A parametric kind of the poly-Bernoulli polynomials

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Abstract

In this paper, we propose a parametric kind of poly-Bernoulli polynomials by defining the two specific generating functions. We also investigate some analytical properties (for example, summation formulae, differential formulae and relations with other well-known polynomials and numbers) for our introduced polynomials in a systematic way. Furthermore, we derive two interesting connections of our parametric poly-Bernoulli polynomials with the familiar Voigt functions.

1. Introduction

Throughout in this paper, let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{R}^+$ be the sets of natural numbers, integers, real numbers and positive real numbers, respectively, and let

$$\mathbb{N}_0 := \{0, 1, 2, 3, \ldots\} = \mathbb{N} \cup \{0\}.$$

The Appell polynomials $A_n(u)$ are described as follows:

$$f(t)e^{ut} = \sum_{n=0}^{\infty} A_n(u)\frac{t^n}{n!}, \tag{1.1}$$

where f is a formal power series in t. These polynomials have found notable applications in several branches of mathematics, physics and engineering. In particular, the most important and useful polynomials of this Appell family (so-called Bernoulli polynomials) are obtained by taking $f(t) = \frac{t}{e^t - 1}$ in (1.1). Hence the Bernoulli polynomials are defined by (see [24]; see also [28])

$$\frac{t}{e^t - 1}e^{ut} = \sum_{n=0}^{\infty} \mathbb{B}_n(u)\frac{t^n}{n!}, \quad (|t| < 2\pi). \tag{1.2}$$

For $u = 0$, (1.2) reduces to

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \mathbb{B}_n \frac{t^n}{n!}, \tag{1.3}$$

where $\mathbb{B}_n$ are the familiar Bernoulli numbers, which are directly connected with several combinatorial numbers such as Cauchy numbers, harmonic numbers and Stirling numbers of the second kind.

Due to notable applications of Bernoulli polynomials, Apostol [1] defined a new generalized Bernoulli polynomials $\mathbb{B}_n(u; \beta)$ so-called Apostol-Bernoulli polynomials by introducing a complex parameter β in the generating function of classical Bernoulli polynomials as follows:

$$\frac{t}{\beta e^t - 1} e^{ut} = \sum_{n=0}^{\infty} \mathbb{B}_n(u; \beta) \frac{t^n}{n!}, \quad (\beta \in \mathbb{C}, \ |t| < |\log \beta|). \tag{1.4}$$

For $\beta = 1$, (1.4) easily reduces to (1.2). Moreover, we have

$$\mathbb{B}_n(\beta) = \mathbb{B}_n(0; \beta),$$

where $\mathbb{B}_n(\beta)$ denotes the corresponding Apostol-Bernoulli numbers (see [1], see also [26]).

The higher order Bernoulli polynomials and numbers are given, respectively, by (see [21], [22])

$$\left(\frac{t}{e^t - 1}\right)^p e^{ut} = \sum_{n=0}^{\infty} \mathbb{B}_n^{(p)}(u) \frac{t^n}{n!} \tag{1.5}$$

and

$$\left(\frac{t}{e^t - 1}\right)^p = \sum_{n=0}^{\infty} \mathbb{B}_n^{(p)} \frac{t^n}{n!}, \quad (|t| < 2\pi; \ 1^p := 1). \tag{1.6}$$

If we set $p = 1$ in (1.5) and (1.6) then we get the classical Bernoulli polynomials and numbers defined by (1.2) and (1.3), respectively.

Motivated by the useful extensions of classical Bernoulli polynomials, Luo and Srivastava in [20] introduced and investigated the following Apostol-Bernoulli polynomials $\mathbb{B}_n^{(p)}(u; \beta)$ of order p:

$$\left(\frac{t}{\beta e^t - 1}\right)^p e^{ut} = \sum_{n=0}^{\infty} \mathbb{B}_n^{(p)}(u; \beta) \frac{t^n}{n!}, \quad (\beta \in \mathbb{C}, \ |t| < |\log \beta|; \ 1^p := 1).$$

A further generalization of Bernoulli polynomials so-called poly-Bernoulli polynomials are defined by Bayad and Hamahata [3] as follows:

$$\frac{Li_k(1 - e^{-t})}{1 - e^{-t}} e^{ut} = \sum_{n=0}^{\infty} \mathbb{B}_n^{(k)}(u) \frac{t^n}{n!}, \tag{1.7}$$

where Li_k denotes the polylogarithmic functions defined by

$$Li_k(z) = \sum_{n=0}^{\infty} \frac{z^n}{n^k}, \quad (k \in \mathbb{Z}).$$

For more details about the above type polynomials and numbers, we refer to [2, 6–23] and the references cited therein.

Recently, by using the exponential as well as trigonometric functions, Kim and Ryoo [17] proposed the two parametric kinds of Bernoulli polynomials as follows:

$$\left(\frac{t}{e^t - 1}\right) e^{ut} \cos vt = \sum_{n=0}^{\infty} \mathbb{B}_n^{(c)}(u, v) \frac{t^n}{n!} \quad \text{and} \quad \left(\frac{t}{e^t - 1}\right) e^{ut} \sin vt = \sum_{n=0}^{\infty} \mathbb{B}_n^{(s)}(u, v) \frac{t^n}{n!}.$$

Moreover, they also introduced and investigated the two parametric kinds of Euler polynomials in [17].

Afterwards, Srivastava et al. [29] presented the following two parametric kinds of Apostol-Bernoulli polynomials:

$$\left(\frac{t}{\beta e^t - 1}\right) e^{ut} \cos vt = \sum_{n=0}^{\infty} \mathbb{B}_n^{(c)}(u, v; \beta) \frac{t^n}{n!} \tag{1.8}$$

and

$$\left(\frac{t}{\beta e^t - 1}\right) e^{ut} \sin vt = \sum_{n=0}^{\infty} \mathbb{B}_n^{(s)}(u, v; \beta) \frac{t^n}{n!}. \tag{1.9}$$

More recently, Srivastava and Kizilates [27] considered a further generalization of above parametric Apostol-Bernoulli polynomials as follows:

$$\left(\frac{t}{\beta e^t - 1}\right)^p e^{ut} \cos vt = \sum_{n=0}^{\infty} \mathbb{B}_n^{(c,p)}(u, v; \beta) \frac{t^n}{n!} \tag{1.10}$$

and

$$\left(\frac{t}{\beta e^t - 1}\right)^p e^{ut} \sin vt = \sum_{n=0}^{\infty} \mathbb{B}_n^{(s,p)}(u, v; \beta) \frac{t^n}{n!}. \tag{1.11}$$

It is clear that the case $p = 1$ in (1.10) and (1.11), yields the known parametric kinds of Apostol-Bernoulli polynomials defined by (1.8) and (1.9), respectively.

The main object of this paper is to present the two parametric kinds of poly-Bernoulli polynomials by means of the two specific generating functions. Some fundamental properties, such as summation formulae, differential formulae and relations with other polynomials and numbers for our introduced polynomials are also investigated. Furthermore, we derive the connections of our parametric poly-Bernoulli polynomials with the familiar Voigt functions $K(u, v)$ and $L(u, v)$. The Voigt functions $K(u, v)$ and $L(u, v)$ are defined by the following Laplace type integral transforms (see [25]):

$$K(u, v) = \frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{-vt - t^2/4} \cos(ut) dt, \quad (u \in \mathbb{R}, v \in \mathbb{R}^+) \tag{1.12}$$

and

$$L(u, v) = \frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{-vt - t^2/4} \sin(ut) dt, \quad (u \in \mathbb{R}, v \in \mathbb{R}^+).$$

2. Parametric kinds of poly-Bernoulli polynomials

In this section, we present two parametric kinds of poly-Bernoulli polynomials by defining the two specific generating functions.

By virtue of (1.7), we consider

$$\sum_{n=0}^{\infty} \mathbb{B}_n^{(k)}(u + iv) \frac{t^n}{n!} = \frac{\text{Li}_k(1 - e^{-t})}{1 - e^{-t}} e^{(u+iv)t} = \frac{\text{Li}_k(1 - e^{-t})}{1 - e^{-t}} e^{ut} [\cos vt + i \sin vt], \quad (u, v \in \mathbb{R}) \tag{2.1}$$

and

$$\sum_{n=0}^{\infty} \mathbb{B}_n^{(k)}(u - iv) \frac{t^n}{n!} = \frac{\text{Li}_k(1 - e^{-t})}{1 - e^{-t}} e^{ut} [\cos vt - i \sin vt]. \tag{2.2}$$

From (2.1) and (2.2), we have

$$\frac{\text{Li}_k(1 - e^{-t})}{1 - e^{-t}} e^{ut} \cos vt = \sum_{n=0}^{\infty} \left[\frac{\mathbb{B}_n^{(k)}(u + iv) + \mathbb{B}_n^{(k)}(u - iv)}{2} \right] \frac{t^n}{n!}$$

and

$$\frac{\text{Li}_k(1-e^{-t})}{1-e^{-t}}e^{ut}\sin vt = \sum_{n=0}^{\infty}\left[\frac{\mathbb{B}_n^{(k)}(u+iv)+\mathbb{B}_n^{(k)}(u-iv)}{2i}\right]\frac{t^n}{n!}.$$

Hence, we define here the two parametric kinds (cosine and sine) of poly-Bernoulli polynomials, as follow.

Definition 2.1. Two parametric kinds of poly-Bernoulli polynomials or the cosine-poly-Bernoulli polynomials $\mathbb{B}_n^{(c,k)}(u,v)$ and sine-poly-Bernoulli polynomials $\mathbb{B}_n^{(s,k)}(u,v)$ for nonnegative integer n are defined, respectively, by

$$\sum_{n=0}^{\infty}\mathbb{B}_n^{(c,k)}(u,v)\frac{t^n}{n!} = \frac{\text{Li}_k(1-e^{-t})}{1-e^{-t}}e^{ut}\cos vt, \quad (u,v\in\mathbb{R},|t|<2\pi) \tag{2.3}$$

and

$$\sum_{n=0}^{\infty}\mathbb{B}_n^{(s,k)}(u,v)\frac{t^n}{n!} = \frac{\text{Li}_k(1-e^{-t})}{1-e^{-t}}e^{ut}\sin vt, \quad (u,v\in\mathbb{R},|t|<2\pi).$$

It is noticed that

$$\mathbb{B}_n^{(c,k)}(u,0) = \mathbb{B}_n^{(c,k)}(u), \qquad \mathbb{B}_n^{(s,k)}(u,0) = 0, \qquad \mathbb{B}_n^{(c,k)}(0,v) = \mathbb{B}_n^{(c,k)}(v),$$

$$\mathbb{B}_n^{(s,k)}(0,v) = \mathbb{B}_n^{(s,k)}(v), \qquad \mathbb{B}_n^{(c,k)}(0,0) = \mathbb{B}_n^{(c,k)}, \qquad \mathbb{B}_n^{(s,k)}(0,0) = 0.$$

Moreover, we consider

$$e^{ut}\cos vt = \sum_{n=0}^{\infty}C_n(u,v)\frac{t^n}{n!}, \quad \text{and} \quad e^{ut}\sin vt = \sum_{n=0}^{\infty}S_n(u,v)\frac{t^n}{n!}. \tag{2.4}$$

The Taylor-Maclaurin expansions of $e^{ut}\cos vt$ and $e^{ut}\sin vt$ for real u and v are given, respectively, by

$$e^{ut}\cos vt = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{[\frac{n}{2}]}\binom{n}{2m}(-1)^m u^{n-2m}v^{2m}\right)\frac{t^n}{n!} \tag{2.5}$$

and

$$e^{ut}\sin vt = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{[\frac{n-1}{2}]}\binom{n}{2m+1}(-1)^m u^{n-2m-1}v^{2m+1}\right)\frac{t^n}{n!}, \tag{2.6}$$

where $[\,.\,]$ denotes the greatest integer.

Hence, from (2.4)-(2.6), we have

$$C_n(u,v) = \sum_{m=0}^{[\frac{n}{2}]}\binom{n}{2m}(-1)^m u^{n-2m}v^{2m} \quad \text{and} \quad S_n(u,v) = \sum_{m=0}^{[\frac{n-1}{2}]}\binom{n}{2m+1}(-1)^m u^{n-2m-1}v^{2m+1}.$$

3. Properties of $\mathbb{B}_n^{(c,k)}(u,v)$ and $\mathbb{B}_n^{(s,k)}(u,v)$

In this section, we derive some fundamental properties (like summation formulae, differential formulae and relations with other standard polynomials and numbers) of our two parametric kinds poly-Bernoulli polynomials.

Theorem 3.1. *For $n \in \mathbb{N}_0$, each of the following identities holds true:*

$$\mathbb{B}_n^{(c,k)}(u,v) = \sum_{r=0}^{n} \binom{n}{r} \mathbb{B}_{n-r}^{(k)} \, C_r(u,v) \tag{3.1}$$

and

$$\mathbb{B}_n^{(s,k)}(u,v) = \sum_{r=0}^{n} \binom{n}{r} \mathbb{B}_{n-r}^{(k)} \, S_r(u,v). \tag{3.2}$$

Proof. Taking

$$\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} e^{ut} \cos vt = \sum_{n=0}^{\infty} \mathbb{B}_n^{(k)} \frac{t^n}{n!} \sum_{r=0}^{\infty} C_r(u,v) \frac{t^r}{r!},$$

$$\sum_{n=0}^{\infty} \mathbb{B}_n^{(c,k)}(u,v) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \mathbb{B}_n^{(k)} C_r(u,v) \frac{t^{n+r}}{n!r!} = \sum_{n=0}^{\infty} \sum_{r=0}^{n} \mathbb{B}_{n-r}^{(k)} C_r(u,v) \frac{t^n}{(n-r)!r!},$$

which on equating the like powers of t gives our needed identity (3.1). Similarly, we can prove our second identity (3.2). $\qquad\square$

Theorem 3.2. *For $n \in \mathbb{N}_0$ and $w \in \mathbb{R}$, we have*

$$\mathbb{B}_n^{(c,k)}(u+w,v) = \sum_{r=0}^{n} \binom{n}{r} w^r \mathbb{B}_{n-r}^{(c,k)}(u,v) \tag{3.3}$$

and

$$\mathbb{B}_n^{(s,k)}(u+w,v) = \sum_{r=0}^{n} \binom{n}{r} w^r \mathbb{B}_{n-r}^{(s,k)}(u,v). \tag{3.4}$$

Proof. From (2.3), we get

$$\sum_{n=0}^{\infty} \mathbb{B}_n^{(c,k)}(u+w,v) \frac{t^n}{n!} = \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} e^{(u+w)t} \cos vt$$

$$= \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} e^{ut} \cos vt \; e^{wt}$$

$$= \sum_{n=0}^{\infty} \mathbb{B}_n^{(c,k)}(u,v) \frac{t^n}{n!} \sum_{r=0}^{\infty} \frac{(wt)^r}{r!} = \sum_{n=0}^{\infty} \sum_{r=0}^{n} \binom{n}{r} w^r \mathbb{B}_{n-r}^{(c,k)}(u,v) \frac{t^n}{n!},$$

which on equating the coefficients of $\frac{t^n}{n!}$, yields our claimed result (3.3). Similarly, we can derive our second result (3.4). $\qquad\square$

Corollary 3.3. *On setting $w = u$ in Theorem 3.2, we have*

$$\mathbb{B}_n^{(c,k)}(2u,v) = \sum_{r=0}^{n} \binom{n}{r} u^r \mathbb{B}_{n-r}^{(c,k)}(u,v) \quad and \quad \mathbb{B}_n^{(s,k)}(2u,v) = \sum_{r=0}^{n} \binom{n}{r} u^r \mathbb{B}_{n-r}^{(s,k)}(u,v).$$

Corollary 3.4. *On setting $w = 1$ in Theorem 3.2, we have*

$$\mathbb{B}_n^{(c,k)}(u+1,v) = \sum_{r=0}^{n} \binom{n}{r} \mathbb{B}_{n-r}^{(c,k)}(u,v) \quad and \quad \mathbb{B}_n^{(s,k)}(u+1,v) = \sum_{r=0}^{n} \binom{n}{r} \mathbb{B}_{n-r}^{(s,k)}(u,v).$$

Theorem 3.5. *For* $n \in \mathbb{N}_0$ *and* $w \in \mathbb{R}$, *we have*

$$\mathbb{B}_n^{(c,k)}(u, v+w) = \sum_{m=0}^{\left[\frac{n}{2}\right]} \binom{n}{2m} (-1)^m w^{2m} \mathbb{B}_{n-2m}^{(c,k)}(u,v) - \sum_{m=0}^{\left[\frac{n-1}{2}\right]} \frac{(-1)^m n! \, w^{2m+1} \mathbb{B}_{n-2m-1}^{(s,k)}(u,v)}{m!(n-2m-1)! 2^{2m} \left(\frac{3}{2}\right)_m}$$

and

$$\mathbb{B}_n^{(s,k)}(u, v+w) = \sum_{m=0}^{\left[\frac{n}{2}\right]} \binom{n}{2m} (-1)^m w^{2m} \mathbb{B}_{n-2m}^{(s,k)}(u,v) + \sum_{m=0}^{\left[\frac{n-1}{2}\right]} \frac{(-1)^m n! \, w^{2m+1} \mathbb{B}_{n-2m-1}^{(c,k)}(u,v)}{m!(n-2m-1)! 2^{2m} \left(\frac{3}{2}\right)_m}.$$

Proof. We have

$$\sum_{n=0}^{\infty} \mathbb{B}_n^{(c,k)}(u, v+w) \frac{t^n}{n!}$$

$$= \frac{Li_k(1-e^{-t})}{1-e^{-t}} e^{ut} \cos(v+w)t$$

$$= \frac{Li_k(1-e^{-t})}{1-e^{-t}} e^{ut} [\cos vt \cos wt - \sin vt \sin wt]$$

$$= \frac{Li_k(1-e^{-t})}{1-e^{-t}} e^{ut} \cos vt \sum_{m=0}^{\infty} \frac{(-1)^m (wt)^{2m}}{2m!} - \frac{Li_k(1-e^{-t})}{1-e^{-t}} e^{ut} \sin vt \sum_{m=0}^{\infty} \frac{(-1)^m (wt)^{2m+1}}{2^{2m} m! \left(\frac{3}{2}\right)_m}$$

$$= \sum_{n=0}^{\infty} \mathbb{B}_n^{(c,k)}(u,v) \frac{t^n}{n!} \sum_{m=0}^{\infty} \frac{(-1)^m (wt)^{2m}}{2m!} - \sum_{n=0}^{\infty} \mathbb{B}_n^{(s,k)}(u,v) \frac{t^n}{n!} \sum_{m=0}^{\infty} \frac{(-1)^m (wt)^{2m+1}}{2^{2m} m! \left(\frac{3}{2}\right)_m}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m (w)^{2m} \mathbb{B}_n^{(c,k)}(u,v) t^{n+2m}}{2m! \, n!} - \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m w^{2m+1} \mathbb{B}_n^{(s,k)}(u,v) t^{n+2m+1}}{2^{2m} m! \, n! \left(\frac{3}{2}\right)_m}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^m (w)^{2m} \mathbb{B}_{n-2m}^{(c,k)}(u,v) t^n}{2m! \, (n-2m)!} - \sum_{n=0}^{\infty} \sum_{m=0}^{\left[\frac{n-1}{2}\right]} \frac{(-1)^m w^{2m+1} \mathbb{B}_{n-2m-1}^{(s,k)}(u,v) t^n}{2^{2m} m! \, (n-2m-1)! \left(\frac{3}{2}\right)_m}.$$

Finally, by comparing the coefficients of t^n, we get our assertion (3.5). In a similar way, we can derive our second assertion (3.5). □

Corollary 3.6. *If we replace* w *by* $-w$ *in Theorem* 3.5*, then we have*

$$\mathbb{B}_n^{(c,k)}(u, v-w) = \sum_{m=0}^{\left[\frac{n}{2}\right]} \binom{n}{2m} (-1)^m w^{2m} \mathbb{B}_{n-2m}^{(c,k)}(u,v) + \sum_{m=0}^{\left[\frac{n-1}{2}\right]} \frac{(-1)^m n! \, w^{2m+1} \mathbb{B}_{n-2m-1}^{(s,k)}(u,v)}{m!(n-2m-1)! 2^{2m} \left(\frac{3}{2}\right)_m}$$

and

$$\mathbb{B}_n^{(s,k)}(u, v-w) = \sum_{m=0}^{\left[\frac{n}{2}\right]} \binom{n}{2m} (-1)^m w^{2m} \mathbb{B}_{n-2m}^{(s,k)}(u,v) - \sum_{m=0}^{\left[\frac{n-1}{2}\right]} \frac{(-1)^m n! \, w^{2m+1} \mathbb{B}_{n-2m-1}^{(c,k)}(u,v)}{m!(n-2m-1)! 2^{2m} \left(\frac{3}{2}\right)_m}.$$

Theorem 3.7. *Let* $n \in \mathbb{N}$. *Then we have*

$$\frac{\partial}{\partial u} \mathbb{B}_n^{(c,k)}(u,v) = n \mathbb{B}_{n-1}^{(c,k)}(u,v), \tag{3.5}$$

$$\frac{\partial}{\partial v} \mathbb{B}_n^{(c,k)}(u,v) = -n \mathbb{B}_{n-1}^{(s,k)}(u,v), \tag{3.6}$$

$$\frac{\partial}{\partial u}\mathbb{B}_n^{(s,k)}(u,v) = n\mathbb{B}_{n-1}^{(s,k)}(u,v),\tag{3.7}$$

and

$$\frac{\partial}{\partial v}\mathbb{B}_n^{(s,k)}(u,v) = n\mathbb{B}_{n-1}^{(c,k)}(u,v).\tag{3.8}$$

Proof. From (2.3), we have

$$\sum_{n=0}^{\infty}\mathbb{B}_n^{(c,k)}(u,v)\frac{t^n}{n!} = \frac{Li_k(1-e^{-t})}{1-e^{-t}}e^{ut}\cos vt.$$

On partially differentiating with respect to u, we get

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial u}\mathbb{B}_n^{(c,k)}(u,v)\frac{t^n}{n!} = t\frac{Li_k(1-e^{-t})}{1-e^{-t}}e^{ut}\cos vt = \sum_{n=0}^{\infty}\mathbb{B}_n^{(c,k)}(u,v)\frac{t^{n+1}}{n!},$$

which, upon comparing the coefficients of t^n on both sides, yields the desired result (3.5). Similarly, we can derive (3.6)-(3.8). □

The rasing factorial, denoted by $(u)^\gamma$, is defined by

$$(u)^\gamma = u(u+1)\cdots(u+\gamma-1).$$

We also note that the negative binomial expansion is given by

$$(\beta+\gamma)^{-k} = \sum_{m=0}^{\infty}(-1)^m\binom{k+m-1}{m}\beta^m\gamma^{-k-m}.$$

Theorem 3.8. *For $n, m \in \mathbb{N}_0$, each of the following identities holds true:*

$$\mathbb{B}_n^{(c,k)}(u,v) = \sum_{m=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}(u)^{(m)}S_2(l,m)\mathbb{B}_{n-l}^{(c,k)}(-m,v)\tag{3.9}$$

and

$$\mathbb{B}_n^{(s,k)}(u,v) = \sum_{m=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}(u)^{(m)}S_2(l,m)\mathbb{B}_{n-l}^{(s,k)}(-m,v),\tag{3.10}$$

where $S_2(l,m)$ denotes the Stirling numbers of second kind and is defined by (see [4])

$$\sum_{l=0}^{\infty}S_2(l,m)\frac{t^l}{l!} = \frac{(e^t-1)^m}{m!}.$$

Proof. From (2.3), we have

$$\sum_{n=0}^{\infty}\mathbb{B}_n^{(c,k)}(u,v)\frac{t^n}{n!} = \frac{Li_k(1-e^{-t})}{1-e^{-t}}e^{ut}\cos vt$$

$$= \frac{Li_k(1-e^{-t})}{1-e^{-t}}[1-(1-e^{-t})]^{-u}\cos vt$$

$$= \frac{Li_k(1-e^{-t})}{1-e^{-t}}\cos vt\sum_{m=0}^{\infty}\binom{u+m-1}{m}(1-e^{-t})^m$$

$$= \sum_{m=0}^{\infty} (u)^{(m)} \frac{(e^t - 1)^m}{m!} \frac{\text{Li}_k(1 - e^{-t})}{1 - e^{-t}} e^{-mt} \cos vt$$

$$= \sum_{m=0}^{\infty} (u)^{(m)} \sum_{l=0}^{\infty} S_2(l, m) \frac{t^l}{l!} \sum_{n=0}^{\infty} \mathbb{B}_n^{(c,k)}(-m, v) \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{l=0}^{n} \binom{n}{l} (u)^m S_2(l, m) \mathbb{B}_{n-l}^{(c,k)}(-m, v) \frac{t^n}{n!},$$

which gives our needed result (3.9). Similarly, we can prove our second relation (3.10). ☐

Theorem 3.9. *For* $n \in \mathbb{N}$, *we have*

$$\mathbb{B}_n^{(c,k)}(u+1, v) - \mathbb{B}_n^{(c,k)}(u, v) = \sum_{m=0}^{n-1} \frac{n!}{m!(n-1-m)!} \mathbb{B}_{n-1-m}^{(c,k)}(u, v) \mathbb{B}_m^{(-1)} \qquad (3.11)$$

and

$$\mathbb{B}_n^{(s,k)}(u+1, v) - \mathbb{B}_n^{(s,k)}(u, v) = \sum_{m=0}^{n-1} \frac{n!}{m!(n-1-m)!} \mathbb{B}_{n-1-m}^{(s,k)}(u, v) \mathbb{B}_m^{(-1)}, \qquad (3.12)$$

where $\mathbb{B}_m^{(-1)}$ *are the generalized Bernoulli numbers given in* (1.6).

Proof. Taking

$$\frac{\text{Li}_k(1 - e^{-t})}{1 - e^{-t}} e^{ut} \cos vt (e^t - 1) = t \frac{\text{Li}_k(1 - e^{-t})}{1 - e^{-t}} e^{ut} \cos vt \left(\frac{t}{e^t - 1} \right)^{-1}$$

$$= t \sum_{n=0}^{\infty} \mathbb{B}_n^{(c,k)}(u, v) \frac{t^n}{n!} \sum_{m=0}^{\infty} \mathbb{B}_m^{(-1)} \frac{t^m}{m!},$$

$$\sum_{n=0}^{\infty} \mathbb{B}_n^{(c,k)}(u+1, v) \frac{t^n}{n!} - \sum_{n=0}^{\infty} \mathbb{B}_n^{(c,k)}(u, v) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \mathbb{B}_n^{(c,k)}(u, v) \mathbb{B}_m^{(-1)} \frac{t^{n+m+1}}{n!m!}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \mathbb{B}_{n-m}^{(c,k)}(u, v) \mathbb{B}_m^{(-1)} \frac{t^{n+1}}{(n-m)!m!}.$$

By comparing the coefficients of t^n on both sides, we get our claimed result (3.11). In a similar way, we can derive our second result (3.12). ☐

Theorem 3.10. *For* $n, m \in \mathbb{N}_0$, *each of the following relations holds true:*

$$\mathbb{B}_n^{(c,k)}(u+1, v) + \mathbb{B}_n^{(c,k)}(u, v) = 2 \sum_{m=0}^{n} \binom{n}{m} \mathbb{B}_{n-m}^{(c,k)}(u, v) \mathbb{E}_m^{(-1)} \qquad (3.13)$$

and

$$\mathbb{B}_n^{(s,k)}(u+1, v) + \mathbb{B}_n^{(s,k)}(u, v) = 2 \sum_{m=0}^{n} \binom{n}{m} \mathbb{B}_{n-m}^{(s,k)}(u, v) \mathbb{E}_m^{(-1)}, \qquad (3.14)$$

where $\mathbb{E}_m^{(-1)}$ *are the generalized Euler numbers defined by (see [6])*

$$\left(\frac{2}{e^t + 1} \right)^{\lambda} = \sum_{m=0}^{\infty} \mathbb{E}_m^{(\lambda)} \frac{t^m}{m!}.$$

Proof. Taking

$$\frac{Li_k(1-e^{-t})}{1-e^{-t}}e^{ut}\cos vt(e^t+1) = 2\frac{Li_k(1-e^{-t})}{1-e^{-t}}e^{ut}\cos vt\left(\frac{2}{e^t+1}\right)^{-1}$$

$$= 2\sum_{n=0}^{\infty}\mathbb{B}_n^{(c,k)}(u,v)\frac{t^n}{n!}\sum_{m=0}^{\infty}\mathbb{E}_m^{(-1)}\frac{t^m}{m!},$$

$$\sum_{n=0}^{\infty}\mathbb{B}_n^{(c,k)}(u+1,v)\frac{t^n}{n!} + \sum_{n=0}^{\infty}\mathbb{B}_n^{(c,k)}(u,v)\frac{t^n}{n!} = 2\sum_{n=0}^{\infty}\sum_{m=0}^{n}\mathbb{B}_{n-m}^{(c,k)}(u,v)\mathbb{E}_m^{(-1)}\frac{t^n}{(n-m)!m!},$$

which, on equating the like powers of t on both sides, yields the desired result (3.13). In a similar way, the assertion (3.14) can be derived. $\square$

Theorem 3.11. *Let* $n \in \mathbb{N}$. *Then we have*

$$\mathbb{B}_n^{(c,k)}(u+1,v)+\mathbb{B}_n^{(c,k)}(u,v) = 2\sum_{m=0}^{n-1}\frac{n!}{m!(n-1-m)!}\mathbb{B}_{n-1-m}^{(c,k)}(u,v)\mathbb{G}_m^{(-1)}$$

and

$$\mathbb{B}_n^{(s,k)}(u+1,v)+\mathbb{B}_n^{(s,k)}(u,v) = 2\sum_{m=0}^{n-1}\frac{n!}{m!(n-1-m)!}\mathbb{B}_{n-1-m}^{(s,k)}(u,v)\mathbb{G}_m^{(-1)},$$

where $\mathbb{G}_m^{(-1)}$ *are the generalized Genocchi numbers defined by (see [6])*

$$\left(\frac{2t}{e^t+1}\right)^{\lambda} = \sum_{m=0}^{\infty}\mathbb{G}_m^{(\lambda)}\frac{t^m}{m!}.$$

Proof. Proof of this theorem is similar to that of Theorems 3.9 and 3.10. $\square$

Theorem 3.12. *For* $n, m \in \mathbb{N}_0$, *each of the following relations holds true:*

$$\mathbb{B}_n^{(c,k)}(u,v) = \sum_{m=0}^{n}\binom{n}{m}\mathbb{B}_{n-m}^{(c,k)}(v)[(1+v)\mathbb{F}_m(u,v)-v\,\mathbb{F}_m(u+1,v)] \tag{3.15}$$

and

$$\mathbb{B}_n^{(s,k)}(u,v) = \sum_{m=0}^{n}\binom{n}{m}\mathbb{B}_{n-m}^{(s,k)}(v)[(1+v)\mathbb{F}_m(u,v)-v\,\mathbb{F}_m(u+1,v)], \tag{3.16}$$

where $\mathbb{F}_m(u,v)$ *are the classical two-variable Fubini polynomials defined by (see [4])*

$$\sum_{m=0}^{\infty}\mathbb{F}_m(u,v)\frac{t^m}{m!} = \frac{e^{ut}}{1-v(e^t-1)}.$$

Proof. From (2.3), we have

$$\sum_{n=0}^{\infty}\mathbb{B}_n^{(c,k)}(u,v)\frac{t^n}{n!} = \frac{Li_k(1-e^{-t})}{1-e^{-t}}e^{ut}\cos vt$$

$$= \frac{Li_k(1-e^{-t})}{1-e^{-t}}\cos vt\frac{e^{ut}}{[1-v(e^t-1)]}[1-v(e^t-1)]$$

$$= (1+v)\frac{Li_k(1-e^{-t})}{1-e^{-t}}\cos vt\frac{e^{ut}}{[1-v(e^t-1)]} - v\frac{Li_k(1-e^{-t})}{1-e^{-t}}\cos vt\frac{e^{(u+1)t}}{[1-v(e^t-1)]}$$

$$= (1+v) \sum_{n=0}^{\infty} \mathbb{B}_n^{(c,k)}(v) \frac{t^n}{n!} \sum_{m=0}^{\infty} \mathbb{F}_m(u,v) \frac{t^m}{m!} - v \sum_{n=0}^{\infty} \mathbb{B}_n^{(c,k)}(v) \frac{t^n}{n!} \sum_{m=0}^{\infty} \mathbb{F}_m(u+1,v) \frac{t^m}{m!}$$

$$= (1+v) \sum_{n=0}^{\infty} \sum_{m=0}^{n} \binom{n}{m} \mathbb{B}_{n-m}^{(c,k)}(v) \mathbb{F}_m(u,v) \frac{t^n}{n!} - v \sum_{n=0}^{\infty} \sum_{m=0}^{n} \binom{n}{m} \mathbb{B}_{n-m}^{(c,k)}(v) \mathbb{F}_m(u+1,v) \frac{t^n}{n!},$$

which, on equating the coefficients of $\frac{t^n}{n!}$ on both sides, gives our first claimed result (3.15). Similarly, we can derive our second result (3.16). $\qquad \square$

Theorem 3.13. *For* $n, m \in \mathbb{N}_0$, *each of the following relations holds true:*

$$\mathbb{B}_n^{(c,k)}(u,v) = \frac{n!m!}{2(n+m)!} \sum_{r=0}^{n+m} \binom{n+m}{r} \mathbb{B}_{n+m-r}^{(c,k)}(v) \mathbb{E}_{m,r}(u) + \frac{n!m!}{2} \sum_{r=0}^{n} \sum_{j=0}^{n} \frac{\mathbb{B}_{n-r-j}^{(c,k)}(v) \mathbb{E}_{m,r}(u)}{r!(n-r-j)!(j+m)!} \quad (3.17)$$

and

$$\mathbb{B}_n^{(s,k)}(u,v) = \frac{n!m!}{2(n+m)!} \sum_{r=0}^{n+m} \binom{n+m}{r} \mathbb{B}_{n+m-r}^{(s,k)}(v) \mathbb{E}_{m,r}(u) + \frac{n!m!}{2} \sum_{r=0}^{n} \sum_{j=0}^{n} \frac{\mathbb{B}_{n-r-j}^{(s,k)}(v) \mathbb{E}_{m,r}(u)}{r!(n-r-j)!(j+m)!}, \quad (3.18)$$

where $\mathbb{E}_{m,r}(u)$ *denotes the truncated Euler polynomials defined by (see [18])*

$$\sum_{r=0}^{\infty} \mathbb{E}_{m,r}(u) \frac{t^r}{r!} = \frac{2t^m/m!}{e^t + 1 - \sum\limits_{j=0}^{m-1} t^j/j!} e^{ut}.$$

Proof. We have

$$\sum_{n=0}^{\infty} \mathbb{B}_n^{(c,k)}(u,v) \frac{t^n}{n!} = \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} e^{ut} \cos vt \frac{2t^m/m!}{\left(e^t + 1 - \sum\limits_{j=0}^{m-1} t^j/j!\right)} \frac{\left(e^t + 1 - \sum\limits_{j=0}^{m-1} t^j/j!\right)}{2t^m/m!}$$

$$= \frac{m!}{2t^m} \sum_{n=0}^{\infty} \mathbb{B}_n^{(c,k)}(v) \frac{t^n}{n!} \sum_{r=0}^{\infty} \mathbb{E}_{m,r}(u) \frac{t^r}{r!} \left(\sum_{j=m}^{\infty} \frac{t^j}{j!} + 1 \right)$$

$$= \frac{m!}{2} \sum_{n=0}^{\infty} \sum_{r=0}^{n} \mathbb{B}_{n-r}^{(c,k)}(v) \mathbb{E}_{m,r}(u) \frac{t^{n-m}}{r!(n-r)!} \left(\sum_{j=0}^{\infty} \frac{t^{j+m}}{(j+m)!} + 1 \right)$$

$$= \frac{m!}{2} \sum_{n=0}^{\infty} \sum_{r=0}^{n} \mathbb{B}_{n-r}^{(c,k)}(v) \mathbb{E}_{m,r}(u) \frac{t^{n-m}}{r!(n-r)!}$$

$$+ \frac{m!}{2} \sum_{n=0}^{\infty} \sum_{r=0}^{n} \sum_{j=0}^{n} \mathbb{B}_{n-r-j}^{(c,k)}(v) \mathbb{E}_{m,r}(u) \frac{t^n}{r!(n-r-j)!(j+m)!}.$$

By comparing the coefficients of t^n on both sides in the above expression, we easily get our claimed result (3.17). Similarly, we can derive our second result (3.18). $\qquad \square$

Theorem 3.14. *For* $n, m \in \mathbb{N}_0$, *we have*

$$\mathbb{B}_n^{(c,k)}(u,v) = n!m! \sum_{r=0}^{n} \sum_{j=0}^{n} \frac{\mathbb{B}_{n-j-r}^{(c,k)}(v) \mathbb{B}_{m,r}(u)}{r!(n-j-r)!(j+m)!} \quad (3.19)$$

and

$$\mathbb{B}_n^{(s,k)}(u,v) = n!m! \sum_{r=0}^{n} \sum_{j=0}^{n} \frac{\mathbb{B}_{n-j-r}^{(s,k)}(v)\mathbb{B}_{m,r}(u)}{r!(n-j-r)!(j+m)!}, \tag{3.20}$$

where $\mathbb{B}_{m,r}(u)$ denotes the truncated Bernoulli polynomials defined by (see [5])

$$\sum_{r=0}^{\infty} \mathbb{B}_{m,r}(u)\frac{t^r}{r!} = \frac{t^m/m!}{\left(e^t - \sum_{j=0}^{m-1} t^j/j!\right)}.$$

Proof. From (2.3), we have

$$\sum_{n=0}^{\infty} \mathbb{B}_n^{(c,k)}(u,v)\frac{t^n}{n!} = \frac{Li_k(1-e^{-t})}{1-e^{-t}}e^{ut}\cos vt \frac{t^m/m!}{\left(e^t - \sum_{j=0}^{m-1} t^j/j!\right)} \frac{\left(e^t - \sum_{j=0}^{m-1} t^j/j!\right)}{t^m/m!}$$

$$= \frac{m!}{t^m} \sum_{n=0}^{\infty} \mathbb{B}_n^{(c,k)}(v)\frac{t^n}{n!} \sum_{r=0}^{\infty} \mathbb{B}_{m,r}(u)\frac{t^r}{r!} \left(\sum_{j=m}^{\infty} \frac{t^j}{j!}\right)$$

$$= \frac{m!}{t^m} \sum_{n=0}^{\infty} \sum_{r=0}^{n} \mathbb{B}_{n-r}^{(c,k)}(v)\mathbb{B}_{m,r}(u)\frac{t^n}{r!(n-r)!} \sum_{j=0}^{\infty} \frac{t^{j+m}}{(j+m)!}$$

$$= m! \sum_{n=0}^{\infty} \sum_{r=0}^{n} \sum_{j=0}^{n} \mathbb{B}_{n-j-r}^{(c,k)}(v)\mathbb{B}_{m,r}(u)\frac{t^n}{r!(n-j-r)!(j+m)!},$$

which, upon comparing the coefficients of t^n, yields our needed result (3.19). Similarly, we can establish our second results (3.20). □

4. Connections with Voigt functions

In this section, we establish the following connections of our introduced polynomials with the well known Voigt functions $K(u,v)$ and $L(u,v)$.

Theorem 4.1. *For $n \in \mathbb{N}_0$ and $u, v \in \mathbb{R}$, each of the following connections holds true:*

$$\sum_{n=0}^{\infty} \frac{2^n \Gamma(\frac{n+1}{2})\mathbb{B}_n^{(c,k)}(-u,v)}{n!} = \sqrt{\pi} \sum_{n=1}^{\infty} \sum_{r=0}^{n-1} \binom{n-1}{r} \frac{(-1)^r}{n^k} K(v, u+r) \tag{4.1}$$

and

$$\sum_{n=0}^{\infty} \frac{2^n \Gamma(\frac{n+1}{2})\mathbb{B}_n^{(s,k)}(-u,v)}{n!} = \sqrt{\pi} \sum_{n=1}^{\infty} \sum_{r=0}^{n-1} \binom{n-1}{r} \frac{(-1)^r}{n^k} L(v, u+r). \tag{4.2}$$

Proof. By virtue of (2.3), we have

$$\sum_{n=0}^{\infty} \mathbb{B}_n^{(c,k)}(-u,v)\frac{t^n}{n!} = \frac{Li_k(1-e^{-t})}{1-e^{-t}}e^{-ut}\cos vt$$

$$= Li_k(1-e^{-t})(1-e^{-t})^{-1}e^{-ut}\cos vt$$

$$= \sum_{n=1}^{\infty} \frac{(1-e^{-t})^{n-1}}{n^k}e^{-ut}\cos vt = \sum_{n=1}^{\infty} \frac{1}{n^k} \sum_{r=0}^{n-1} (-1)^r \binom{n-1}{r}e^{-rt}e^{-ut}\cos vt.$$

Multiplying by $e^{-t^2/4}$ on both sides of last expression and integrating with respect to t from 0 to ∞ (after interchanging the order of integration and summations which is permissible due to the uniform convergence), and by using (1.12), we get

$$\sum_{n=0}^{\infty} \frac{\mathbb{B}_n^{(c,k)}(-u,v)}{n!} \int_0^{\infty} e^{-\frac{t^2}{4}} t^n dt = \sqrt{\pi} \sum_{n=1}^{\infty} \sum_{r=0}^{n-1} \binom{n-1}{r} \frac{(-1)^r}{n^k} K(v, u+r).$$

Finally, by applying the definition of gamma function on the left hand side of last expression, we easily get our first connection (4.1). Similarly, we can establish our second connection (4.2). $\qquad \square$

5. Conclusions

In the present article, we have proposed two parametric kinds of poly-Bernoulli polynomials by means of two special generating functions. We have also derived some interesting identities for these polynomials by using the series manipulation technique. Furthermore, we have established two connections of our proposed parametric polynomials with the familiar Voigt functions $K(u,v)$ and $L(u,v)$.

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