) & 0 & \dots & 0 & 0 \\ 0 & \epsilon_3(x_1) & \rho_3(x_1) & \tau_3(x_1) & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \epsilon_3(x_n) & \rho_3(x_n) & \tau_3(x_n) \\ \cdot & \cdot & \cdot & \cdot & \frac{4-\lambda_2}{24} & \frac{8+\lambda_2}{12} & \frac{4-\lambda_2}{24} \end{pmatrix}_{(n+3) \times (n+3)}, \\
 H_4 &= \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \epsilon_4(x_0) & \rho_4(x_0) & \tau_4(x_0) & 0 & \dots & 0 & 0 \\ 0 & \epsilon_4(x_1) & \rho_4(x_1) & \tau_4(x_1) & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \epsilon_4(x_n) & \rho_4(x_n) & \tau_4(x_n) \\ \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 \end{pmatrix}_{(n+3) \times (n+3)}.
 \end{aligned}$$

The input of H_1, H_2, H_3 , and H_4 are given below for $i = 0, 1, \dots, n$.

$$\begin{aligned}
 \epsilon_1(x_i) &= \frac{2+\lambda_1}{2h^2} - a_1(x_i) \frac{1}{2h} + a_2(x_i) \frac{4-\lambda_1}{24}, \\
 \epsilon_2(x_i) &= a_3(x_i) \frac{2+\lambda_2}{2h^2} - a_4(x_i) \frac{1}{2h} + a_5(x_i) \frac{4-\lambda_2}{24}, \\
 \epsilon_3(x_i) &= \frac{2+\lambda_2}{2h^2} - b_1(x_i) \frac{1}{2h} + b_2(x_i) \frac{4-\lambda_2}{24}, \\
 \epsilon_4(x_i) &= b_3(x_i) \frac{2+\lambda_1}{2h^2} - b_4(x_i) \frac{1}{2h} + b_5(x_i) \frac{4-\lambda_1}{24},
 \end{aligned}$$

$$\begin{aligned}
\rho_1(x_i) &= \frac{-2-\lambda_1}{h^2} + a_1(x_i)\frac{0}{h} + a_2(x_i)\frac{8+\lambda_1}{12}, \\
\rho_2(x_i) &= a_3(x_i)\frac{-2-\lambda_2}{h^2} + a_4(x_i)\frac{0}{h} + a_5(x_i)\frac{8+\lambda_2}{12}, \\
\rho_3(x_i) &= \frac{-2-\lambda_2}{h^2} + b_1(x_i)\frac{0}{h} + b_2(x_i)\frac{8+\lambda_2}{12}, \\
\rho_4(x_i) &= b_3(x_i)\frac{-2-\lambda_1}{h^2} + b_4(x_i)\frac{0}{h} + b_5(x_i)\frac{8+\lambda_1}{12}, \\
\tau_1(x_i) &= \frac{2+\lambda_1}{2h^2} + a_1(x_i)\frac{1}{2h} + a_2(x_i)\frac{4-\lambda_1}{24}, \\
\tau_2(x_i) &= a_3(x_i)\frac{2+\lambda_2}{2h^2} + a_4(x_i)\frac{1}{2h} + a_5(x_i)\frac{4-\lambda_2}{24}, \\
\tau_3(x_i) &= \frac{2+\lambda_2}{2h^2} + b_1(x_i)\frac{1}{2h} + b_2(x_i)\frac{4-\lambda_2}{24}, \\
\tau_4(x_i) &= b_3(x_i)\frac{2+\lambda_1}{2h^2} + b_4(x_i)\frac{1}{2h} + b_5(x_i)\frac{4-\lambda_1}{24}.
\end{aligned}$$

To find the values of α_i 's and β_i 's in terms of λ_1 and λ_2 , we can use equation (3.2) by taking $M = Q^{-1}R$. Lastly, the values of λ_1 and λ_2 can be determined by optimizing the one norm as discussed in [17]. Hence, the approximate analytical solution can be generated.

4. Truncation error

The truncation error for the system in (2.4) has already been calculated in [17] to be of order h^2 . The errors are also dependent on the values of λ_1 and λ_2 .

5. Results and discussions

A number of examples are solved using ECBSM and the results are compared with the results of He's homotopy perturbation, Laplace homotopy analysis, variational iteration, homotopy perturbation-reproducing kernel, reproducing kernel, sinc collocation, Chebyshev finite difference, spline collocation, and cubic B-spline scaling methods [9, 11, 13, 14, 20, 21, 23, 25, 26]. Calculations were carried out using Wolfram Mathematica 10.2 with Intel(R) Core(TM) i7 CPU 2.4GHz processor, 8.00GB RAM. The results can only be generated for $n \leq 10$ due to the computational limit of the computer. Numerical errors are calculated using infinite and two norms, as respectively follows:

$$\begin{aligned}
L_\infty &= \max_i |u(x_i) - U(x_i)| \quad \text{or} \quad L_\infty = \max_i |v(x_i) - V(x_i)|, \\
L_2 &= \sqrt{\sum_{i=1}^n (u(x_i) - U(x_i))^2} \quad \text{or} \quad L_2 = \sqrt{\sum_{i=1}^n (v(x_i) - V(x_i))^2}.
\end{aligned}$$

Example 5.1. Consider the system of nonlinear equation [20],

$$\begin{cases}
u''(x) - xv'(x) + u(x) = f_1(x), \\
v''(x) + xu'(x) + u(x)v(x) = f_2(x), \\
u(0) = u(1) = 0, v(0) = v(1) = 0,
\end{cases}$$

where $0 \leq x \leq 1$, $f_1(x) = x^3 - 2x^2 + 6x$, and $f_2(x) = x^5 - x^4 + 2x^3 + x^2 - x + 2$. The exact solutions are $u(x) = x^3 - x$ and $v(x) = x^2 - x$.

Table 2 displays the optimized values of λ_1 and λ_2 for $n = 3, 5, 7$ and 9 together with their corresponding computational time and norms. The numerical results for Example 5.1 for $n = 5$ are shown in Table 3 and compared with spline collocation method [20]. However, the results of He's and Laplace

homotopy methods [23, 25] cannot be compared with the results of ECBSM because the solutions are of different class. The former produce solutions from the truncation of series whereas the latter depends on the number of domain partition. However, for $n = 5$, the results of ECBSM are comparable with He’s and Laplace homotopy methods. From the tables, ECBSM produces better approximation than the other methods. The plots of the numerical and exact solutions are shown in Figures 1 and 2.

Table 2: The optimized values of λ_1 and λ_2 in the last iteration, computational time and norms for $n = 3, 5, 7$, and 9 .

| n | λ_1 | λ_2 | Computational time(s) | L_∞ | L_2 |
|-----|----------------|----------------|-----------------------|---------------|---------------|
| 3 | 0.00000 | 0.00000 | 4.09603E - 01 | 7.21645E - 12 | 1.07855E - 11 |
| 5 | -1.30927E - 13 | -8.81209E - 14 | 5.46864E + 00 | 2.54089E - 11 | 4.34444E - 11 |
| 7 | -8.18702E - 15 | -5.47207E - 15 | 4.36838E + 01 | 1.68754E - 10 | 2.25381E - 10 |
| 9 | -7.00458E - 14 | -4.67703E - 14 | 2.56199E + 02 | 3.06838E - 10 | 4.40696E - 10 |

Table 3: Absolute errors and norms of spline collocation approach [20] and ECBSM for Example 5.1 with $n = 5$ for $u(x)$ and $v(x)$.

| x | spline collocation approach | | ECBSM ($\lambda_1 = \lambda_2 = 0$) | | ECBSM ($\lambda_1 = -1.30927E - 13, \lambda_2 = -8.81209E - 14$) | |
|------------|-----------------------------|-----------|---------------------------------------|-----------|--|-----------|
| | $u(x)$ | $v(x)$ | $u(x)$ | $v(x)$ | $u(x)$ | $v(x)$ |
| 0.2 | 0.00 | 2.2E - 09 | 4.7E - 10 | 1.6E - 10 | 1.5E - 11 | 6.6E - 12 |
| 0.4 | 0.00 | 4.3E - 09 | 5.1E - 10 | 2.7E - 10 | 2.5E - 11 | 9.7E - 12 |
| 0.6 | 3.0E - 10 | 5.4E - 09 | 2.4E - 10 | 2.3E - 10 | 2.4E - 11 | 1.0E - 11 |
| 0.8 | 3.0E - 10 | 3.9E - 09 | 1.9E - 10 | 1.4E - 10 | 6.8E - 12 | 1.2E - 11 |
| L_∞ | 3.0E - 10 | 5.4E - 09 | 5.1E - 10 | 2.7E - 10 | 2.5E - 11 | 1.2E - 11 |
| L_2 | 4.2E - 10 | 8.2E - 09 | 7.6E - 10 | 4.1E - 10 | 3.8E - 11 | 2.0E - 11 |

Table 4: Norms of He’s homotopy [25], Laplace homotopy [23], and ECBSM for Example 5.1 with $n = 5$ for $u(x)$ and $v(x)$.

| x | $u(x)$ | | $v(x)$ | |
|-------------------|------------|-----------|------------|-----------|
| | L_∞ | L_2 | L_∞ | L_2 |
| ECBSM ($n = 5$) | 2.5E - 11 | 3.8E - 11 | 1.2E - 11 | 2.0E - 11 |
| He’s homotopy | 7.2E - 03 | 1.2E - 02 | 9.4E - 03 | 1.4E - 02 |
| Laplace homotopy | 9.2E - 06 | 1.5E - 05 | 5.5E - 05 | 8.7E - 05 |

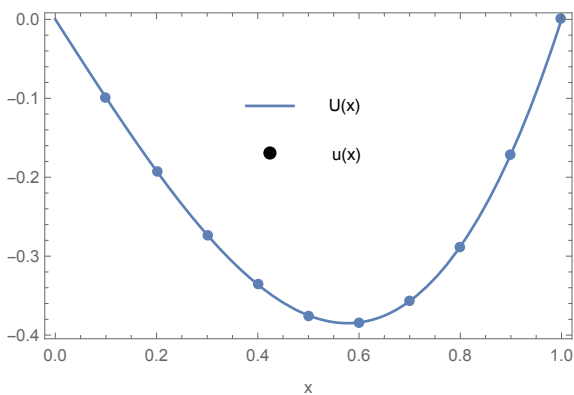


Figure 1: Numerical solution $U(x)$ and exact solution $u(x)$ for Example 5.1 using ECBSM with $\lambda_1 = -1.30927E - 13$, $\lambda_2 = -8.81209E - 14$, and $n = 5$.

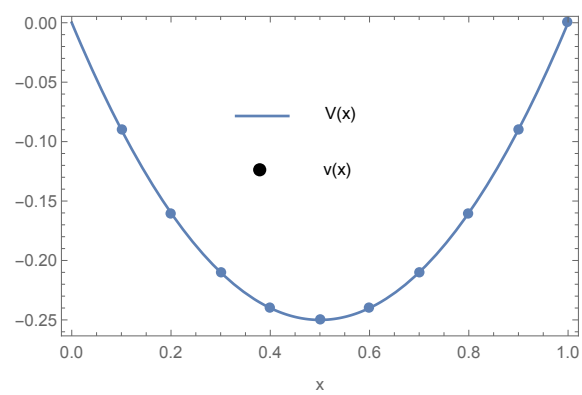


Figure 2: Numerical solution $V(x)$ and exact solution $v(x)$ for Example 5.1 using ECBSM with $\lambda_1 = -1.30927E - 13$, $\lambda_2 = -8.81209E - 14$, and $n = 5$.

Example 5.2. Consider the system of nonlinear equation includes the nonlinear $u'v'$ -term [20].

$$\begin{cases} 3u''(x) + 3xu(x) + 3v(x) - u'(x)v'(x) = f_1(x), \\ v''(x) + x^3u'(x) + x^2u(x) - \frac{x}{2}u'(x)v'(x) = f_2(x), \\ u(0) = 0, u(1) = 3, v(0) = v(1) = 0, \end{cases}$$

where $0 \leq x \leq 1$, $f_1(x) = 8 - x$, and $f_2(x) = x^3 + x^2 + 7x$. The exact solutions are $u(x) = x^2 + 2x$, and $v(x) = x^3 - x$.

Table 5 displays the optimized values of λ_1 and λ_2 for $n = 3, 5, 7$ and 9 together with their corresponding computational time and norms. The numerical results for Example 5.2 for $n = 5$ are shown in Table 6 and compared with spline collocation method [20]. From the tables, ECBSM produces better approximation than the other methods. The plots of the numerical and exact solutions are shown in Figures 3 and 4.

Table 5: The optimized values of λ_1 and λ_2 in the last iteration, computational time and norms for $n = 3, 5, 7$, and 9 .

| n | λ_1 | λ_2 | Computational time(s) | L_∞ | L_2 |
|-----|------------------|------------------|-----------------------|-----------------|-----------------|
| 3 | $-3.08195E - 15$ | $-3.08432E - 15$ | $6.46373E - 01$ | $1.28786E - 11$ | $1.00254E - 10$ |
| 5 | $-1.48737E - 15$ | $-7.01692E - 15$ | $7.20707E + 00$ | $7.77156E - 11$ | $1.03060E - 10$ |
| 7 | $-4.37203E - 20$ | $-4.46526E - 20$ | $5.08717E + 01$ | $8.12683E - 11$ | $9.20049E - 10$ |
| 9 | $-9.85542E - 13$ | $-1.07684E - 12$ | $2.83140E + 02$ | $2.76223E - 10$ | $5.66209E - 09$ |

Table 6: Absolute errors and norms of spline collocation approach [20] and ECBSM for Example 5.2 with $n = 5$ for $u(x)$ and $v(x)$.

| x | spline collocation approach | | ECBSM ($\lambda_1 = \lambda_2 = 0$) | | ECBSM ($\lambda_1 = -1.48737E - 15, \lambda_2 = -2.24845E - 15$) | |
|------------|-----------------------------|-------------|---------------------------------------|-------------|--|-------------|
| | $u(x)$ | $v(x)$ | $u(x)$ | $v(x)$ | $u(x)$ | $v(x)$ |
| | 0.2 | $3.8E - 08$ | $2.3E - 08$ | $1.5E - 09$ | $1.1E - 10$ | $1.7E - 11$ |
| 0.4 | $5.3E - 08$ | $4.3E - 08$ | $3.4E - 09$ | $3.9E - 10$ | $2.2E - 12$ | $3.9E - 12$ |
| 0.6 | $6.3E - 08$ | $6.2E - 08$ | $2.2E - 09$ | $2.2E - 10$ | $4.4E - 11$ | $7.8E - 12$ |
| 0.8 | $6.8E - 08$ | $7.6E - 08$ | $3.4E - 09$ | $3.8E - 10$ | $4.9E - 11$ | $7.8E - 11$ |
| L_∞ | $6.8E - 08$ | $7.6E - 08$ | $3.4E - 09$ | $4.9E - 10$ | $4.9E - 11$ | $7.8E - 11$ |
| L_2 | $1.1E - 07$ | $1.1E - 07$ | $5.5E - 09$ | $7.1E - 10$ | $6.8E - 11$ | $7.8E - 11$ |

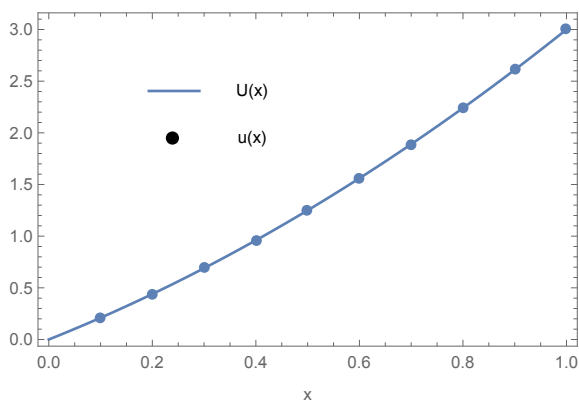


Figure 3: Numerical solution $U(x)$ and exact solution $u(x)$ for Example 5.2 using ECBSM with $\lambda_1 = -1.48737E - 15$, $\lambda_2 = -2.24845E - 15$, and $n = 5$.

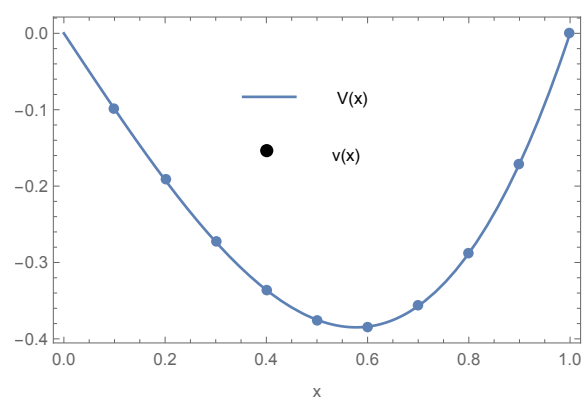


Figure 4: Numerical solution $V(x)$ and exact solution $v(x)$ for Example 5.2 using ECBSM with $\lambda_1 = -1.48737E - 15$, $\lambda_2 = -2.24845E - 15$, and $n = 5$.

Example 5.3. Consider a system of nonlinear equation [14],

$$\begin{cases} u''(x) + xu'(x) + \cos(\pi x)v'(x) = f_1(x), \\ v''(x) + xu'(x) + xu^2(x) = f_2(x), \\ u(0) = u(1) = 0, v(0) = v(1) = 0, \end{cases}$$

where $0 \leq x \leq 1$, $f_1(x) = \sin(x) + (x^2 - x + 2) \cos(x) + (1 - 2x) \cos(\pi x)$, and $f_2(x) = -2 + x \sin(x) + x(x - 1)^2 \sin^2(x) + (x^2 - x) \cos(x)$. The exact solutions are $u(x) = (x - 1) \sin(x)$, and $v(x) = x - x^2$.

Table 7 displays the optimized values of λ_1 and λ_2 for $n = 3, 5, 7$ and 10 together with their corresponding computational time and norms. The numerical results for Example 5.3 for $n = 10$ are shown in Tables 8 and 9 and compared with VIM and homotopy perturbation-reproducing kernel methods [14, 21]. From the tables, ECBSM produces better approximation than the other methods. The plots of the numerical and exact solutions are shown in Figures 5 and 6.

Table 7: The optimized values of λ_1 and λ_2 in the last iteration, computational time and norms for $n = 3, 5, 7$, and 10.

| n | λ_1 | λ_2 | Computational time(s) | L_∞ | L_2 |
|----|------------------|-----------------|-----------------------|-----------------|-----------------|
| 3 | $-3.57648E - 02$ | $3.07651E - 05$ | $4.26617E - 01$ | $3.04790E - 05$ | $3.79185E - 05$ |
| 5 | $-1.27073E - 02$ | $2.98753E - 06$ | $2.61117E + 00$ | $3.42265E - 06$ | $4.37469E - 06$ |
| 7 | $-6.48231E - 03$ | $6.63778E - 07$ | $1.76063E + 01$ | $2.37250E - 06$ | $3.15425E - 06$ |
| 10 | $-3.16155E - 03$ | $1.67076E - 07$ | $2.75975E + 02$ | $1.23232E - 06$ | $1.67593E - 06$ |

Table 8: Absolute errors and norms of ECBSM for Example 5.4 with $n = 10$ for $v(x)$.

| x | VIM | HP-RKM | ECBSM | |
|------------|-------------|-------------|-------------------------------|---|
| | | | $(\lambda_1 = \lambda_2 = 0)$ | $(\lambda_1 = -3.16155E - 03, \lambda_2 = 1.67076E - 07)$ |
| 0.1 | $3.0E - 04$ | $4.4E - 05$ | $1.4E - 04$ | $1.2E - 06$ |
| 0.3 | $7.8E - 03$ | $1.0E - 04$ | $3.2E - 04$ | $1.0E - 06$ |
| 0.5 | $2.7E - 02$ | $1.2E - 04$ | $3.6E - 04$ | $1.6E - 07$ |
| 0.7 | $4.6E - 02$ | $9.6E - 05$ | $2.7E - 04$ | $1.9E - 07$ |
| 0.9 | $3.1E - 02$ | $3.8E - 05$ | $1.0E - 04$ | $4.5E - 07$ |
| L_∞ | $4.6E - 02$ | $1.2E - 04$ | $3.6E - 04$ | $1.2E - 06$ |
| L_2 | $6.2E - 02$ | $1.9E - 04$ | $5.8E - 04$ | $1.6E - 06$ |

Table 9: Absolute errors and norms of ECBSM for Example 5.3 with $n = 10$ for $v(x)$.

| x | ECBSM | |
|------------|-------------------------------|---|
| | $(\lambda_1 = \lambda_2 = 0)$ | $(\lambda_1 = -3.16155E - 03, \lambda_2 = 1.67076E - 07)$ |
| 0.1 | $3.2E - 06$ | $2.7E - 08$ |
| 0.3 | $1.3E - 05$ | $7.7E - 08$ |
| 0.5 | $2.7E - 05$ | $6.6E - 08$ |
| 0.7 | $3.4E - 05$ | $1.6E - 08$ |
| 0.9 | $2.0E - 05$ | $1.0E - 08$ |
| L_∞ | $3.4E - 05$ | $7.7E - 08$ |
| L_2 | $5.0E - 05$ | $1.1E - 07$ |

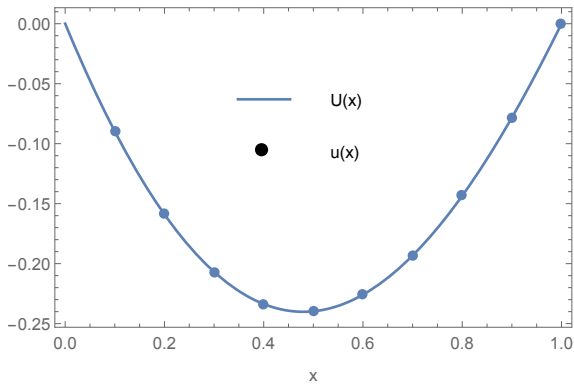


Figure 5: Numerical solution $U(x)$ and exact solution $u(x)$ for Example 5.3 using ECBSM with $\lambda_1 = -3.16155E - 03$, $\lambda_2 = 1.67076E - 07$, and $n = 10$.

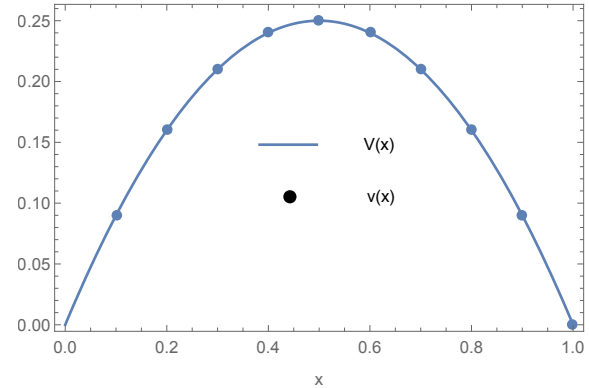


Figure 6: Numerical solution $V(x)$ and exact solution $v(x)$ for Example 5.3 using ECBSM with $\lambda_1 = -3.16155E - 03$, $\lambda_2 = 1.67076E - 07$, and $n = 10$.

Example 5.4. Consider a system of nonlinear equation [9],

$$\begin{cases} u''(x) + xu(x) + 2xv(x) + xu^2(x) = f_1(x), \\ v''(x) + v(x) + x^2u(x) + \sin(x)v^2(x) = f_2(x), \\ u(0) = u(1) = 0, v(0) = v(1) = 0, \end{cases}$$

where $0 \leq x \leq 1$, $f_1(x) = 2x \sin(\pi x) + x^5 - 2x^4 + x^2 - 2$, and $f_2(x) = x^3(1 - x) + (1 - \pi^2) \sin(\pi x) + \sin(x) \sin^2(\pi x)$. The exact solutions are $u(x) = x - x^2$, and $v(x) = \sin(\pi x)$.

Table 10 displays the optimized values of λ_1 and λ_2 for $n = 3, 5, 7$ and 10 together with their corresponding computational time and norms. The numerical results for Example 5.4 for $n = 10$ are shown in Tables 11 and 12 and compared with homotopy perturbation-reproducing kernel, reproducing kernel, sinc collocation, Chebyshev finite difference, and cubic B-spline scaling methods [9, 11, 13, 14, 26]. From the tables, ECBSM produces better approximation than the other methods. The plots of the numerical and exact solutions are shown in Figures 7 and 8.

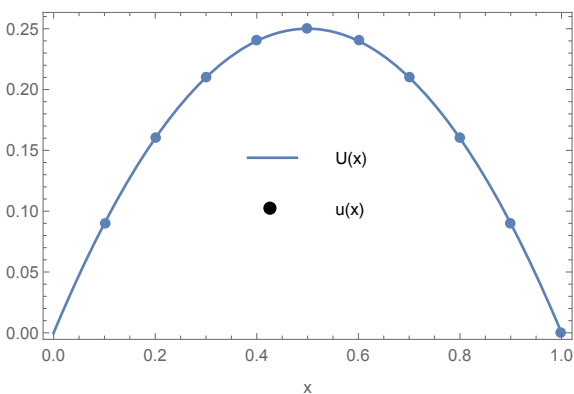


Figure 7: Numerical solution $U(x)$ and exact solution $u(x)$ for Example 5.4 using ECBSM with $\lambda_1 = 1.67646E - 08$, $\lambda_2 = -1.64832E - 02$, and $n = 10$.

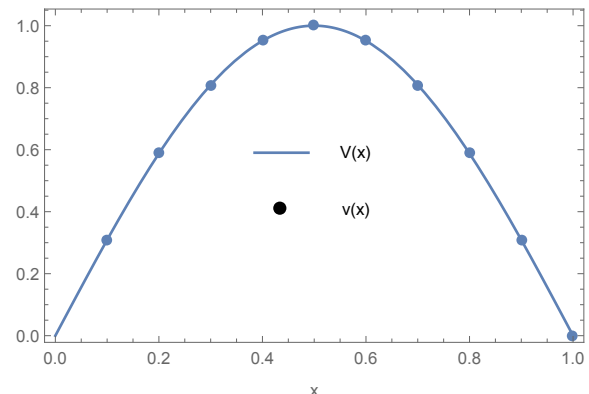


Figure 8: Numerical solution $V(x)$ and exact solution $v(x)$ for Example 5.4 using ECBSM with $\lambda_1 = 1.67646E - 08$, $\lambda_2 = -1.64832E - 02$, and $n = 10$.

Table 10: The optimized values of λ_1 and λ_2 in the last iteration, computational time and norms for $n = 3, 5, 7$, and 10 .

| n | λ_1 | λ_2 | Computational time(s) | L_∞ | L_2 |
|----|---------------|----------------|-----------------------|---------------|---------------|
| 3 | 1.32459E - 06 | -1.87092E - 01 | 2.37289E - 01 | 1.34147E - 03 | 2.31416E - 03 |
| 5 | 3.11779E - 07 | -6.63416E - 02 | 5.34669E + 00 | 1.93619E - 04 | 3.22282E - 04 |
| 7 | 5.19719E - 08 | -3.37116E - 02 | 2.96829E + 01 | 5.06215E - 05 | 8.21971E - 05 |
| 10 | 1.67646E - 08 | -1.64832E - 02 | 4.01543E + 02 | 1.21298E - 05 | 2.21764E - 05 |

Table 11: Absolute errors and norms of RKM [13], sinc collocation [11], HP-RKM [14], ChFD [26], cubic B-spline scaling [9], and ECBSM for Example 5.4 with $\lambda_1 = 1.67646E - 08$, $\lambda_2 = -1.64832E - 02$, and $n = 10$ for $u(x)$.

| x | RKM | sinc collocation | HP-RKM | ChFD | cubic B-spline scaling | ECBSM $\lambda_1 = 0, \lambda_2 = 0,$ | ECBSM |
|------------|-----------|------------------|-----------|-----------|------------------------|---------------------------------------|-----------|
| 0.08 | 5.0E - 04 | 1.4E - 04 | 7.7E - 05 | 2.8E - 05 | 2.1E - 04 | 2.2E - 04 | 2.8E - 07 |
| 0.24 | 1.4E - 03 | 4.4E - 05 | 2.2E - 04 | 7.1E - 05 | 1.6E - 04 | 6.5E - 04 | 8.1E - 07 |
| 0.40 | 2.1E - 03 | 6.7E - 05 | 3.3E - 04 | 1.1E - 04 | 8.5E - 05 | 9.8E - 04 | 1.2E - 06 |
| 0.56 | 2.2E - 03 | 9.3E - 05 | 3.7E - 04 | 1.4E - 04 | 1.3E - 04 | 1.1E - 03 | 1.4E - 06 |
| 0.72 | 1.8E - 03 | 4.9E - 05 | 3.1E - 04 | 1.3E - 04 | 8.8E - 05 | 9.4E - 04 | 1.2E - 06 |
| 0.88 | 9.0E - 04 | 8.6E - 05 | 1.5E - 04 | 7.1E - 05 | 2.3E - 04 | 4.7E - 04 | 5.9E - 07 |
| 0.96 | 3.0E - 04 | 7.1E - 05 | 5.4E - 05 | 2.4E - 05 | 1.3E - 04 | 1.6E - 04 | 2.0E - 07 |
| L_∞ | 2.2E - 03 | 1.4E - 04 | 3.7E - 04 | 1.4E - 04 | 2.3E - 04 | 1.1E - 03 | 1.4E - 06 |
| L_2 | 3.9E - 03 | 2.2E - 04 | 6.5E - 04 | 2.5E - 04 | 4.1E - 04 | 1.9E - 03 | 2.4E - 06 |

Table 12: Absolute errors and norms of RKM [13], sinc collocation [11], HP-RKM [14], ChFD [26], cubic B-spline scaling [9], and ECBSM for Example 5.4 with $\lambda_1 = 1.67646E - 08$, $\lambda_2 = -1.64832E - 02$, and $n = 10$ for $v(x)$.

| x | RKM | sinc collocation | HP-RKM | ChFD | cubic B-spline scaling | ECBSM $\lambda_1 = 0, \lambda_2 = 0,$ | ECBSM |
|------------|-----------|------------------|-----------|-----------|------------------------|---------------------------------------|-----------|
| 0.08 | 2.0E - 03 | 2.4E - 04 | 7.1E - 04 | 4.0E - 03 | 8.1E - 04 | 2.5E - 03 | 3.6E - 06 |
| 0.24 | 5.6E - 03 | 2.3E - 03 | 1.9E - 03 | 1.0E - 03 | 8.3E - 04 | 6.8E - 03 | 8.1E - 06 |
| 0.40 | 7.9E - 03 | 8.9E - 04 | 2.7E - 03 | 2.2E - 04 | 7.0E - 04 | 9.5E - 03 | 1.2E - 05 |
| 0.56 | 8.2E - 03 | 1.4E - 03 | 2.8E - 03 | 1.5E - 03 | 3.5E - 04 | 9.9E - 03 | 1.2E - 05 |
| 0.72 | 6.5E - 03 | 3.1E - 03 | 2.2E - 03 | 1.2E - 03 | 1.7E - 04 | 7.8E - 03 | 1.0E - 05 |
| 0.88 | 3.1E - 03 | 1.6E - 03 | 1.7E - 03 | 5.1E - 04 | 7.4E - 04 | 3.7E - 03 | 4.0E - 06 |
| 0.96 | 1.0E - 03 | 9.8E - 04 | 3.6E - 04 | 3.7E - 04 | 4.6E - 04 | 1.3E - 03 | 1.2E - 06 |
| L_∞ | 8.2E - 03 | 3.1E - 03 | 2.8E - 03 | 4.0E - 03 | 8.3E - 04 | 9.9E - 03 | 1.2E - 05 |
| L_2 | 1.5E - 02 | 4.6E - 03 | 5.2E - 03 | 4.6E - 03 | 1.7E - 03 | 1.8E - 02 | 2.2E - 05 |

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