J. Math. Computer Sci., 21 (2020), 286–295

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

Several Euler-type integrals involving Exton's quadruple hypergeometric series

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Abstract

The quadruple hypergeometric functions are introduced by H. Exton and their various applications are studied by many authors. In this line, we introduce new integral representations of Euler-type for certain Exton's hypergeometric functions of four variables.

Keywords: Beta and gamma functions, Euler integrals, triple hypergeometric functions, Exton's hypergeometric functions of four variables, Horn's functions.

2020 MSC: 33C20, 33C65.

1. Introduction

The hypergeometric functions have many useful properties in diverse areas of mathematics such as number theory, partition theory, group theory, combinatorics, difference equations, algebraic geometry, etc. Moreover, the hypergeometric functions of several variables play an important role to solve many problems in the field of science and engineering [2, 6–9, 11–14].

We begin by recalling the classical Gauss hypergeometric function $_2F_1$ which is defined as [17]

$$_2F_1\left(a,b;c;x\right)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!},$$

where $(a)_n$ denotes the Pochhammer symbol given by

$$(a)_n=\frac{\Gamma(a+n)}{\Gamma(a)}=\begin{cases}1, & (n=0),\\ a(a+1)\cdots(a+n-1), & (n\in\mathbb{N}:=\{1,2,\ldots\}).\end{cases}$$

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doi: 10.22436/jmcs.021.04.02

Received: 2020-03-02 *Revised:* 2020-03-17 *Accepted:* 2020-04-10

Appell gave the following hypergeometric function in two variables [1]

$$F_1(a,b,c;d;x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_m(c)_n}{(d)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!}.$$

The Horn's functions of two variables G_1, G_2, G_3, H_3, H_6 are given by [17]

$$\begin{aligned}
G_1(a,c_1,c_2;x,y)&=\sum_{m,n=0}^{\infty}(a)_{m+n}(c_1)_{n-m}(c_2)_{m-n}\frac{x^m}{m!}\frac{y^n}{n!},\\
G_2(a_1,a_2,c_1,c_2;x,y)&=\sum_{m,n=0}^{\infty}(a_1)_m(a_2)_n(c_1)_{n-m}(c_2)_{m-n}\frac{x^m}{m!}\frac{y^n}{n!},\\
G_3(a_1,a_2;x,y)&=\sum_{m,n=0}^{\infty}(a_1)_{2n-m}(a_2)_{2m-n}\frac{x^m}{m!}\frac{y^n}{n!},\\
H_3(a,b;c;x,y)&=\sum_{m,n=0}^{\infty}\frac{(a)_{2m+n}(b)_n}{(c)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!},\\
H_6(a_1,a_2,a_3;x,y)&=\sum_{m,n=0}^{\infty}(a_1)_{2m-n}(a_2)_{n-m}(a_3)_n\frac{x^m}{m!}\frac{y^n}{n!}.
\end{aligned} \tag{1.1}$$

The Lauricella function of three variables $F_D^{(3)}$ [10] is given in the following form:

$$F_D^{(3)}(a,b_1,b_2,b_3;c;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{m+n+p}(b_1)_m(b_2)_n(b_3)_p}{(c)_{m+n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p}.$$

Pandey [15] defined the following two hypergeometric functions of three variables G_A, G_B:

$$\begin{aligned}
G_A(a_1,a_2,c_1,c_2;x,y,z)&=\sum_{m,n,p=0}^{\infty}(a_1)_{n+p}(a_2)_m(c_1)_{m+n-p}(c_2)_{p-m-n}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},\\
G_B(a_1,a_2,a_3,c_1,c_2;x,y,z)&=\sum_{m,n,p=0}^{\infty}(a_1)_m(a_2)_n(a_3)_p(c_1)_{m+n-p}(c_2)_{p-m-n}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}.
\end{aligned}$$

Exton [4, 5] introduced the following five quadruple hypergeometric functions:

$$D_1(a_1,a_2,c_1,c_2;x,y,z,u)=\sum_{m,n,p,q=0}^{\infty}(a_1)_{m+n}(a_2)_{p+q}(c_1)_{m+p-n-q}(c_2)_{n+q-m-p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!},$$

which generalizes the Pandey's function G_A.

$$D_2(a_1,a_2,a_3,c_1,c_2;x,y,z,u)=\sum_{m,n,p,q=0}^{\infty}(a_1)_{m+n}(a_2)_p(a_3)_q(c_1)_{m-n-p-q}(c_2)_{n+p+q-m}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!},$$

which generalizes the Lauricella's function $F_D^{(3)}$ and the Pandey's functions G_A and G_B.

$$D_3(a_1,a_2,a_3,c_1,c_2;x,y,z,u)=\sum_{m,n,p,q=0}^{\infty}(a_1)_{m+n}(a_2)_p(a_3)_q(c_1)_{n+q-m-p}(c_2)_{m+p-n-q}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!},$$

which generalizes the Pandey's functions G_A and G_B.

$$D_4\left(a_1,a_2,a_3,a_4,c_1,c_2;x,y,z,u\right)=\sum_{m,n,p,q=0}^{\infty}(a_1)_m(a_2)_n(a_3)_p(a_4)_q(c_1)_{n+p+q-m}(c_2)_{m-n-p-q}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!},$$

which generalizes the Lauricella's function $F_D^{(3)}$ and the Pandey's function G_B.

$$D_5\left(a_1,a_2,a_3,a_4,c_1,c_2;x,y,z,u\right)=\sum_{m,n,p,q=0}^{\infty}(a_1)_m(a_2)_n(a_3)_p(a_4)_q(c_1)_{p+q-m-n}(c_2)_{m+n-p-q}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!},$$

which generalizes the Pandey's G_B.

Here, we aim at investigating various integral representations of Euler-type which involve Gaussian hypergeometric series ${}_2F_1$, Appell's double hypergeometric function F_1, the Horn's functions of two variables G_1, G_2, G_3, H_3 and H_6, the Lauricella's triple series $F_D^{(3)}$ and the Pandey's functions of three variables G_A and G_B for the Exton's functions of four variables D_1, D_2, D_3, D_4 and D_5.

2. Main results

In this section, we establish five integral representations of Euler-type for each quadruple hypergeometric functions $D_i (i = 1, 2, 3, 4, 5)$.

Theorem 2.1. *The following integral representations hold true:*

$$\begin{aligned}D_1\left(a_1,a_2,c_1,c_2;x,y,z,u\right)=&\frac{(1+M)^{a_1}\Gamma(a_1+a_2)}{\Gamma(a_1)\Gamma(a_2)}\int_0^1\alpha^{a_1-1}\left(1-\alpha\right)^{a_2-1}\left(1+M\alpha\right)^{-(a_1+a_2)}\\&\times G_1\left(a_1+a_2,c_2,c_1;\frac{(1+M)\alpha x}{(1+M\alpha)}+\frac{(1-\alpha)\,z}{(1+M\alpha)},\frac{(1+M)\alpha y}{(1+M\alpha)}+\frac{(1-\alpha)\,u}{(1+M\alpha)}\right)d\alpha,\\&\left(\Re(a_1)>0,\Re(a_2)>0,M>-1\right),\end{aligned}\tag{2.1}$$

$$\begin{aligned}D_1\left(a_1,a_2,c_1,c_2;x,y,z,u\right)=&\frac{\Gamma(a_1+a_2)}{\Gamma(a_1)\Gamma(a_2)}\int_0^{\infty}\left(e^{-\alpha}\right)^{a_1}\left(1-e^{-\alpha}\right)^{a_2-1}\\&\times G_1\left(a_1+a_2,c_1,c_2;ye^{-\alpha}+\left(1-e^{-\alpha}\right)u,xe^{-\alpha}+\left(1-e^{-\alpha}\right)z\right)d\alpha,\\&\left(\Re(a_1)>0,\Re(a_2)>0\right),\end{aligned}$$

$$\begin{aligned}&D_1\left(a_1,a_2,c_1,c_2;x,y,z,u\right)\\&=\frac{\Gamma(a_1+c_2)\Gamma(a_2+c_1)(S_1-R_1)(S_2-R_2)}{\Gamma(a_1)\Gamma(a_2)\Gamma(c_1)\Gamma(c_2)}\int_{R_1}^{S_1}\int_{R_2}^{S_2}\left(\alpha-R_1\right)^{a_1-1}\left(S_1-\alpha\right)^{a_2+c_1+c_2-1}\\&\times\left(\beta-R_2\right)^{a_2-1}\left(S_2-\beta\right)^{a_1+c_1+c_2-1}\left[\left(S_2-R_2\right)\left(S_1-\alpha\right)-\left(\alpha-R_1\right)\left(S_2-\beta\right)x\right]^{-(a_2+c_1)}\\&\times\left[\left(S_1-R_1\right)\left(S_2-\beta\right)-\left(S_1-\alpha\right)\left(\beta-R_2\right)u\right]^{-(a_1+c_2)}G_3\left(a_1+c_2,a_2+c_1;\right.\\&\frac{\left(S_1-\alpha\right)\left(\beta-R_2\right)\left[\left(S_1-R_1\right)\left(S_2-\beta\right)-\left(S_1-\alpha\right)\left(\beta-R_2\right)u\right]z}{\left[\left(S_2-R_2\right)\left(S_1-\alpha\right)-\left(\alpha-R_1\right)\left(S_2-\beta\right)x\right]^2},\\&\left.\frac{\left(\alpha-R_1\right)\left(S_2-\beta\right)\left[\left(S_2-R_2\right)\left(S_1-\alpha\right)-\left(\alpha-R_1\right)\left(S_2-\beta\right)x\right]y}{\left[\left(S_1-R_1\right)\left(S_2-\beta\right)-\left(S_1-\alpha\right)\left(\beta-R_2\right)u\right]^2}\right)d\alpha d\beta,\\&\left(\Re(a_1)>0,\Re(a_2)>0,\Re(c_1)>0,\Re(c_2)>0,R_1<S_1,R_2<S_2\right),\end{aligned}\tag{2.2}$$

$$\begin{aligned}D_1\left(a_1,a_2,c_1,c_2;x,y,z,u\right)=&\frac{4\Gamma(a_1+c_1)\Gamma(a_2+c_2)}{\Gamma(a_1)\Gamma(a_2)\Gamma(c_1)\Gamma(c_2)}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}(\sin^2\alpha)^{a_1-\frac12}(\cos^2\alpha)^{c_1-\frac12}(\sin^2\beta)^{a_2-\frac12}(\cos^2\beta)^{c_2-\frac12}\\&\times\left(1-y\,\tan^2\alpha\,\cos^2\beta\right)^{-(a_1+c_1)}\left(1-z\,\cos^2\alpha\,\tan^2\beta\right)^{-(a_2+c_2)}G_3\left(a_2+c_2,a_1+c_1;\right.\end{aligned}$$

$$\frac{x\ \sin^2 2\alpha\left(1-y\ \tan^2\alpha\ \cos^2\beta\right)}{4\left(\cos^2\beta-z\ \cos^2\alpha\ \sin^2\beta\right)^2},\frac{u\ \sin^2 2\beta\left(1-z\ \cos^2\alpha\ \tan^2\beta\right)}{4\left(\cos^2\alpha-y\ \sin^2\alpha\ \cos^2\beta\right)^2}\Bigg)\, d\alpha d\beta,$$
$$(\Re(a_1)>0,\Re(a_2)>0,\Re(c_1)>0,\Re(c_2)>0),$$

$$D_1\left(a_1,a_2,c_1,c_2;x,y,z,u\right)=\frac{\Gamma(c_1+c_2)}{\Gamma(c_1)\Gamma(c_2)}\int_{-\frac{1}{2}}^{\frac{1}{2}}\left(\frac{1}{2}+\alpha\right)^{a_1+a_2+c_1-1}\left(\frac{1}{2}-\alpha\right)^{a_1+a_2+c_2-1}$$
$$\times\left[\left(\frac{1}{4}-\alpha^2\right)-\left(\frac{1}{2}+\alpha\right)^2x-\left(\frac{1}{2}-\alpha\right)^2y\right]^{-a_1}$$
$$\times\left[\left(\frac{1}{4}-\alpha^2\right)-\left(\frac{1}{2}+\alpha\right)^2z-\left(\frac{1}{2}-\alpha\right)^2u\right]^{-a_2}d\alpha,\quad(\Re(c_1)>0,\Re(c_2)>0).$$

Proof. First of all, we recall the following integral representations of the Beta function (see [3, 16, 18]):

$$B(a,b)=\begin{cases}\int_0^1 t^{a-1}(1-t)^{b-1}\,dt, & (\Re(a)>0,\Re(b)>0),\\ \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, & (a,b\in\mathbb{C}\backslash\mathbb{Z}_0^-),\end{cases}\tag{2.3}$$

$$B(a,b)=2\int_0^{\frac{\pi}{2}}(\sin\alpha)^{2a-1}(\cos\alpha)^{2b-1}\,d\alpha=\int_0^\infty\frac{\alpha^{a-1}}{(1+\alpha)^{a+b}}\,d\alpha,(\Re(a)>0,\Re(b)>0),\tag{2.4}$$

$$B(a,b)=2^{1-a-b}\int_{-1}^{1}(1+\alpha)^{a-1}(1-\alpha)^{b-1}\,d\alpha$$
$$=2M^a\int_0^\infty\frac{\cosh\alpha\,(\sinh\alpha)^{2a-1}}{\left(1+M\sinh^2\alpha\right)^{a+b}}\,d\alpha,(\Re(a)>0,\Re(b)>0,M>0),\tag{2.5}$$

$$B(a,b)=\frac{(S-T)^a(R-T)^b}{(S-R)^{a+b-1}}\int_R^S\frac{(\alpha-R)^{a-1}(S-\alpha)^{b-1}}{(\alpha-T)^{a+b}}\,d\alpha$$
$$=(M+1)^a\int_0^1\frac{\alpha^{a-1}(1-\alpha)^{b-1}}{(1+M\alpha)^{a+b}}\,d\alpha,\quad(T<R<S,M>-1,\Re(a)>0,\Re(b)>0).\tag{2.6}$$

For convenience and simplicity, we denote the right side of the relation (2.1) by Υ. Then, by applying the expression of the Horn's function G_1 (1.1) to the right hand side of (2.1) and using (2.6), we find that

$$\Upsilon=\sum_{m,n,p,q=0}^{\infty}(a_1+a_2)_{m+n+p+q}(c_1)_{m+p-n-q}(c_2)_{n+q-m-p}$$
$$\times\frac{(1+M)^{a_1+m+n}\Gamma(a_1+a_2)}{\Gamma(a_1)\Gamma(a_2)}\int_0^1\frac{\alpha^{a_1+m+n-1}(1-\alpha)^{a_2+p+q-1}}{(1+M\alpha)^{a_1+a_2+m+n+p+q}}d\alpha\times\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}$$
$$=\sum_{m,n,p,q=0}^{\infty}(a_1+a_2)_{m+n+p+q}(c_1)_{m+p-n-q}(c_2)_{n+q-m-p}$$
$$\times\frac{\Gamma(a_1+a_2)\Gamma(a_1+m+n)\Gamma(a_2+p+q)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_1+a_2+m+n+p+q)}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}$$
$$=\sum_{m,n,p,q=0}^{\infty}(a_1)_{m+n}(a_2)_{p+q}(c_1)_{m+p-n-q}(c_2)_{n+q-m-p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}$$
$$=D_1\left(a_1,a_2,c_1,c_2;x,y,z,u\right),$$

which yields the desired relation (2.1). Similarly as in the proof of relation (2.1), we obtain the other integral representations. □

Corollary 2.1. *If we set $x=u=0$ in (2.2), we get the following integral representation:*

$$
\begin{aligned}
&G_2\left(a_1,a_2,c_1,c_2;y,z\right)\\
&=\frac{\Gamma(a_1+c_2)\Gamma(a_2+c_1)}{(S_1-R_1)^{a_1+c_2-1}(S_2-R_2)^{a_2+c_1-1}\Gamma(a_1)\Gamma(a_2)\Gamma(c_1)\Gamma(c_2)}\int_{R_1}^{S_1}\int_{R_2}^{S_2}(\alpha-R_1)^{a_1-1}\\
&\quad\times(S_1-\alpha)^{c_2-1}(\beta-R_2)^{a_2-1}(S_2-\beta)^{c_1-1}\\
&\quad\times G_3\left(a_1+c_2,a_2+c_1;\frac{(\beta-R_2)(S_1-R_1)(S_2-\beta)z}{(S_2-R_2)^2(S_1-\alpha)},\frac{(\alpha-R_1)(S_2-R_2)(S_1-\alpha)y}{(S_1-R_1)^2(S_2-\beta)}\right)d\alpha d\beta,\\
&\left(\Re(a_1)>0,\Re(a_2)>0,\Re(c_1)>0,\Re(c_2)>0,R_1<S_1,R_2<S_2\right).
\end{aligned}
$$

By means of the formulas (2.3)-(2.6), one can get the the following theorems without proof.

Theorem 2.2. *The following integral representations hold true:*

$$
\begin{aligned}
D_2\left(a_1,a_2,a_3,c_1,c_2;x,y,z,u\right)=&\frac{\Gamma(a_3+c_1)}{\Gamma(a_3)\Gamma(c_1)}\int_0^\infty\alpha^{a_3-1}(1+\alpha)^{-(a_3+c_1)}\left(1-\alpha u\right)^{-c_2}\\
&\times G_A\left(a_1,a_2,c_2,a_3+c_1;\frac{(1+\alpha)z}{(1-\alpha u)},\frac{(1+\alpha)y}{(1-\alpha u)},\frac{(1-\alpha u)x}{(1+\alpha)}\right)d\alpha,\\
&\left(\Re(a_3)>0,\Re(c_1)>0\right),
\end{aligned}\tag{2.7}
$$

$$
\begin{aligned}
&D_2\left(a_1,a_2,a_3,c_1,c_2;x,y,z,u\right)\\
&=\frac{\Gamma(a_2+a_3)}{2^{a_2+a_3-2}\Gamma(a_2)\Gamma(a_3)}\int_{-1}^1\left[(1+\alpha)^2\right]^{a_2-\frac12}\left[(1-\alpha)^2\right]^{a_3-\frac12}\left(1+\alpha^2\right)^{-(a_2+a_3)}\\
&\quad\times G_A\left(a_1,a_2+a_3,c_2,c_1;\frac{(1+\alpha)^2z+(1-\alpha)^2u}{2(1+\alpha^2)},y,x\right)d\alpha,\quad\left(\Re(a_2)>0,\Re(a_3)>0\right),\\
&D_2\left(a_1,a_2,a_3,c_1,c_2;x,y,z,u\right)\\
&=\frac{\Gamma(a_1+a_2)\Gamma(a_3+c_1)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(c_1)}\int_0^1\int_0^1\alpha^{a_1-1}\left(1-\alpha\right)^{a_2-1}\beta^{a_3-1}\left(1-\beta\right)^{c_1+c_2-1}\\
&\quad\times\left[(1-\beta)-\beta u\right]^{-c_2}G_1\left(a_1+a_2,a_3+c_1,c_2;\frac{\alpha(y-z)+z}{\left[(1-\beta)-\beta u\right]},\alpha\left[(1-\beta)-\beta u\right]x\right)d\alpha d\beta,\\
&\left(\Re(a_i)>0,(i=1,2,3),\Re(c_1)>0\right),\\
&D_2\left(a_1,a_2,a_3,c_1,c_2;x,y,z,u\right)\\
&=\frac{4M_1^{a_1}M_2^{a_2}\Gamma(a_1+c_1)\Gamma(a_2+a_3)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(c_1)}\int_0^\infty\int_0^\infty\cosh\alpha\left(\sinh^2\alpha\right)^{a_1-\frac12}\cosh\beta\left(\sinh^2\beta\right)^{a_2-\frac12}\\
&\quad\times\left(1+M_1\sinh^2\alpha\right)^{-(a_1+c_1)}\left(1+M_2\sinh^2\beta\right)^{-(a_2+a_3)}\left(1-M_1y\sinh^2\alpha\right)^{-c_2}\\
&\quad\times H_6\left(a_1+c_1,c_2,a_2+a_3;\frac{M_1x\ \sinh^2\alpha\left(1-M_1y\sinh^2\alpha\right)}{\left(1+M_1\sinh^2\alpha\right)^2},\frac{\left(M_2z\ \sinh^2\beta+u\right)\left(1-M_1\sinh^2\alpha\right)}{\left(1+M_2\sinh^2\beta\right)\left(1-M_1y\sinh^2\alpha\right)}\right)d\alpha d\beta,\\
&\left(\Re(a_1)>0,\Re(a_2)>0,\Re(a_3)>0,\Re(c_1)>0,M_1>0,M_2>0\right),\\
&D_2\left(a_1,a_2,a_3,c_1,c_2;x,y,z,u\right)\\
&=\frac{\Gamma(a_2+a_3+c_1)\Gamma(a_1+c_2)}{2^{a_1+a_2+a_3+c_1+c_2-1}\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(c_1)\Gamma(c_2)}\int_{-1}^1\int_{-1}^1\int_{-1}^1(1+\alpha)^{a_2-1}(1-\alpha)^{a_3-1}\\
&\quad\times(1+\beta)^{a_2+a_3-1}(1-\beta)^{a_1+c_1+c_2-1}(1+\gamma)^{a_1-1}(1-\gamma)^{a_1+a_2+c_1+c_2-1}\\
&\quad\times\left[(1-\beta)-\frac14(1+\alpha)(1+\beta)(1-\gamma)z-\frac14(1-\alpha)(1+\beta)(1-\gamma)u\right]^{-(a_1+c_2)}\\
&\quad\times\left[(1-\gamma)-\frac12(1-\beta)(1+\gamma)x\right]^{-(a_2+a_3+c_1)}{}_2F_1\left(\frac{a_1+c_2}{2},\frac{a_1+c_2+1}{2};1-a_2-a_3;\right.
\end{aligned}
$$

$$\left.\frac{-2\left(1-\beta\right)\left(1+\gamma\right)\left[\left(1-\gamma\right)-\frac{1}{2}\left(1-\beta\right)\left(1-\gamma\right)x\right]y}{\left[\left(1-\beta\right)-\frac{1}{4}\left(1+\alpha\right)\left(1+\beta\right)\left(1-\gamma\right)z-\frac{1}{4}\left(1-\alpha\right)\left(1+\beta\right)\left(1-\gamma\right)u\right]^{2}}\right)d\alpha d\beta d\gamma,$$

$$\left(\Re(a_i)>0,(i=1,2,3),\Re(c_1)>0,\Re(c_2)>0\right).$$

Corollary 2.3. *In (2.7), if we put $z=0$, we have the following integral representation for G_A:*

$$G_A\left(a_1,a_3,c_2,c_1;u,y,x\right)=\frac{\Gamma(a_3+c_1)}{\Gamma(a_3)\Gamma(c_1)}\int_0^{\infty}\alpha^{a_3-1}(1+\alpha)^{-(a_3+c_1)}\left(1-\alpha u\right)^{-c_2}$$
$$\times G_1\left(a_1,c_2,a_3+c_1;\frac{\left(1-\alpha u\right)x}{\left(1+\alpha\right)},\frac{\left(1+\alpha\right)y}{\left(1-\alpha u\right)}\right)d\alpha,\left(\Re(a_3)>0,\Re(c_1)>0\right).$$

Theorem 2.4. *The quadruple hypergeometric function D_3 satisfies the following integral representations:*

$$D_3\left(a_1,a_2,a_3,c_1,c_2;x,y,z,u\right)$$
$$=\frac{\Gamma(a_2+c_1)\Gamma(a_3+c_2)}{\Gamma(a_2)\Gamma(a_3)\Gamma(c_1)\Gamma(c_2)}\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}}\left(\frac{1}{2}+\alpha\right)^{a_2-1}\left(\frac{1}{2}-\alpha\right)^{a_3+c_1+c_2-1}$$
$$\times\left(\frac{1}{2}+\beta\right)^{a_3-1}\left(\frac{1}{2}-\beta\right)^{a_2+c_1+c_2-1}\left[\left(\frac{1}{2}-\alpha\right)-\left(\frac{1}{2}+\alpha\right)\left(\frac{1}{2}-\beta\right)z\right]^{-(a_3+c_2)}$$
$$\times\left[\left(\frac{1}{2}-\beta\right)-\left(\frac{1}{2}-\alpha\right)\left(\frac{1}{2}+\beta\right)u\right]^{-(a_2+c_1)}G_1\left(a_1,a_2+c_1,a_3+c_2;\right.$$
$$\left.\frac{\left[\left(\frac{1}{2}-\beta\right)-\left(\frac{1}{2}-\alpha\right)\left(\frac{1}{2}+\beta\right)u\right]x}{\left[\left(\frac{1}{2}-\alpha\right)-\left(\frac{1}{2}+\alpha\right)\left(\frac{1}{2}-\beta\right)z\right]},\frac{\left[\left(\frac{1}{2}-\alpha\right)-\left(\frac{1}{2}+\alpha\right)\left(\frac{1}{2}-\beta\right)z\right]y}{\left[\left(\frac{1}{2}-\beta\right)-\left(\frac{1}{2}-\alpha\right)\left(\frac{1}{2}+\beta\right)u\right]}\right)d\alpha d\beta,$$
$$\left(\Re(a_2)>0,\Re(a_3)>0,\Re(c_1)>0,\Re(c_2)>0\right),$$

$$D_3\left(a_1,a_2,a_3,c_1,c_2;x,y,z,u\right)$$
$$=\frac{4(1+M_1)^{a_1}(1+M_2)^{a_1+a_2}\Gamma(a_2+a_2+c_1)}{\Gamma(a_1)\Gamma(a_2)\Gamma(c_1)}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\left(\sin^2\alpha\right)^{a_1-\frac{1}{2}}\left(\cos^2\alpha\right)^{a_2-\frac{1}{2}}$$
$$\times\left(1+M_1\sin^2\alpha\right)^{c_2-a_1-a_2}\left(\sin^2\beta\right)^{a_1+a_2-\frac{1}{2}}\left(\cos^2\beta\right)^{c_1-\frac{1}{2}}\left(1+M_2\sin^2\beta\right)^{-(a_1+a_2+c_1)}$$
$$\times\left[(1+M_1\sin^2\alpha)-(1+M_1)(1+M_2)x\ \sin^2\alpha\tan^2\beta-(1+M_2)z\ \cos^2\alpha\tan^2\beta\right]^{-c_2}$$
$$\times H_3\left(a_1+a_2+c_1,a_3;1-c_2;\frac{-(1+M_1)(1+M_2)y\Omega\sin^2\alpha\sin^2 2\beta}{4\left(1+M_1\sin^2\alpha\right)^2\left(1+M_2\sin^2\beta\right)^2},\right.$$
$$\left.\frac{-u\Omega\cos^2\beta}{\left(1+M_1\sin^2\alpha\right)\left(1+M_2\sin^2\beta\right)}\right)d\alpha d\beta,$$
$$\left(\Omega=\left[(1+M_1\sin^2\alpha)-(1+M_1)(1+M_2)x\ \sin^2\alpha\tan^2\beta-(1+M_2)z\ \cos^2\alpha\tan^2\beta\right]\right),$$
$$\left(\Re(a_1)>0,\Re(a_2)>0,\Re(c_1)>0,M_1>-1,M_2>-1\right),$$

$$D_3\left(a_1,a_2,a_3,c_1,c_2;x,y,z,u\right)$$
$$=\frac{\Gamma(a_1+a_2+c_2)}{2^{2(a_1+a_2-1)+c_2}\Gamma(a_1)\Gamma(a_2)\Gamma(c_2)}\int_{-1}^{1}\int_{-1}^{1}\left[(1+\alpha)^2\right]^{a_1-\frac{1}{2}}\left[(1-\alpha)^2\right]^{a_2-\frac{1}{2}}$$
$$\times\left(1+\alpha^2\right)^{c_1-a_1-a_2}\left[(1+\beta)^2\right]^{a_1+a_2-\frac{1}{2}}\left[(1-\beta)^2\right]^{c_1+c_2-\frac{1}{2}}$$
$$\times\left(1+\beta^2\right)^{-(a_1+a_2+c_2)}\left[\left(1+\alpha^2\right)\left(1-\beta\right)^2-\frac{1}{2}\left(1+\alpha\right)^2\left(1+\beta\right)^2y\right]^{-c_1}H_6\left(a_1+a_2+c_2,c_1,a_3;\right.$$
$$\frac{\left(1+\beta\right)^2\left[\left(1+\alpha\right)^2x+\left(1-\alpha\right)^2z\right]\left[\left(1+\alpha^2\right)\left(1-\beta\right)^2-\frac{1}{2}\left(1+\alpha\right)^2\left(1+\beta\right)^2y\right]}{8\left(1+\alpha^2\right)^2\left(1+\beta^2\right)^2},$$

$$\left.\frac{2\left(1+\alpha^2\right)\left(1+\beta^2\right)u}{\left[\left(1+\alpha^2\right)(1-\beta)^2-\frac{1}{2}(1+\alpha)^2(1+\beta)^2y\right]}\right)d\alpha d\beta,$$

$$(\Re(a_1)>0,\Re(a_2)>0,\Re(c_2)>0),$$

$$\begin{aligned}&D_3(a_1,a_2,a_3,c_1,c_2;x,y,z,u)\\&=\frac{\Gamma(a_1+a_2+a_3+c_1)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(c_1)}\int_0^\infty\int_0^\infty\int_0^\infty \alpha^{a_1-1}(1+\alpha)^{c_2-a_1-a_2}\beta^{a_1+a_2-1}\\&\times(1+\beta)^{c_2-a_1-a_2-a_3}\gamma^{a_1+a_2+a_3-1}(1+\gamma)^{-(a_1+a_2+a_3+c_1)}\left[(1+\alpha)(1+\beta)-\alpha\beta\gamma x-\beta\gamma z\right]^{-c_2}\\&\times{}_2F_1\left(\frac{a_1+a_2+a_3+c_1}{2},\frac{a_1+a_2+a_3+c_1+1}{2};1-c_2;\right.\\&\left.\frac{-4\gamma\left[\alpha\beta y+(1+\alpha)u\right]\left[(1+\alpha)(1+\beta)-\alpha\beta\gamma x-\beta\gamma z\right]}{(1+\alpha)^2(1+\beta)^2(1+\gamma)^2}\right)d\alpha d\beta d\gamma,\end{aligned}$$

$$(\Re(a_i)>0,(i=1,2,3),\Re(c_1)>0),$$

$$\begin{aligned}&D_3(a_1,a_2,a_3,c_1,c_2;x,y,z,u)\\&=\frac{\Gamma(c_1+c_2)(S-T)^{a_1+a_2+c_1}(R-T)^{a_1+a_3+c_2}}{\Gamma(c_1)\Gamma(c_2)(S-R)^{c_1+c_2-1}}\int_R^S(\alpha-R)^{a_1+a_2+c_1-1}(S-\alpha)^{a_1+a_3+c_2-1}(\alpha-T)^{-(c_1+c_2)}\\&\times\left[(S-T)(R-T)(\alpha-R)(S-\alpha)-(R-T)^2(S-\alpha)^2x-(S-T)^2(\alpha-R)^2y\right]^{-a_1}\\&\times\left[(S-T)(\alpha-R)-(R-T)(S-\alpha)z\right]^{-a_2}\left[(R-T)(S-\alpha)-(S-T)(\alpha-R)u\right]^{-a_3}d\alpha,\end{aligned}$$

$$(\Re(c_1)>0,\Re(c_2)>0,T<R<S).$$

Theorem 2.5. *The following integral representations hold true:*

$$\begin{aligned}&D_4(a_1,a_2,a_3,a_4,c_1,c_2;x,y,z,u)\\&=\frac{2M^{a_1}\Gamma(a_1+c_1)}{\Gamma(a_1)\Gamma(c_1)}\int_0^{\frac{\pi}{2}}\left(\sin^2\alpha\right)^{a_1-\frac{1}{2}}\left(\cos^2\alpha\right)^{c_1-\frac{1}{2}}\left(\cos^2\alpha+M\sin^2\alpha\right)^{-(a_1+c_1)}\\&\times\left[1-Mx\tan^2\alpha\right]^{-c_2}F_D^{(3)}\left(a_1+c_1,a_2,a_3,a_4;1-c_2;\right.\\&\left.-\frac{\left[\cos^2\alpha-Mx\sin^2\alpha\right]y}{\left(\cos^2\alpha+M\sin^2\alpha\right)},-\frac{\left[\cos^2\alpha-Mx\sin^2\alpha\right]z}{\left(\cos^2\alpha+M\sin^2\alpha\right)},-\frac{\left[\cos^2\alpha-Mx\sin^2\alpha\right]u}{\left(\cos^2\alpha+M\sin^2\alpha\right)}\right)d\alpha,\end{aligned}$$

$$(\Re(a_1)>0,\Re(c_1)>0,M>0),$$

$$\begin{aligned}&D_4(a_1,a_2,a_3,a_4,c_1,c_2;x,y,z,u)\\&=\frac{2M^{a_3}\Gamma(a_3+a_4)}{\Gamma(a_3)\Gamma(a_4)}\int_0^\infty\cosh\alpha\left(\sinh^2\alpha\right)^{a_3-\frac{1}{2}}\left(1+M\sinh^2\alpha\right)^{-(a_3+a_4)}\\&\times G_B\left(a_2,a_3+a_4,a_1,c_1,c_2;y,\frac{Mz\sinh^2\alpha+u}{\left(1+M\sinh^2\alpha\right)},x\right)d\alpha,\end{aligned}$$

$$(\Re(a_3)>0,\Re(a_4)>0,M>0),$$

$$\begin{aligned}&D_4(a_1,a_2,a_3,a_4,c_1,c_2;x,y,z,u)\\&=\frac{\Gamma(a_1+c_1)\Gamma(a_2+c_2)}{2^{a_1+a_2+c_1+c_2-1}\Gamma(a_1)\Gamma(a_2)\Gamma(c_1)\Gamma(c_2)}\int_{-1}^1\int_{-1}^1(1+\alpha)^{a_1-1}(1-\alpha)^{a_1+2c_1-1}\\&\times(1+\beta)^{a_2-1}(1-\beta)^{a_2+2c_2-1}\left[(1-\alpha)-\frac{1}{2}(1+\alpha)(1-\beta)x\right]^{-(a_1+c_1)}\end{aligned}$$

$$\times\left[(1-\beta)-\frac{1}{2}(1-\alpha)(1+\beta)y\right]^{-(a_2+c_2)}$$
$$\times F_1\left(a_1+c_1,a_3,a_4,1-a_2-c_2;-\frac{\left[(1-\alpha)-\frac{1}{2}(1+\alpha)(1-\beta)x\right]}{\left[(1-\beta)-\frac{1}{2}(1-\alpha)(1+\beta)y\right]}z,\right.$$
$$\left.-\frac{\left[(1-\alpha)-\frac{1}{2}(1+\alpha)(1-\beta)x\right]}{\left[(1-\beta)-\frac{1}{2}(1-\alpha)(1+\beta)y\right]}u\right)d\alpha d\beta,$$
$$(\Re(a_1)>0,\Re(a_2)>0,\Re(c_1)>0,\Re(c_2)>0),$$

$$D_4(a_1,a_2,a_3,a_4,c_1,c_2;x,y,z,u)$$
$$=\frac{\Gamma(a_2+a_3+c_2)}{\Gamma(a_2)\Gamma(a_3)\Gamma(c_2)}\int_0^\infty\int_0^\infty(e^{-\alpha})^{a_2}(1-e^{-\alpha})^{a_3-1}\left(e^{-\beta}\right)^{a_2+a_3}\left(1-e^{-\beta}\right)^{c_1+c_2-1}$$
$$\times\left[(1-e^{-\beta})-e^{-(\alpha+\beta)}z-\left(e^{-\beta}-e^{-(\alpha+\beta)}\right)u\right]^{-c_1}G_2\left(a_1,a_2,c_1,a_2+a_3+c_2;\right.$$
$$\left[(1-e^{-\beta})-e^{-(\alpha+\beta)}z-\left(e^{-\beta}-e^{-(\alpha+\beta)}\right)u\right]x,$$
$$\left.\frac{y}{\left[(1-e^{-\beta})-e^{-(\alpha+\beta)}z-\left(e^{-\beta}-e^{-(\alpha+\beta)}\right)u\right]}\right)d\alpha d\beta,$$
$$(\Re(a_2)>0,\Re(a_3)>0,\Re(c_2)>0),$$

$$D_4(a_1,a_2,a_3,a_4,c_1,c_2;x,y,z,u)$$
$$=\frac{\Gamma(c_1+c_2)}{2^{c_1+c_2-2}\Gamma(c_1)\Gamma(c_2)}\int_{-1}^{1}\left[(1+\alpha)^2\right]^{a_1+c_1-\frac{1}{2}}\left[(1-\alpha)^2\right]^{a_2+a_3+a_4+c_2-\frac{1}{2}}$$
$$\times\left(1+\alpha^2\right)^{-(c_1+c_2)}\left[(1+\alpha)^2-\frac{1}{2}(1-\alpha)^2x\right]^{-a_1}\left[(1-\alpha)^2-\frac{1}{2}(1+\alpha)^2y\right]^{-a_2}$$
$$\times\left[(1-\alpha)^2-\frac{1}{2}(1+\alpha)^2z\right]^{-a_3}\left[(1-\alpha)^2-\frac{1}{2}(1+\alpha)^2u\right]^{-a_4}d\alpha,\quad(\Re(c_1)>0,\Re(c_2)>0).$$

Theorem 2.6. *The hypergeometric function* D_5 *has the following integral representations of Euler-type:*

$$D_5(a_1,a_2,a_3,a_4,c_1,c_2;x,y,z,u)$$
$$=\frac{\Gamma(a_1+c_1)\Gamma(a_2+a_3)}{(S_1-R_1)^{a_1+c_1-1}(S_2-R_2)^{a_2+a_3-1}\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(c_1)}\int_{R_1}^{S_1}\int_{R_2}^{S_2}(\alpha-R_1)^{a_1-1}$$
$$\times(S_1-\alpha)^{c_1+c_2-1}(\beta-R_2)^{a_2-1}(S_2-\beta)^{a_3-1}[(S_1-\alpha)-(\alpha-R_1)x]^{-c_2}$$
$$\times G_A\left(a_4,a_2+a_3,a_1+c_1,c_2;\frac{[(S_1-\alpha)-(\alpha-R_1)x]u}{(S_1-R_1)},\right.$$
$$\left.\frac{(S_2-\beta)[(S_1-\alpha)-(\alpha-R_1)]z}{(S_1-R_1)(S_2-R_2)},\frac{(S_1-R_1)(\beta-R_2)y}{(S_2-R_2)[(S_1-\alpha)-(\alpha-R_1)x]}\right)d\alpha d\beta,$$
$$(\Re(a_i)>0,(i=1,2,3),\Re(c_1)>0,R_1<S_1,R_2<S_2),$$

$$D_5(a_1,a_2,a_3,a_4,c_1,c_2;x,y,z,u)$$
$$=\frac{\Gamma(a_1+c_1)(S-T)^{a_1}(R-T)^{c_1+c_2}}{\Gamma(a_1)\Gamma(c_1)(S-R)^{a_1+c_1-1}}\int_R^S(\alpha-R)^{a_1-1}(S-\alpha)^{c_1+c_2-1}$$
$$\times(\alpha-T)^{-(a_1+c_2)}[(R-T)(S-\alpha)-(S-T)(\alpha-R)x]^{-c_2}$$
$$\times G_B\left(a_3,a_4,a_2,a_1+c_1,c_2;\frac{[(R-T)(S-\alpha)-(S-T)(\alpha-R)x]z}{(S-R)(\alpha-T)},\right.$$
$$\left.\frac{[(R-T)(S-\alpha)-(S-T)(\alpha-R)x]u}{(S-R)(\alpha-T)},\frac{(S-R)(\alpha-T)y}{[(R-T)(S-\alpha)-(S-T)(\alpha-R)x]}\right)d\alpha,$$

$$(\Re(a_1)>0,\Re(c_1)>0,T<R<S),$$

$$\begin{aligned}&D_5\left(a_1,a_2,a_3,a_4,c_1,c_2;x,y,z,u\right)\\&=\frac{4\Gamma(a_1+a_2+c_1)}{\Gamma(a_1)\Gamma(a_2)\Gamma(c_1)}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}(\sin^2\alpha)^{a_1-\frac{1}{2}}(\cos^2\alpha)^{a_2-\frac{1}{2}}\\&\times(\sin^2\beta)^{a_1+a_2-\frac{1}{2}}(\cos^2\beta)^{c_1+c_2-\frac{1}{2}}\left(\cos^2\beta-x\sin^2\alpha\sin^2\beta-y\cos^2\alpha\sin^2\beta\right)^{-c_2}\\&\times F_1\left(a_1+a_2+c_1,a_3,a_4;1-c_2;-\left(\cos^2\beta-x\sin^2\alpha\sin^2\beta-y\cos^2\alpha\sin^2\beta\right)z,\right.\\&\left.-\left(\cos^2\beta-x\sin^2\alpha\sin^2\beta-y\cos^2\alpha\sin^2\beta\right)u\right)d\alpha d\beta,\\&(\Re(a_1)>0,\Re(a_2)>0,\Re(c_1)>0),\end{aligned}$$

$$\begin{aligned}&D_5\left(a_1,a_2,a_3,a_4,c_1,c_2;x,y,z,u\right)\\&=\frac{\Gamma(a_1+c_1)\Gamma(a_3+c_2)}{\Gamma(a_1)\Gamma(a_3)\Gamma(c_1)\Gamma(c_2)}\int_0^1\int_0^1\alpha^{a_1-1}\left(1-\alpha\right)^{a_3+c_1+c_2-1}\beta^{a_3-1}\left(1-\beta\right)^{a_1+c_1+c_2-1}\\&\times\left[(1-\alpha)-\alpha\left(1-\beta\right)x\right]^{-(a_3+c_2)}\left[(1-\beta)-\beta\left(1-\alpha\right)z\right]^{-(a_1+c_1)}G_2\Big(a_2,a_4,a_1+c_1,\\&a_3+c_2;\frac{\left[(1-\beta)-\beta\left(1-\alpha\right)z\right]y}{\left[(1-\alpha)-\alpha\left(1-\beta\right)x\right]},\frac{\left[(1-\alpha)-\alpha\left(1-\beta\right)x\right]u}{\left[(1-\beta)-\beta\left(1-\alpha\right)z\right]}\Big)d\alpha d\beta,\\&(\Re(a_1)>0,\Re(a_3)>0,\Re(c_1)>0,\Re(c_2)>0),\end{aligned}\tag{2.8}$$

$$\begin{aligned}&D_5\left(a_1,a_2,a_3,a_4,c_1,c_2;x,y,z,u\right)\\&=\frac{8M_1^{a_1}M_2^{a_3}M_3^{a_3+a_4}\Gamma(a_1+a_2)\Gamma(a_3+a_4+c_1)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a_4)\Gamma(c_1)}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\left(\sin^2\alpha\right)^{a_1-\frac{1}{2}}\left(\cos^2\alpha\right)^{a_2-\frac{1}{2}}\\&\times\left(\cos^2\alpha+M_1\sin^2\alpha\right)^{-(a_1+a_2)}\left(\sin^2\beta\right)^{a_3-\frac{1}{2}}\left(\cos^2\beta\right)^{a_4-\frac{1}{2}}\left(\cos^2\beta+M_2\sin^2\beta\right)^{-(a_3+a_4)}\\&\times\left(\sin^2\gamma\right)^{a_3+a_4-\frac{1}{2}}\left(\cos^2\gamma\right)^{c_1-\frac{1}{2}}\left(\cos^2\gamma+M_3\sin^2\gamma\right)^{-(a_3+a_4+c_1)}\\&\times H_6\left(a_3+a_4+c_1,c_2,a_1+a_2;\frac{M_3\left(M_2z\sin^2\beta+y\cos^2\beta\right)\sin^2 2\gamma}{4\left(\cos^2\beta+M_2\sin^2\beta\right)\left(\cos^2\gamma+M_3\sin^2\gamma\right)^2},\right.\\&\left.\frac{\left(M_1x\sin^2\alpha+u\cos^2\alpha\right)\left(\cos^2\gamma+M_3\sin^2\gamma\right)}{\left(\cos^2\alpha+M_1\sin^2\alpha\right)\cos^2\gamma}\right)d\alpha d\beta d\gamma,\\&(\Re(a_i)>0,(i=1,2,3,4),\Re(c_1)>0,M_1>0,M_3>0,M_3>0).\end{aligned}$$

Corollary 2.7. *Taking* $y=0$ *in* (2.8), *we obtain the following integral representation:*

$$\begin{aligned}&G_B\left(a_3,a_4,a_1,c_1,c_2;z,u,x\right)\\&=\frac{\Gamma(a_1+c_1)\Gamma(a_3+c_2)}{\Gamma(a_1)\Gamma(a_3)\Gamma(c_1)\Gamma(c_2)}\int_0^1\int_0^1\alpha^{a_1-1}\left(1-\alpha\right)^{a_3+c_1+c_2-1}\beta^{a_3-1}\left(1-\beta\right)^{a_1+c_1+c_2-1}\\&\times\left[(1-\alpha)-\alpha\left(1-\beta\right)x\right]^{-(a_3+c_2)}\left[(1-\beta)-\beta\left(1-\alpha\right)z\right]^{-(a_1+c_1)}{}_2F_1\Big(a_4,a_1+c_1;\\&1-a_3-c_2;-\frac{\left[(1-\alpha)-\alpha\left(1-\beta\right)x\right]u}{\left[(1-\beta)-\beta\left(1-\alpha\right)z\right]}\Big)d\alpha d\beta,\\&(\Re(a_1)>0,\Re(a_3)>0,\Re(c_1)>0,\Re(c_2)>0).\end{aligned}$$

3. Conclusion

Several integrals containing the Exton's quadruple hypergeometric series are established in this study. Some of the special cases of the main results are also mentioned.

Acknowledgment

The author K. S. Nisar express his thanks to the Deanship of Scientific Research (DSR), Prince Sattam bin Abdulaziz University, Saudi Arabia for providing facilities and support.

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