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A q-analogue of r-Whitney numbers of the second kind and its Hankel transform

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Abstract

A q-analogue of r-Whitney numbers of the second kind, denoted by $W_{m,r}[n,k]_q$, is defined by means of a triangular recurrence relation. In this paper, several fundamental properties for the said q-analogue are established including other forms of recurrence relations, explicit formulas and generating functions. Moreover, a kind of Hankel transform for $W_{m,r}[n,k]_q$ is obtained.

Keywords: r-Whitney numbers, r-Dowling numbers, generating function, q-exponential function, symmetric function, Hankel transform

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1. Introduction

Several mathematicians developed a way of obtaining a generalization of some special numbers. One generalization is a q-analogue of these special numbers. A q-analogue is a mathematical expression parameterized by a quantity q that generalizes a known expression and reduces to the known expression in the limit, as $q \to 1$. For instance, a polynomial $a_k(q)$ is a q-analogue of the polynomial a_k if $\lim_{q\to 1} a_k(q) = a_k$. The q-analogue of n, $n!$, $(n)_k$, and $\binom{n}{k}$ are respectively given by

$$[n]_q = 1 + q + q^2 + \cdots + q^{n-1} = \frac{1-q^n}{1-q},$$
$$[n]_q! = [n]_q[n-1]_q \cdots [2]_q[1]_q,$$
$$[n]_{k,q} = [n]_q[n-1]_q \cdots [n-k+1]_q,$$
$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!} = \frac{[n]_{k,q}}{k!},$$

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where n and k are integers and that $n \geqslant k$. The polynomials $\begin{bmatrix} n \\ k \end{bmatrix}_q$ are usually called the q-*binomial coefficients*. These can be viewed as a formal polynomial in q of degree $k(n-k)$, where the coefficient of q^j counts the number of k-subsets of $\{1, \ldots, n\}$ with element sum $j + \frac{k(k+1)}{2}$. Thus, $\begin{bmatrix} n \\ k \end{bmatrix}_q$ are polynomials with positive coefficients.

The q-binomial coefficients $\begin{bmatrix} n \\ k \end{bmatrix}_q$, also known as Gaussian coefficients, count subspaces of a finite vector space. That is, if q is the number of elements in a finite field (the number q is then a power of a prime number, $q = p^e$), then the number of k-dimensional subspaces of the n-dimensional vector space over the q-element field equals $\begin{bmatrix} n \\ k \end{bmatrix}_q$. We observe that, when q approaches 1, we get the binomial coefficient $\binom{n}{k}$, which counts the number of k-element subsets of an n-element set. Thus, one can regard a finite vector space as a q-generalization of a set, and the subspaces as the q-generalization of the subsets of the set. This has been a fruitful point of view in finding interesting new theorems.

The q-binomial coefficients possess several properties including the q-binomial inversion formula

$$f_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q g_k \Longleftrightarrow g_n = \sum_{k=0}^{n} (-1)^{n-k} q^{\binom{n-k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q f_k$$

and the generating function

$$\sum_{k=0}^{n} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k = (1+x)(1+xq)(1+xq^2)\cdots(1+xq^{n-1}). \tag{1.1}$$

Again, when q approaches 1, the q-binomial inversion formula reduces to the binomial inversion formula [5]

$$f_n = \sum_{k=0}^{n} \binom{n}{k} g_k \Longleftrightarrow g_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f_k$$

and the generating function in (1.1) reduces to the following binomial expansion

$$\sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n.$$

It is worth-mentioning that q-analogues have applications in the study of fractals and multi-fractal measures, and expressions for the entropy of chaotic dynamical systems. These applications of q-analogues to fractals and dynamical systems are based on the fact that many fractal patterns have the symmetries of Fuchsian groups in general (see, for example Indra's pearls and the Apollonian gasket) and the modular group in particular. The connection passes through the concept of q-series, which is closely related to elliptic integrals (see [14, 15]).

q-analogues also appear in the study of quantum groups and in q-deformed superalgebras. The connection here is similar, in that much of string theory is set in the language of Riemann surfaces, resulting in connections to elliptic curves, which in turn relate to q-series.

One can also frequently find q-analogues in exact solutions of many-body problems. For instance, in atomic physics, the model with a q-deformed version of the $SU(2)$ algebra of operators describes the process of the model of molecular condensate creation from an ultra cold fermionic atomic gas during a sweep of an external magnetic field through the Feshbach resonance (see [26]). Moreover, its solution is described by q-deformed exponential and binomial distributions.

Carlitz [3] was the first to define a q-analogue of the Stirling numbers of the first and second kinds. The q-Stirling numbers of the second kind, a q-analogue of the classical Stirling numbers of the second kind $S(n,k)$, was defined in the same paper in terms of the recurrence relation

$$\tilde{S}_q[n,k] = \tilde{S}_q[n-1,k-1] + [k]_q \tilde{S}_q[n-1,k] \tag{1.2}$$

in connection with a problem in Abelian groups. Notice that as $q \to 1$, (1.2) gives the triangular recurrence relation

$$S(n,k) = S(n-1,k-1) + kS(n-1,k).$$

A different way of defining q-analogue of Stirling numbers of the second kind has been adapted in the paper by Ehrenborg [13] which is given as follows

$$S_q[n,k] = q^{k-1} S_q[n-1,k-1] + [k]_q S_q[n-1,k]. \tag{1.3}$$

A slight difference is imposed in (1.3) by multiplying the first term of the right-hand side of (1.2) with the factor q^{k-1}. This type of q-analogue gives the Hankel transform of q-exponential polynomials and numbers which are certain q-analogue of Bell polynomials and numbers. It is worth-noting that the q-Stirling numbers of the second kind in (1.3) appeared as coefficients in the expansion of normally ordered form (see [23])

$$(VU)^n = \sum_{k=0}^{n} S_q[n,k] V^k U^k,$$

where U and V are operators satisfying the q-commutation relation $UV = qVU + 1$. This expansion is a kind of q-analogue of the following expansion in [18]

$$(a^+ a)^n = \sum_{k=0}^{n} S(n,k)(a^+)^k a^k,$$

where $S(n,k)$ are the *Stirling numbers of the second kind*, a an *annihilation operator* that lowers the number of particles in a given state by one, and a^+ a *creation operator* that increases the number of particles in a given state by one. These operators are adjoint to each other that satisfy the commutation relation

$$[a, a^+] := aa^+ - a^+ a = 1.$$

Using the method of Aigner [1] and Mező [25], Corcino and Corcino [7] have successfully established the Hankel transform $H(G_{n,r,\beta})$ of the sequence of generalized Bell numbers $G_{n,r,\beta}$ (also known as (r,β)-Bell numbers) given by

$$H(G_{n,r,\beta}) = \prod_{j=0}^{n} \beta^j j!,$$

where

$$G_{n,r,\beta} = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_{r,\beta}$$

and $\left\{ {n \atop k} \right\}_{r,\beta}$ is the (r,β)-Stirling numbers [6, 8]. These numbers are also known as (r,β)-Bell numbers. The the method of Aigner is carried out in the following steps:

1. obtain generating function for the Bell-type numbers;
2. determine a sequence of numbers $a_{n,k}$ such that the first column entries of the corresponding Hankel matrix $M = [a_{n,k}]$ are exactly those Bell-type numbers;

3. express the Hankel matrix of order n as a product of lower and upper triangular matrices to easily obtain the desired Hankel transform,

while the method of Mező is carried out as follows:

1. obtain generating function for the Bell-type numbers;
2. determine a sequence of numbers $a_{n,k}$ such that the first column entries of the corresponding Hankel matrix $M = [a_{n,k}]$ are those Bell-type numbers with one parameter equal to zero;
3. express the Hankel matrix of order n as a product of lower and upper triangular matrices to easily obtain the Hankel transform Bell-type numbers with one parameter equal to zero;
4. establish the binomial transform of the Bell-type numbers;
5. apply Layman's theorem [21] to finally obtain the desired Hankel transform.

In the same paper [7], the authors have tried to establish the Hankel transform for the q-analogue of Rucinski-Voigt numbers [11], which are also equivalent to (r, β)-Bell numbers. However, the authors were not successful in their attempt. Recently, by introducing a new way of defining the q-analogue of Stirling-type and Bell-type numbers, Corcino et al. [9] were able to establish the Hankel transform for the q-analogue of non-central Bell numbers using the method of Mező. One can easily verify that non-central Bell numbers are special case of (r, β)-Bell numbers (or Rucinski-Voigt numbers or r-Dowling numbers [4]). In the desire to establish the Hankel transform of q-analogue of r-Dowling numbers, the present authors have made the initiative to define and investigate a q-analogue of r-Whitney numbers of the second kind [24] parallel to the q-analogues of noncentral Stirling numbers of the second kind in [9, 20]. This paper is concluded by establishing the Hankel transform of the q-analogue of r-Whitney numbers of the second kind using different method. The derivation of the Hankel transform of the q-analogue of r-Dowling numbers will be included in another paper (see [10]). It is worth-mentioning that r-Whitney numbers of the second kind and r-Dowling numbers are equivalent to (r, β)-Stirling numbers and (r, β)-Bell numbers, respectively. This motivates the present authors to define a q-analogue of r-Whitney numbers of the second kind parallel to that in [9].

2. A q-analogue of $W_{m,r}(n, k)$ and recurrence relations

Now, let us introduce the desired definition of the q-analogue of r-Whitney numbers of the second kind.

Definition 2.1. For non-negative integers n and k, a q-analogue $W_{m,r}[n, k]_q$ of $W_{m,r}(n, k)$ is defined by

$$W_{m,r}[n,k]_q = q^{m(k-1)+r} W_{m,r}[n-1,k-1]_q + [mk+r]_q W_{m,r}[n-1,k]_q, \tag{2.1}$$

where $W_{m,r}[0, 0]_q = 1, W_{m,r}[n, k]_q = 0$ for $n < k$ or $n, k < 0$ and $[t-k]_q = \frac{1}{q^k}([t]_q - [k]_q)$.

Remark 2.2. This definition was motivated by the definition of q-Stirling numbers of the second kind $S_q[n, k]$ in (1.3). The q-analogue $W_{m,r}[n, k]_q$ can then be considered as generalization of $S_q[n, k]$ in the sense that

$$W_{1,0}[n,k]_q = S_q[n,k].$$

Remark 2.3. Clearly, this q-analogue of r-Whitney numbers of the second kind is different from that of Mangontarum and Katriel [22] in the sense that their q-analogue, namely, (q, r)-Whitney numbers of the second kind, denoted by $W_{m,r,q}[n, k]$, satisfy the following recurrence relation

$$W_{m,r,q}(n,k) = q^{k-1} W_{m,r,q}(n-1,k-1) + (m[k]_q + r) W_{m,r,q}(n-1,k). \tag{2.2}$$

However, when $q \to 1$, both equations (2.1) and (2.2) will reduce to the recurrence relation of the r-Whitney numbers of the second kind established by Mező [24] which is given by

$$W_{m,r}(n,k) = W_{m,r}(n-1,k-1) + (mk+r) W_{m,r}(n-1,k).$$

Moreover, an alternative q-analogue of of the Ruciński-Voigt numbers was defined by Bent-Usman et al. [2] as follows

$$[x]^n = \sum_{k=0}^{n} S_q^{n,k}(\mathbf{a}) Q_q^{k,\mathbf{a}}(x),$$

where $\mathbf{a} = (a, a+r, a+2r, a+3r, ...)$ and

$$Q_q^{k,\mathbf{a}}(x) = \prod_{i=0}^{k-1} [x-(a+ir)]_q.$$

Surprisingly, the q-analogue $S_q^{n,k}(\mathbf{a})$ is equivalent to the above q-analogue of r-Whitney numbers of the second kind $W_{m,r}[n,k]_q$ with a and r are replaced by r and m, respectively. That is,

$$S_q^{n,k}(\mathbf{a}) = W_{r,a}[n,k]_q.$$

However, $S_q^{n,k}(\mathbf{a})$ are defined by means of horizontal generating function, while $W_{m,r}[n,k]_q$ are defined in terms of recurrence relation. Unfortunately, relation with $W_{m,r,q}[n,k]$ is difficult to establish.

Remark 2.4. It can easily be verified that

$$W_{m,r}[n,0] = [r]_q^n$$

and

$$W_{m,r}[n,0] = q^{m\binom{n}{2}+nr}. \tag{2.3}$$

By proper application of (2.1), we can easily obtain two other forms of recurrence relations and certain generating function.

Theorem 2.5. *For nonnegative integers n and k, the q-analogue $W_{m,r}[n,k]_q$ satisfies the following vertical and horizontal recurrence relations:*

$$W_{m,r}[n+1,k+1]_q = q^{mk+r} \sum_{j=k}^{n} [m(k+1)+r]_q^{n-j} W_{m,r}[j,k]_q, \tag{2.4}$$

$$W_{m,r}[n,k]_q = \sum_{j=0}^{n-k} (-1)^j q^{-r-m(k+j)} \frac{r_{k+j+1,q}}{r_{k+1,q}} W_{m,r}[n+1,k+j+1]_q, \tag{2.5}$$

respectively, where

$$r_{i,q} = \prod_{h=1}^{i-1} q^{-r-mh+m}[mh+r]_q$$

and initial value $W_{m,r}[0,0]_q = 1$.

Proof. Replacing n by $n+1$ and k by $k+1$ in (2.1) gives

$$W_{m,r}[n+1,k+1]_q = q^{m(k)+r} W_{m,r}[n,k]_q + [m(k+1)+r]_q W_{m,r}[n,k+1].$$

Applying repeatedly by (2.1) gives

$$\begin{aligned} W_{m,r}[n+1,k+1]_q &= q^{m(k)+r} W_{m,r}[n,k]_q \\ &\quad + [m(k+1)+r]_q (q^{mk+r} W_{m,r}[n-1,k]_q + [m(k+1)+r]_q W_{m,r}[n-1,k]_q) \\ &= q^{m(k)+r} W_{m,r}[n,k]_q + [m(k+1)+r]_q W_{m,r}[n-1,k]_q \\ &\quad + [m(k+1)+r]^2 (q^{mk+r} W_{m,r}[n-2,k]_q + [m(k+1)+r]q W_{m,r}[n-2,k]) \end{aligned}$$

$$+\cdots+[m(k+1)+r]_q^{n-k}(q^{mk+r}W_{m,r}[k+1,k+1]_q).$$

Using the fact that $W_{m,r}[k+1,k+1]_q = W_{m,r}[k,k]_q$, we obtain (2.4). Now, to prove (2.5), we have to rewrite the right-hand side (RHS) of the relation using (2.1) as follows

$$\begin{aligned}\mathrm{RHS} &= \sum_{j=0}^{n-k}(-1)^j\frac{r_{k+j+1,q}}{r_{k+1,q}}W_{m,r}[n,k+j]_q\\ &+\sum_{j=1}^{n-k}(-1)^{j-1}q^{-r-m(k+j-1)}\frac{\prod_{h=1}^{k+j-1}q^{-r-mh+m}[mh+r]_q}{\prod_{h=1}^{k}q^{-r-mh+m}[mh+r]_q}[m(k+j)+r]_qW_{m,r}[n,k+j]_q.\end{aligned}$$

Note that

$$\begin{aligned}\prod_{h=1}^{k+j-1}q^{-r-mh+m}[mh+r]_q &= (q^{-r-m+m}[m+r]_q)\\ &\times(q^{-r-m(2)+m}[m(2)+r]_q)\cdots(q^{-r-m(k+j-1)+m}[m(k+j-1)+r]_q).\end{aligned}$$

Now, we have

$$\mathrm{RHS}=\sum_{j=0}^{n-k}(-1)^j\frac{r_{k+j+1,q}}{r_{k+1,q}}W_{m,r}[n,k+j]_q+\sum_{j=0}^{n-k}(-1)^{j-1}\frac{r_{k+j+1,q}}{r_{k+1,q}}W_{m,r}[n,k+j]_q=W_{m,r}[n,k]_q.$$

This proves the theorem. □

Theorem 2.6. *A horizontal generating function of $W_m, r[n,k]_q$ is given by*

$$\sum_{k=0}^{n}W_{m,r}[n,k]_q[t-r|m]_{k,q}=[t]_q^n. \tag{2.6}$$

Proof. We will prove this by induction on n. It is easy to verify that (2.6) holds when $n=0$.

$$W_{m,r}[0,0][t-r|m]_{0,q}=1=[t]_q^0.$$

Now, suppose that it is true for some $n\geqslant 0$. That is,

$$\sum_{k=0}^{n}W_{m,r}[n,k]_q[t-r|m]_{k,q}=[t]_q^n.$$

Then by the definition of $W_{m,r}[n+1,k]_q$ and the fact that $[t-k]_q=\frac{1}{q^k}([t]_q-[k]_q)$, with $k=r+km$, we have,

$$[t-r-km]_q=q^{-(km+r)}([t]_q-[km+r]_q).$$

Finally,

$$\begin{aligned}\sum_{k=0}^{n+1}W_{m,r}[n+1,k]_q[t-r|m]_{k,q} &= \sum_{k=0}^{n}q^{m(k)+r}W_{m,r}[n,k]_q[t-r|m]_{k,q}q^{-(m(k)+r)}([t]_q-[km+r]_q)\\ &+\sum_{k=0}^{n}[mk+r]_qW_{m,r}[n,k]_q[t-r|m]_{k,q}\\ &=\sum_{k=0}^{n}[t]_qW_{m,r}[n,k]_q[t-r|m]_{k,q}\\ &=[t]_q\sum_{k=0}^{n}W_{m,r}[n,k]_q[t-r|m]_{k,q}=[t]_q[t]_q^n=[t]_q^{n+1}.\end{aligned}$$

This proves the theorem. □

3. Explicit formula and generating function

Explicit formulas and generating functions of a given sequence of numbers or polynomials are useful tools in giving combinatorial interpretation of the numbers or polynomials. In the subsequent theorems, we establish the exponential and rational generating functions and explicit formulas for $W_{m,r}[n,k]_q$.

A q-analogue of the difference operator, denoted by $\Delta^n_{q,h}$, also known as q-difference operator of order n, was defined by the rule

$$\Delta^n_{q,h} f(x) = \left[\prod_{j=0}^{n-1}(E_h - q^j)\right] f(x), \quad n \geqslant 1,$$

where E_h is the shift operator defined by $E_h f(x) = f(x+h)$. When $h = 1$, we use the notation

$$\Delta^n_{q,h} = \Delta^n_q.$$

This operator was thoroughly discussed in [5, 19]. By convention, it is defined that $\Delta^0_{q,h} = 1$ (identity map). The following is the explicit formula for the q-difference operator

$$\Delta^k_{q,h} f(x) = \sum_{j=0}^{k} (-1)^{k-j} q^{\binom{k-j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q f(x+jh). \tag{3.1}$$

The new q-analogue of Newton's interpolation formula in [19] states that, for

$$f_q(x) = a_0 + a_1[x-x_0]_q + \cdots + a_k[x-x_0]_q[x-x_1]_q \cdots [x-x_{k-1}]_q,$$

we have

$$f_q(x) = f_q(x_0) + \frac{\Delta_{q^h,h} f_q(x_0)[x-x_0]_q}{[1]_{q^h}![h]_q} + \frac{\Delta^2_{q^h,h} f_q(x_0)[x-x_0]_q[x-x_1]_q}{[2]_{q^h}![h]^2_q}$$
$$+ \cdots + \frac{\Delta^k_{q^h,h} f_q(x_0)[x-x_0]_q[x-x_1]_q \cdots [x-x_{k-1}]_q}{[k]_{q^h}![h]^k_q},$$

where $x_k = x_0 + kh$, $k = 1, 2, \cdots$ such that when $x_0 = 0$ and $h = m$ this can be simplified as

$$f_q(x) = f_q(0) + \frac{\Delta_{q^m,m} f_q(0)[x]_q}{[1]_{q^m}![m]_q} + \frac{\Delta^2_{q^m,m} f_q(0)[x]_q[x-m]_q}{[2]_{q^m}![m]^2_q}$$
$$+ \cdots + \frac{\Delta^k_{q^m,m} f_q(0)[x]_q[x-m]_q \cdots [x-m(k-1)]_q}{[m]_{q^m}![m]^k_q}.$$

Using (2.6) with $t = x$, we get

$$\sum_{k=0}^{n} W_m, r[n,k]_q[x-r|m]_{k,q} = [x]^n_q,$$

which can be expressed further as

$$\sum_{k=0}^{n} W_{m,r}[n,k][x]_q[x-m]_q[x-2m]_q \cdots [x-(k-1)m]_q = [x+r]^n_q.$$

Let $f_q(x) = [x+r]^n_q$ and $W_{m,r}[n,k]_q = \frac{\Delta^k_{q^m,m} f(q(0)}{[k]_{q^m}![m]^k_q}$. By proper application of the above Newton's interpolation formula and the identity in (3.1), we get

$$\Delta^k_{q^m,m} f_q(x) = \sum_{j=0}^{k} (-1)^{k-j} q^{m\binom{k-j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_{q^m} f_q(x+jm) = \sum_{j=0}^{k} (-1)^{k-j} q^{m\binom{k-j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_{q^m} [x-r+jm]^n_q.$$

Evaluating at $x = 0$ yields

$$\Delta^k_{q^m,m} f_q(0) = \sum_{j=0}^{k} (-1)^{k-j} q^{m\binom{k-j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_{q^m} [r + jm]^n_q,$$

which gives the following explicit formula for $W_{m,r}[n, k]_q$.

Theorem 3.1. *The explicit formula for $W_{m,r}[n, k]_q$ is given by*

$$W_{m,r}[n, k]_q = \frac{1}{[k]_{q^m}![m]^k_q} \sum_{j=0}^{k} (-1)^{k-j} q^{m\binom{k-j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_{q^m} [jm + r]^n_q. \tag{3.2}$$

Remark 3.2. The above theorem reduces to

$$W_{m,r}(n, k) = \frac{1}{k!m^k} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (jm + r)^n,$$

when $q \to 1$, which is exactly the explicit formula of the r-Whitney numbers of the second kind.

Remark 3.3. For brevity, (3.2) can be expressed as

$$W_{m,r}[n, k]_q = \frac{1}{[k]_{q^m}![m]^k_q} \left[\Delta^k_{q^m,m} [x + r]^n_q\right]_{x=0}.$$

Theorem 3.4. *For nonnegative integers n and k, the q-analogue $W_{m,r}[n, k]_q$ has a generating function*

$$\sum_{n\geqslant 0} W_{m,r}[n, k]_q \frac{[t]^n_q}{[n]_q!} = \frac{1}{[k]_q m![m]^k_q} \left[\Delta_{q^m,m^k} e_q\left([x + jm + r]_q [t]_q\right)\right]_{x=0}.$$

Proof. Using the formula in Theorem 3.1, we obtain

$$\begin{aligned}
\sum_{n\geqslant 0} W_{m,r}[n, k]_q \frac{[t]^n_q}{[n]_q!} &= \sum_{n\geqslant 0} \frac{1}{[k]_q m![m]^k_q} \sum_{j=0}^{k} (-1)^{k-j} q^{m\binom{k-j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_{q^m} [jm + r]^n_q \frac{[t]^n_q}{[n]_q!} \\
&= \sum_{j=0}^{k} \frac{1}{[k]_q m![m]^k_q} (-1)^{k-j} q^{m\binom{k-j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_{q^m} \sum_{n\geqslant 0} \frac{\left([jm + r]_q [t]_q\right)^n}{[n]_q!} \\
&= \frac{1}{[k]_q m![m]^k_q} \sum_{j=0}^{k} (-1)^{k-j} q^{m\binom{k-j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_{q^m} e_q\left([jm + r]_q [t]_q\right) \\
&= \frac{1}{[k]_q m![m]^k_q} \left[\Delta_{q^m,m^k} e_q\left([x + jm + r]_q [t]_q\right)\right]_{x=0}.
\end{aligned}$$

□

Remark 3.5. When $q \to 1$, the above theorem becomes

$$\begin{aligned}
\sum_{n\geqslant 0} W_{m,r}(n, k)_q \frac{t^n}{n!} &= \frac{1}{k!(m)^k} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} e^{(jm+r)t} \\
&= \frac{e^{rt}}{k!(m)^k} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \left(e^{mt}\right)^j = \frac{e^{rt}}{k!(m)^k} \left(e^{mt} - 1\right)^k,
\end{aligned}$$

which is the exponential generating function of the r-Whitney numbers of the second kind.

Theorem 3.6. *For nonnegative integers n and k, the q-analogue $W_m, r[n,k]_q$ satisfies the rational generating function*

$$\Psi_k(t) = \sum_{n\geqslant k} W_{m,r}[n,k]_q [t]_q^n = \frac{q^{m\binom{k}{2}+kr}[t]_q^k}{\prod_{j=0}^k (1-[mj+r]_q[t]_q)}.$$

Proof. It can easily be verified that, when $k = 0$, we obtain

$$\Psi_0(t) = \sum_{n\geqslant 0} W_{m,r}[n,0]_q [t]_q^n = \sum_{n\geqslant 0} [r]_q^n [t]_q^n = \sum_{n\geqslant 0} ([r]_q[t]_q)^n = \frac{1}{1-[r]_q[t]_q}.$$

Now, for $k \geqslant 0$ and using the recurrence relation in (2.1), we obtain

$$\begin{aligned}
\Psi_k(t) &= \sum_{n\geqslant k} W_{m,r}[n,k]_q [t]_q^n = \sum_{n\geqslant k} \left(q^{m(k-1)+r} W_{m,r}[n-1,k-1]_q + [mk+r]_q W_{m,r}[n-1,k]_q\right)[t]_q^n \\
&= \sum_{n\geqslant k} q^{m(k-1)+r} W_{m,r}[n-1,k-1]_q [t]_q^n + \sum_{n\geqslant k} [mk+r]_q W_{m,r}[n-1,k]_q [t]_q^n \\
&= q^{m(k-1)+r} \sum_{n\geqslant k} W_{m,r}[n-1,k-1]_q [t]_q^n + [mk+r]_q \sum_{n\geqslant k} W_{m,r}[n-1,k]_q [t]_q^n \\
&= q^{m(k-1)+r}[t]_q \sum_{n\geqslant k} W_{m,r}[n-1,k-1]_q [t]_q^{n-1} + [mk+r]_q[t]_q \sum_{n\geqslant k} W_{m,r}[n-1,k]_q [t]_q^{n-1}.
\end{aligned}$$

This can be expressed as

$$\begin{aligned}
\Psi_k(t) &= q^{m(k-1)+r}[t]_q \Psi_{k-1}(t) + [mk+r]_q[t]_q\Psi_k(t), \\
\Psi_k(t) - [mk+r]_q[t]_q\Psi_k(t) &= q^{m(k-1)+r}[t]_q \Psi_{k-1}(t), \\
\Psi_k(t)(1-[mk+r]_q[t]_q) &= q^{m(k-1)+r}[t]_q \Psi_{k-1}(t).
\end{aligned}$$

This gives

$$\Psi_k(t) = \frac{q^{m(k-1)+r}[t]_q}{1-[mk+r]_q[t]_q}\Psi_{k-1}(t).$$

By backward substitution, we get

$$\begin{aligned}
\Psi_k(t) &= \frac{q^{m(k-1)+r}[t]_q}{1-[mk+r]_q[t]_q} \cdot \frac{q^{m(k-2)+r}[t]_q}{1-[m(k-1)+r]_q[t]_q}\Psi_{k-2}(t) \\
&= \frac{q^{m(k-1)+r}[t]_q}{1-[mk+r]_q[t]_q} \cdot \frac{q^{m(k-2)+r}[t]_q}{1-[m(k-1)+r]_q[t]_q} \cdot \frac{q^{m(k-3)+r}[t]_q}{1-[mk-2+r]_q[t]_q}\Psi_{k-3}(t) \\
&\vdots \\
&= \frac{q^{m\binom{k}{2}+kr}[t]_q^k}{\prod_{j=0}^k \left(1-[mj+r]_q[t]_q\right)}.
\end{aligned}$$

□

As a consequence of Theorem 3.6, we have the following explicit formula in symmetric function form.

Theorem 3.7. *For nonnegative integers n and k, the explicit formula for $W_{m,r}[n,k]_q$ in the homogeneous symmetric function form is given by*

$$W_{m,r}[n,k]_q = \sum_{0\leqslant j_1\leqslant j_2\leqslant\cdots\leqslant j_{n-k}\leqslant k} q^{m\binom{k}{2}+kr} \prod_{i=1}^{n-k} [mj_i+r]_q. \tag{3.3}$$

Proof. The rational generating function in Theorem 3.6 can be expressed as

$$\begin{aligned}\sum_{n\geqslant k} W_{m,r}[n,k]_q[t]_q^n &= q^{m\binom{k}{2}+kr}[t]_q^k \prod_{j=0}^{k} \frac{1}{(1-[mj+r]_q[t]_q)} \\ &= q^{m\binom{k}{2}+kr}[t]_q^k \prod_{j=0}^{k}\sum_{n\geqslant 0}[mj+r]_q^n[t]_q^n \\ &= q^{m\binom{k}{2}+kr}[t]_q^k \sum_{n\geqslant k}\sum_{S_1+S_2+\cdots S_k=n-k}\prod_{j=0}^{k}[mj+r]_q^{S_j}\}[t]_q^{n-k} \\ &= q^{m\binom{k}{2}+kr}\sum_{n\geqslant k}\sum_{S_1+S_2+\cdots S_k=n-k}\prod_{j=0}^{k}[mj+r]_q^{S_j}[t]_q^n.\end{aligned}$$

Thus, by comparing the coefficients of $[t]_q^n$, we obtain

$$W_{m,r}[n,k]_q = q^{m\binom{k}{2}+kr}\sum_{S_1+S_2+\cdots S_k=n-k}\prod_{j=0}^{k}[mj+r]_q^{S_j},$$

which is equivalent to the desired explicit formula in (3.3). □

4. The Hankel transform

Now, we define another form of q-analogue of r-Whitney numbers of the second kind, denoted by $W^*_{m,r}[n,k]_q$, as

$$W^*_{m,r}[n,k]_q = q^{-m\binom{k}{2}-kr}W_{m,r}[n,k]_q,$$

which is parallel to that in [9]. Then, using equation (3.3), we have

$$W^*_{m,r}[n,k]_q = \sum_{0\leqslant j_1\leqslant j_2\leqslant\cdots\leqslant j_{n-k}\leqslant k}\prod_{i=1}^{n-k}[mj_i+r]_q.$$

The complete symmetric function of degree n in k variables $x_1,x_2,\dots,x_k$, denoted by $h_n(x_1,x_2,\dots,x_k)$, is defined by

$$h_n(x_1,x_2,\dots,x_k) = \sum_{1\leqslant j_1\leqslant j_2\leqslant\cdots\leqslant j_n\leqslant k}\prod_{i=1}^{n}x_{j_i},$$

with initial condition $h_0(x_1,x_2,\dots,x_k)=1$ and $h_{n-k}(x_1,x_2,\dots,x_k)=0$ if $n<k$.

Consider the case where $x_i = [mi+r]_q$ for $i=0,1,\dots,k$. Hence, we can express the other form of q-analogue of r-Whitney numbers of the second kind as

$$W^*_{m,r}[n+k,k]_q = h_n(x_0,x_1,\dots,x_k).$$

Moreover, it can easily be shown that

$$W^*_{m,r+mk}[s,t-k]_q = h_{s-t+k}(x_k,x_{k+1},\dots,x_t).$$

Note that

$$h_n(x_k) = \sum_{k\leqslant j_1\leqslant j_2\leqslant\cdots\leqslant j_n\leqslant k}\prod_{i=1}^{n}x_{j_i} = x_k^n.$$

So, when $t = k$, we have

$$W^*_{m,r+mk}[s, 0]_q = h_s(x_k) = x_k^s = [mk + r]_q^s. \tag{4.1}$$

An A-*tableau* is defined in [12] as a list ϕ of column c of a Ferrer's diagram of a partition λ(by decreasing order of length) such that the lengths $|c|$ are part of the sequence $A = (r_i)_{i\geqslant 0}$, a strictly increasing sequence of nonnegative integers. Let ω be a function from the set of nonnegative integers $\mathbb{N}$ to a ring K (column weights according to length). Suppose Φ is an A-tableau with l columns of lengths $|c| \leqslant h$. We use $T_r^A(h, l)$ to denote the set of such A-tableaux. Then, we set

$$\omega_A(\Phi) = \prod_{c\in\Phi} \omega(|c|).$$

Note that Φ might contain a finite number of columns whose lengths are zero since $0 \in A = \{0, 1, 2, \dots, k\}$ and if $\omega(0) \neq 0$.

From this point onward, whenever an A-tableau is mentioned, it is always associated with the sequence $A = \{0, 1, 2, \dots, k\}$.

Consider $\omega(|c|) = [m|c| + r]_q$, where r is a complex number, and $|c|$ is the length of column l of an A-tableau in $T_r^A(k, n-k)$. Then

$$W^*_{m,r}[n, k] = \sum_{\phi\in T_r^A(k,n-k)} \prod_{c\in\phi} \omega(|c|).$$

Suppose ϕ_1 is a tableau with $k - l$ columns whose lengths are in the set $\{0, 1, \dots, l\}$, and ϕ_2 be a tableau with $n - k - j$ columns whose lengths are in the set $\{l + 1, l + 2, \dots, l + j + 1\}$. Then

$$\phi_1 \in T^{A_1}(l, k - l) \text{ and } \phi_2 \in T^{A_2}(j, n - k - j),$$

where $A_1 = \{0, 1, \dots, l\}$ and $A_2 = \{l + 1, l + 2, \dots, l + j + 1\}$. Notice that by joining the columns of ϕ_1 and ϕ_2, we obtain an A-tableau ϕ with $n - l - j$ columns whose lengths are in the set $A = A_1 \cup A_2 = \{0, 1, \dots, l + j + 1\}$. That is, $\phi \in T^A(l + j + 1, n - l - j)$. Then,

$$\sum_{\phi\in T^A(l+j+1,n-l-j)} \omega_A(\phi) = \sum_{k=l}^{n-j} \left\{ \sum_{\phi_1\in T^{A_1}(l,\, k-l)} \omega_{A_1}(\phi_1) \right\} \left\{ \sum_{\phi_2\in T^{A_2}(j,\, n-k-j)} \omega_{A_2}(\phi_2) \right\}.$$

Note that

$$\begin{aligned}
\sum_{\phi_2\in T^{A_2}(j,\, n-k-j)} \omega_{A_2}(\phi_2) &= \sum_{\phi_2\in T^{A_2}(j,\, n-k-j)} \prod_{c\in\phi_2} [m|c| + r]_q \\
&= \sum_{l+1\leqslant g_1\leqslant\dots\leqslant g_{n-k-j}\leqslant l+j+1} \prod_{i=1}^{n-k-j} [mg_i + r]_q \\
&= \sum_{0\leqslant g_1\leqslant\dots\leqslant g_{n-k-j}\leqslant j} \prod_{i=1}^{n-k-j} [mg_i + m(l + 1) + r]_q.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\sum_{0\leqslant g_1\leqslant\dots\leqslant g_{n-l-j}\leqslant l+j+1} \prod_{i=1}^{n-l-j} [mg_i + r]_q \\
&= \sum_{k=l}^{n-j} \left\{ \sum_{0\leqslant g_1\leqslant\dots\leqslant g_{k-l}\leqslant l} \prod_{i=1}^{k-l} [mg_i + r]_q \right\} \left\{ \sum_{0\leqslant g_1\leqslant\dots\leqslant g_{n-k-j}\leqslant j} \prod_{i=1}^{n-k-j} [mg_i + m(l + 1) + r]_q \right\}.
\end{aligned}$$

Then the q-analogue $W^*_{m,r}[n,k]$ satisfies the following convolution-type identity

$$W^*_{m,r}[n+1,l+j+1]_q = \sum_{k=0}^{n} W^*_{m,r}[k,l]_q W^*_{m,r+m(l+1)}[n-k,j]_q.$$

Using the same argument above with: ϕ_1 be a tableau with $l-k$ columns whose lengths are in $A_1 = \{0,1,\dots,k\}$, and ϕ_2 be a tableau with $j-n+k$ columns whose lengths are in $A_2 = \{k,k+1,\dots,n\}$, that is, $\phi_1 \in T^{A_1}(k,l-k)$ and $\phi_2 \in T^{A_2}(n-k,j-n+k)$, we can easily obtain the following convolution formula:

$$W^*_{m,r}[l+j,n]_q = \sum_{k=l}^{n-j} W^*_{m,r}[l,k]_q W^*_{m,r+mk}[j,n-k]_q.$$

This can further be written as

$$W^*_{m,r}[s+p,t]_q = \sum_{k=\max\{0,t-p\}}^{\min\{t,s\}} W^*_{m,r}[s,k]_q W^*_{m,r+mk}[p,t-k]_q.$$

Replacing s with $s+i$, p with j, and t with $s+j$, we get

$$W^*_{m,r}[s+i+j,s+j]_q = \sum_{k=s}^{\min\{s+j,s+i\}} W^*_{m,r}[s+i,k]_q W^*_{m,r+mk}[j,s+j-k]_q. \tag{4.2}$$

This gives the following LU factorization of the matrix

$$\begin{bmatrix} W^*_{m,r}[s,s]_q & W^*_{m,r}[s+1,s+1]_q & \dots & W^*_{m,r}[s+n,s+n]_q \\ W^*_{m,r}[s+1,s]_q & W^*_{m,r}[s+2,s+1]_q & \dots & W^*_{m,r}[s+n+1,s+n]_q \\ \vdots & \vdots & \dots & \vdots \\ W^*_{m,r}[s+n,s]_q & W^*_{m,r}[s+n+1,s+1]_q & \dots & W^*_{m,r}[s+2n,s+n]_q \end{bmatrix}$$
$$= \begin{bmatrix} W^*_{m,r}[s,s]_q & 0 & \dots & 0 \\ W^*_{m,r}[s+1,s]_q & W^*_{m,r}[s+1,s+1]_q & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ W^*_{m,r}[s+n,s]_q & W^*_{m,r}[s+n,s+1]_q & \dots & W^*_{m,r}[s+n,s+n]_q \end{bmatrix}$$
$$\times \begin{bmatrix} W^*_{m,r+ms}[0,0]_q & W^*_{m,r+ms}[1,1]_q & \dots & W^*_{m,r+ms}[n,n]_q \\ 0 & W^*_{m,r+m(s+1)}[1,0]_q & \dots & W^*_{m,r+m(s+1)}[n,n-1]_q \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & W^*_{m,r+m(s+n)}[n,0]_q \end{bmatrix}.$$

This implies that

$$\det\left(W^*_{m,r}[s+i+j,s+j]_q\right)_{0\leqslant i,j\leqslant n} = \left(\prod_{k=0}^{n} W^*_{m,r}[s+k,s+k]_q\right)\left(\prod_{k=0}^{n} W^*_{m,r+m(s+k)}[k,0]_q\right).$$

Using equation (4.1) and the fact that $W^*_{m,r}[n,n]_q = 1$, we have the following theorem.

Theorem 4.1. *For nonnegative integers* n *and* k, *the Hankel transform for* $W_{m,r}[n,k]_q$ *is given by*

$$\det\left(W^*_{m,r}[s+i+j,s+j]_q\right)_{0\leqslant i,j\leqslant n} = \prod_{k=0}^{n} [m(s+k)+r]_q^k.$$

When $q \to 1$, the Hankel transform in Theorem 4.1 yields the Hankel transform for the r-Whitney numbers of the second kind. More precisely,

$$\det\left(W_{m,r}(s+i+j,s+j)\right)_{0\leqslant i,j\leqslant n} = \prod_{k=0}^{n}(m(s+k)+r)^k.$$

This further gives the Hankel transform for the classical Stirling numbers of the second kind when $(m,r) = (1,0)$. That is,

$$\det\left(S(s+i+j,s+j)\right)_{0\leqslant i,j\leqslant n} = \prod_{k=0}^{n}(s+k)^k.$$

We recall that

$$W_{m,r}[n,k]_q = q^{m\binom{k}{2}+kr}W^*_{m,r}[n,k]_q.$$

Then, multiplying both sides of (4.2) by $q^{m\binom{s+j}{2}+(s+j)r}$, we obtain

$$W_{m,r}[s+i+j,s+j]_q = \sum_{k=s}^{\min\{s+j,s+i\}} q^{m\binom{s+j}{2}+(s+j)r}W^*_{m,r}[s+i,k]_q W^*_{m,r+mk}[j,s+j-k]_q.$$

Note that

$$q^{m\binom{k}{2}+kr}q^{m\binom{s+j-k}{2}+(s+j-k)(r+mk)} = q^{m\left[\binom{k}{2}+\binom{s+j-k}{2}\right]+kr+(s+j-k)(r+mk)},$$

$$\begin{aligned} m\left[\binom{k}{2}+\binom{s+j-k}{2}\right] &= m\frac{k(k-1)+(s+j)^2-(k+1)(s+j)-k(s+j)+k(k+1)}{2} \\ &= \frac{m[(s+j)^2-(s+j)]+m[k^2-k-2(s+j)k+k^2+k]}{2} \\ &= m\binom{s+j}{2}+mk^2-mk(s+j). \end{aligned}$$

Hence,

$$\begin{aligned} q^{m\binom{k}{2}+kr}q^{m\binom{s+j-k}{2}+(s+j-k)(r+mk)} &= q^{m\binom{s+j}{2}+mk^2-mk(s+j)+kr+(s+j-k)(r+mk)} \\ &= q^{m\binom{s+j}{2}+kr+mk^2-mk(s+j)+(s+j)r-kr+mk(s+j)-mk^2} = q^{m\binom{s+j}{2}+(s+j)r}. \end{aligned}$$

Thus, we have

$$\begin{aligned} &W_{m,r}[s+i+j,s+j]_q \\ &= \sum_{k=s}^{\min\{s+j,s+i\}} q^{m\binom{k}{2}+kr}W^*_{m,r}[s+i,k]_q\, q^{m\binom{s+j-k}{2}+(s+j-k)(r+mk)}W^*_{m,r+mk}[j,s+j-k]_q \\ &= \sum_{k=s}^{\min\{s+j,s+i\}} W_{m,r}[s+i,k]_q W_{m,r+mk}[j,s+j-k]_q. \end{aligned}$$

This implies that

$$\det\left(W_{m,r}[s+i+j,s+j]_q\right)_{0\leqslant i,j\leqslant n} = \left(\prod_{k=0}^{n} W_{m,r}[s+k,s+k]_q\right)\left(\prod_{k=0}^{n} W_{m,r+m(s+k)}[k,0]_q\right).$$

Since $W^*_{m,r+mk}[s,0]_q = h_s(x_k) = x_k^s = [mk+r]_q^s$, we have

$$W_{m,r+m(s+k)}[k,0]_q = q^{m\binom{0}{2}} W^*_{m,r+m(s+k)}[k,0]_q = [m(s+k)+r]_q^k.$$

Also, using (2.3),

$$W_{m,r}[s+k,s+k]_q = q^{m\binom{s+k}{2}+(s+k)r}.$$

Thus, we have

$$\det\left(W_{m,r}[s+i+j,s+j]_q\right)_{0\leqslant i,j\leqslant n} = \left(\prod_{k=0}^{n} q^{m\binom{s+k}{2}+(s+k)r}\right)\left(\prod_{k=0}^{n}[m(s+k)+r]_q^k\right),$$

which is equivalent to

$$\det\left(W_{m,r}[s+i+j,s+j]_q\right)_{0\leqslant i,j\leqslant n} = \prod_{k=0}^{n} q^{m\binom{s+k}{2}+(s+k)r}[m(s+k)+r]_q^k.$$

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