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Estimation of f-divergence and Shannon entropy by Levinson type inequalities for higher order convex functions via Taylor polynomial

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Abstract

In this paper, Levinson-type inequalities are generalized by using Taylor polynomial for the class of k-convex ($k \geqslant 3$) functions. Bounds for the remainders in new generalized identities involving data points of two types are given by using Čebyšev, Grüss and Ostrowski-type inequalities. In seek of applications of our results to information theory, new generalizations based on f-divergence estimates are also proven. Moreover, some inequalities for Shannon entropies are deduced as well.

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1. Introduction and Preliminaries

The study of convex functions are used as a major tool to solve optimization problems in analysis. However the soil of inequalities involving convex functions is quite fertile as it increases the growth of many branches of mathematics with considerable high rate. That is why the study of such inequalities have been given great importance in literature (see [22]).

Divided differences are observed to be exceptionally useful when we are dealing with capacities having diverse degrees of smoothness. Definition of divided difference is given in [22, p. 14] as follows. The kth-order divided difference of a function $\mathbf{f} : [\zeta_1, \zeta_2] \to \mathbb{R}$ at mutually distinct points $x_0, \dots, x_k \in [\zeta_1, \zeta_2]$ is defined recursively by

$$[x_\rho; \mathbf{f}] = \mathbf{f}(x_\rho), \quad \rho = 0, \dots, k, \quad [x_0, \dots, x_k; \mathbf{f}] = \frac{[x_1, \dots, x_k; \mathbf{f}] - [x_0, \dots, x_{k-1}; \mathbf{f}]}{x_k - x_0}. \tag{1.1}$$

It is easy to see that (1.1) is equivalent to

$$[x_0, \ldots, x_k; \mathbf{f}] = \sum_{\rho=0}^{k} \frac{\mathbf{f}(x_\rho)}{q'(x_\rho)}, \quad \text{where } q(x) = \prod_{j=0}^{k} (x - x_j).$$

The following definition of a real valued convex function is characterized by k^{th}-order divided difference (see [22, p. 15]).

Definition 1.1. A function $\mathbf{f} : [\zeta_1, \zeta_2] \to \mathbb{R}$ is said to be k-convex ($k \geqslant 0$) if and only if for all choices of $(k+1)$ distinct points $x_0, \ldots, x_k \in [\zeta_1, \zeta_2]$, $[x_0, \ldots, x_k; \mathbf{f}] \geqslant 0$ holds.

If this inequality is reversed, then $\mathbf{f}$ is said to be k-concave. If the inequality is strict, then $\mathbf{f}$ is said to be a strictly k-convex (k-concave) function. Note that 0-convex functions, 1-convex functions, and 2-convex functions are non-negative functions, increasing functions and are simply the convex functions, respectively. In [22, p. 16], criteria to examine the k-convexity of a function $\mathbf{f}$ is given as follows.

Theorem 1.2. If $\mathbf{f}^{(k)}$ exists, then $\mathbf{f}$ is k-convex if and only if $\mathbf{f}^{(k)} \geqslant 0$.

In [15], (see also [18, p. 32, Theorem 1]) Ky Fan's inequality is generalized by Levinson for 3-convex functions as follows.

Theorem 1.3. Let $\mathbf{f} : I = (0, 2\alpha) \to \mathbb{R}$ with $\mathbf{f}^{(3)}(t) \geqslant 0$, $x_k \in (0, \alpha)$ and $p_\rho > 0$. Then

$$\frac{1}{P_n} \sum_{\rho=1}^{n} p_\rho \mathbf{f}(x_\rho) - \mathbf{f}\left(\frac{1}{P_n} \sum_{\rho=1}^{n} p_\rho x_\rho\right) \leqslant \frac{1}{P_n} \sum_{\rho=1}^{n} p_\rho \mathbf{f}(2\alpha - x_\rho) - \mathbf{f}\left(\frac{1}{P_n} \sum_{\rho=1}^{n} p_\rho(2\alpha - x_\rho)\right). \tag{1.2}$$

Working with the divided differences, assumptions of differentiability on $\mathbf{f}$ can be weakened. In [23], Popoviciu noted that (1.2) is valid on $(0, 2a)$ for 3-convex functions, while in [7], (see also [18, p. 32, Theorem 2]) Bullen gave different proof of Popoviciu's result and also the converse of (1.2).

Theorem 1.4. Let $\mathbf{f} : I = [\zeta_1, \zeta_2] \to \mathbb{R}$ be 3-convex function and $x_\rho, y_\rho \in [\zeta_1, \zeta_2]$ for $\rho = 1, 2, \ldots, n$ such that

$$\max\{x_1, \ldots, x_n\} \leqslant \min\{y_1, \ldots, y_n\}, \quad x_1 + y_1 = \cdots = x_n + y_n \tag{1.3}$$

and $p_\rho > 0$, then

$$\frac{1}{P_n} \sum_{\rho=1}^{n} p_\rho \mathbf{f}(x_\rho) - \mathbf{f}\left(\frac{1}{P_n} \sum_{\rho=1}^{n} p_\rho x_\rho\right) \leqslant \frac{1}{P_n} \sum_{\rho=1}^{n} p_\rho \mathbf{f}(y_\rho) - \mathbf{f}\left(\frac{1}{P_n} \sum_{\rho=1}^{n} p_\rho y_\rho\right). \tag{1.4}$$

If $\mathbf{f}$ is continuous and $p_\rho > 0$, (1.4) holds for all x_ρ, y_ρ satisfying (1.3), then f is 3-convex.

Functional form of (1.4) is defined as follows:

$$J(\mathbf{f}(\cdot)) = \frac{1}{P_n} \sum_{\rho=1}^{n} p_\rho \mathbf{f}(y_\rho) - \mathbf{f}\left(\frac{1}{P_n} \sum_{\rho=1}^{n} q_\rho y_\rho\right) - \frac{1}{P_n} \sum_{\rho=1}^{n} p_\rho \mathbf{f}(x_\rho) + \mathbf{f}\left(\frac{1}{P_n} \sum_{\rho=1}^{n} p_\rho x_\rho\right). \tag{1.5}$$

Remark 1.5. It is important to note that under the assumptions of Theorem 1.4, if the function $\mathbf{f}$ is **3**-convex then $J(\mathbf{f}(\cdot)) \geqslant 0$ and $J(\mathbf{f}(\cdot)) = 0$ for $\mathbf{f}(x) = x$ or $\mathbf{f}(x) = x^2$ or $\mathbf{f}$ is a constant function.

In [19], (see also [18, p. 32, Theorem 4]) Pečarić weakened the assumption (1.3) and proved that inequality (1.4) still holds, i.e., the following result holds.

Theorem 1.6. Let $\mathbf{f} : I = [\zeta_1, \zeta_2] \to \mathbb{R}$ be a 3-convex function. Also, let $p_\rho > 0$ and $x_\rho, y_\rho \in [\zeta_1, \zeta_2]$ be such that $x_\rho + y_\rho = 2\check{c}$, for $\rho = 1, \ldots, n$, $x_\rho + x_{n-\rho+1} \leqslant 2\check{c}$ and $\frac{p_\rho x_\rho + p_{n-\rho+1} x_{n-\rho+1}}{p_\rho + p_{n-\rho+1}} \leqslant \check{c}$. The (1.4) holds.

In [17], Mercer made a notable work by replacing the condition of symmetric distribution of points x_ρ and y_ρ with symmetric variances of points x_ρ and y_ρ, second condition is weaker condition.

Theorem 1.7. *Let f be a 3-convex function on $[\zeta_1, \zeta_2]$, p_ρ are positive such that $\sum_{\rho=1}^n p_\rho = 1$. Also, let x_ρ, y_ρ satisfy $\max\{x_1 \ldots x_n\} \leqslant \min\{y_1 \ldots y_n\}$ and*

$$\sum_{\rho=1}^n p_\rho \left(x_\rho - \sum_{\rho=1}^n p_\rho x_\rho \right)^2 = \sum_{\rho=1}^n p_\rho \left(y_\rho - \sum_{\rho=1}^n p_\rho y_\rho \right)^2, \tag{1.6}$$

then (1.4) holds.

In [20], Pečarić et al. gave probabilistic version of Levinson's inequality (1.2) under Mercer's assumption of equal variances for the family of 3-convex functions at a point. They showed that this is the largest family of continuous functions for which the inequality (1.2) holds. Operator version of probabilistic Levinson's inequality discussed in [21].

Let us define the real valued function

$$(x - t)_+ = \begin{cases} (x - t), & t \leqslant x, \\ 0, & t > x. \end{cases}$$

The Taylor's formula given in [6], as follows. Let k be a positive integer and $f : [\zeta_1, \zeta_2] \to R$ be such that $f^{(k-1)}$ is absolutely continuous, then for all $x \in [\zeta_1, \zeta_2]$ the Taylor's formula at the point $c \in [\zeta_1, \zeta_2]$ is

$$f(x) = T_{k-1}(f; c, x) + R_{k-1}(f; c, x),$$

where $T_{k-1}(f; c, x)$ is a Taylor's polynomial of degree $(k-1)$, i.e.,

$$T_{k-1}(f; c, x) = \sum_{j=0}^{k-1} \frac{f^{(j)}(c)}{j!} (x - c)^j$$

and the remainder is given by

$$R_{k-1}(f; c, x) = \frac{1}{(k-1)!} \int_c^x f^{(k)}(t)(x - t)^{k-1} dt.$$

Applying Taylor's formula at the points ζ_1 and ζ_2, respectively, we get

$$f(x) = \sum_{j=0}^{k-1} \frac{f^{(j)}(\zeta_1)}{j!} (x - \zeta_1)^j + \frac{1}{(k-1)!} \int_{\zeta_1}^{\zeta_2} f^{(k)}(t)((x - t)_+^{k-1}) dt, \tag{1.7}$$

$$f(x) = \sum_{j=0}^{k-1} \frac{f^{(j)}(\zeta_2)}{j!} (\zeta_2 - x)^j (-1)^j - \frac{1}{(k-1)!} \int_{\zeta_1}^{\zeta_2} (-1)^{k-1} f^{(k)}(t)((t - x)_+^{k-1}) dt. \tag{1.8}$$

In [8], Butt et al. generalized Popoviciu inequality via Taylor polynomial for higher order convex functions. Bounds, exponential convexity and Cauchy means were presented for linear functionals coming from the general inequality. All generalizations exist in litrature are only for one type of data points. But in this pattern and motivated by above discussion Levinson type inequalities are generalized via Taylor polynomial involving two type of data points for higher order convex functions. Čebyšev, Grüss and Ostrowski-type new bounds are also found for the functionals involving data points of two types.

2. Main results

Motivated by identity (1.5), we construct the following identities with help of (1.7) and (1.8), coming from Taylor polynomial.

2.1. Bullen type inequalities for higher order convex functions

First we define the following functional $\mathcal{F}$:

Let $\mathbf{f} : I = [\zeta_1, \zeta_2] \to \mathbb{R}$ be a function. Also, let $(p_1, \ldots, p_n) \in \mathbb{R}^n$ and $(q_1, \ldots, q_m) \in \mathbb{R}^m$ be such that $\sum_{\rho=1}^{n} p_\rho = 1$, $\sum_{\rho=1}^{m} q_\rho = 1$ and $x_\rho, y_\rho, \sum_{\rho=1}^{n} p_\rho x_\rho, \sum_{\rho=1}^{m} q_\rho y_\rho \in I$. Then

$$\check{J}(\mathbf{f}(\cdot)) = \sum_{\rho=1}^{m} q_\rho f(y_\rho) - \mathbf{f}\left(\sum_{\rho=1}^{m} q_\rho y_\rho \right) - \sum_{\rho=1}^{n} p_\rho f(x_\rho) + \mathbf{f}\left(\sum_{\rho=1}^{n} p_\rho x_\rho \right). \tag{2.1}$$

Theorem 2.1. *Assume $\mathcal{F}$ and $\mathbf{f} : I = [\zeta_1, \zeta_2] \to \mathbb{R}$ be such that $\mathbf{f}^{(k-1)}$ is absolutely continuous. Then*
(i)

$$\check{J}(\mathbf{f}(\cdot)) = \check{J}(T_{k-1}(\mathbf{f}; \zeta_1, \cdot)) + \check{J}(R_{k-1}(\mathbf{f}; \zeta_1, \cdot)), \tag{2.2}$$

where $\check{J}(\mathbf{f}(\cdot))$ is given by (2.1),

$$\begin{aligned}
\check{J}(T_{k-1}(\mathbf{f}; \zeta_1, \cdot)) = & \sum_{\rho=1}^{m} q_\rho T_{k-1}(\mathbf{f}; \zeta_1, y_\rho) - T_{k-1}(\mathbf{f}; \zeta_1, \sum_{\rho=1}^{m} q_\rho y_\rho) \\
& - \sum_{\rho=1}^{n} p_\rho T_{k-1}(\mathbf{f}; \zeta_1, x_\rho) + T_{k-1}(\mathbf{f}; \zeta_1, \sum_{\rho=1}^{n} p_\rho x_\rho)
\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}
\check{J}(R_{k-1}(\mathbf{f}; \zeta_1, \cdot)) = & \sum_{\rho=1}^{m} q_\rho R_{k-1}(\mathbf{f}; \zeta_1, y_\rho) - R_{k-1}(\mathbf{f}; \zeta_1, \sum_{\rho=1}^{m} q_\rho y_\rho) \\
& - \sum_{\rho=1}^{n} p_\rho R_{k-1}(\mathbf{f}; \zeta_1, x_\rho) + R_{k-1}(\mathbf{f}; \zeta_1, \sum_{\rho=1}^{n} p_\rho x_\rho).
\end{aligned} \tag{2.4}$$

(ii)

$$\check{J}(\mathbf{f}(\cdot)) = \check{J}(T_{k-1}(\mathbf{f}; \zeta_2, \cdot)) + \check{J}(R_{k-1}(\mathbf{f}; \zeta_2, \cdot)), \tag{2.5}$$

where

$$\begin{aligned}
\check{J}(T_{k-1}(\mathbf{f}; \zeta_2, \cdot)) = & \sum_{j=0}^{k-1} \frac{f^j(\zeta_2)}{j!} (-1)^j \left(\sum_{\rho=1}^{m} q_\rho \left(\zeta_2 - y_\rho \right)^j - \left(\zeta_2 - \sum_{\rho=1}^{m} q_\rho y_\rho \right)^j \right. \\
& \left. - \left(\sum_{\rho=1}^{n} p_\rho (\zeta_2 - x_\rho)^j \right) + \left(\zeta_2 - \sum_{\rho=1}^{n} p_\rho x_\rho \right)^j \right)
\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}
\check{J}(R_{k-1}(\mathbf{f}; \zeta_2, \cdot)) = & \frac{(-1)^{k-1}}{(k-1)!} \int_{\zeta_1}^{\zeta_2} \left(\sum_{\rho=1}^{m} q_\rho (t - y_\rho)_+^{k-1} - (t - \sum_{\rho=1}^{m} q_\rho y_\rho)_+^{k-1} \right. \\
& \left. - \sum_{\rho=1}^{n} p_\rho (t - x_\rho)_+^{k-1} + (t - \sum_{\rho=1}^{n} p_\rho x_\rho)_+^{k-1} \right) f^{(k)}(t) dt.
\end{aligned} \tag{2.7}$$

Proof.
(i) Using (1.7) in (2.1) and by linearity of $\check{J}(\cdot)$, we get (2.3).

(ii) Same as proof of (i). □

In the following theorem we obtain generalizations of Bullen type inequality for k-convex functions.

Theorem 2.2. *Assume $\mathcal{F}$ and let* $\mathbf{f} : I = [\zeta_1, \zeta_2] \to \mathbb{R}$ *be such that* $\mathbf{f}^{(k-1)}$ *is absolutely continuous. Then*

(i) *if* $\mathbf{f}$ *is k-convex function and*

$$\check{J}(R_{k-1}(\mathbf{f}; \zeta_1, \cdot)) \geqslant 0, \tag{2.8}$$

then

$$\check{J}(\mathbf{f}(\cdot)) \geqslant \check{J}(T_{k-1}(\mathbf{f}; \zeta_1, \cdot)); \tag{2.9}$$

(ii) *if* $\mathbf{f}$ *is k-convex function and*

$$\check{J}(R_{k-1}(\mathbf{f}; \zeta_2, \cdot)) \geqslant 0, \tag{2.10}$$

then

$$\check{J}(\mathbf{f}(\cdot)) \geqslant \check{J}(T_{k-1}(\mathbf{f}; \zeta_2, \cdot)). \tag{2.11}$$

Proof. Since $\mathbf{f}^{(k-1)}$ is absolutely continuous on $[\zeta_1, \zeta_2]$, $\mathbf{f}^{(k)}$ exists almost everywhere. As $\mathbf{f}$ is k-convex function, therefore applying Theorem 1.2, we have, $\mathbf{f}^{(k)}(x) \geqslant 0$ for all $x \in [\zeta_1, \zeta_2]$. Hence we can apply Theorem 2.1 to obtain (2.9) and (2.11) respectively. □

Remark 2.3.

(i) In Theorem 2.2, if inequalities in (2.8) and (2.10) are in reverse direction then inequalities in (2.9) and (2.11) are also reversed.

(ii) Inequalities (2.9) and (2.11) are also hold in reverse direction if the function $\mathbf{f}$ is k-concave.

If we put $m = n$, $p_\rho = q_\rho$ and by using positive weights in (2.1), then $\check{J}(\cdot)$ converted to the functional $J(\cdot)$ defined in (1.5), also in this case, (2.2), (2.5), (2.8), (2.9), (2.10) and (2.11) become

$$J(\mathbf{f}(\cdot)) = J(T_{k-1}(\mathbf{f}; \zeta_1, \cdot)) + J(R_{k-1}(\mathbf{f}; \zeta_1, \cdot)), \tag{2.12}$$

where

$$J(T_{k-1}(\mathbf{f}; \zeta_1, \cdot)) = \sum_{\rho=1}^{n} p_\rho T_{k-1}(\mathbf{f}; \zeta_1, y_\rho) - T_{k-1}(\mathbf{f}; \zeta_1, \sum_{\rho=1}^{n} p_\rho y_\rho) - \sum_{\rho=1}^{n} p_\rho T_{k-1}(\mathbf{f}; \zeta_1, x_\rho) + T_{k-1}(\mathbf{f}; \zeta_1, \sum_{\rho=1}^{n} p_\rho x_\rho)$$

and

$$J(R_{k-1}(\mathbf{f}; \zeta_1, \cdot)) = \sum_{\rho=1}^{n} p_\rho R_{k-1}(\mathbf{f}; \zeta_1, y_\rho) - R_{k-1}(\mathbf{f}; \zeta_1, \sum_{\rho=1}^{n} p_\rho y_\rho)$$
$$- \sum_{\rho=1}^{n} p_\rho R_{k-1}(\mathbf{f}; \zeta_1, x_\rho) + R_{k-1}(\mathbf{f}; \zeta_1, \sum_{\rho=1}^{n} p_\rho x_\rho),$$

$$J(\mathbf{f}(\cdot)) = J(T_{k-1}(\mathbf{f}; \zeta_2, \cdot)) + J(R_{k-1}(\mathbf{f}; \zeta_2, \cdot)), \tag{2.13}$$

where

$$J(T_{k-1}(\mathbf{f}; \zeta_2, \cdot)) = \sum_{j=0}^{k-1} \frac{\mathbf{f}^j(\zeta_2)}{j!} (-1)^j \left(\sum_{\rho=1}^{n} p_\rho \left(\zeta_2 - y_\rho \right)^j - \left(\zeta_2 - \sum_{\rho=1}^{n} p_\rho y_\rho \right)^j \right.$$
$$\left. - \left(\sum_{\rho=1}^{n} p_\rho (\zeta_2 - x_\rho)^j \right) + \left(\zeta_2 - \sum_{\rho=1}^{n} p_\rho x_\rho \right)^j \right),$$

and

$$J(R_{k-1}(\mathbf{f}; \zeta_2, \cdot)) = \frac{(-1)^{k-1}}{(k-1)!} \int_{\zeta_1}^{\zeta_2} \left(\sum_{\rho=1}^{n} p_\rho (t - y_\rho)_+^{k-1} - (t - \sum_{\rho=1}^{n} p_\rho y_\rho)_+^{k-1} \right.$$

$$\left. - \sum_{\rho=1}^{n} p_\rho (t - x_\rho)_+^{k-1} + (t - \sum_{\rho=1}^{n} p_\rho x_\rho)_+^{k-1} \right) \mathbf{f}^{(k)}(t) dt,$$

$$J(R_{k-1}(\mathbf{f}; \zeta_1, \cdot)) \geqslant 0, \tag{2.14}$$

$$J(\mathbf{f}(\cdot)) \geqslant J(T_{k-1}(\mathbf{f}; \zeta_1, \cdot)), \tag{2.15}$$

$$J(R_{k-1}(\mathbf{f}; \zeta_2, \cdot)) \geqslant 0, \tag{2.16}$$

$$J(\mathbf{f}(\cdot)) \geqslant J(T_{k-1}(\mathbf{f}; \zeta_2, \cdot)), \tag{2.17}$$

respectively.

Theorem 2.4. *Let $\mathbf{f}^{(k-1)}$ be absolutely continuous on I, $\mathbf{p} = (p_1, ..., p_n)$ be positive n-tuple such that $\sum_{\rho=1}^{n} p_\rho = 1$. Also, let $x_\rho, y_\rho \in I$ such that (1.3) is valid for $\rho = 1, \dots, n$. Then for the functional $J(\cdot)$ defined in (1.5), we have the following:*

(i) *if $\mathbf{f}$ is k-convex function, then (2.15) holds, moreover if $\mathbf{f}^{(j)}(\zeta_1) \geqslant 0$ for $j = 3, \dots, k-1$, the R.H.S. of (2.15) will be non negative;*

(ii) *if k is even $\mathbf{f}$ k-convex function, then (2.17) holds, moreover if $\mathbf{f}^{(j)}(\zeta_2) \geqslant 0$ for $j = 4, \dots, k-2$ and $\mathbf{f}^{(j)}(\zeta_2) \leqslant 0$ for $j = 3, \dots, k-1$ the R.H.S. of (2.17) will be non negative,*

(iii) *if k is odd $\mathbf{f}$ is $k-$convex function, then (2.17) holds in reverse direction, moreover if $\mathbf{f}^{(j)}(\zeta_2) \leqslant 0$ for $j = 4, \dots, k-2$ and $\mathbf{f}^{(j)}(\zeta_2) \geqslant 0$ for $j = 3, \dots, k-1$ the R.H.S. of (2.17) will be non positive.*

Proof.

(i) Since

$$\frac{d^3}{dy_\rho^3} (y_\rho - t)_+^{k-1} = \begin{cases} (k-1)(k-2)(k-3)(y_{\rho-t})^{k-4} \geqslant 0, & \text{for } t \leqslant y_\rho, \\ 0, & t \leqslant y_\rho. \end{cases}$$

and also, $x \to (x-t)_+^{k-1}$ is 3-convex, therefore $(\cdot - t)_+^{k-1}$ is 3-convex. Since weights are positive, therefore using Remark 1.5 for n points, we get (2.14). By following Theorem 2.2 (for $n = m$ and $p_\rho = q_\rho$), we get (2.15). Also by using Remark 1.5 and given conditions, $J(T_{k-1}(\mathbf{f}; \zeta_1, \cdot)) \geqslant 0$ for $j = 2, \dots, k-1$, we get R.H.S. of (2.15) is non negative.

The pfoors of (ii) and (iii) are similar to (i). $\square$

Corollary 2.5. *Let $\mathbf{f} : I = [\zeta_1, \zeta_2] \to \mathbb{R}$ be such that $\mathbf{f}^{(k-1)}$ is absolutely continuous. Also, let $(p_1, \dots, p_n)$ and $(q_1, \dots, q_m)$ be positive real numbers such that $\sum_{\rho=1}^{n} p_\rho = 1$, $\sum_{\rho=1}^{m} q_\rho = 1$ and $x_\rho, y_\rho, \sum_{\rho=1}^{n} p_\rho x_\rho, \sum_{\rho=1}^{m} q_\rho y_\rho \in I$. Moreover if (2.14) and (2.16) are valid then (2.15) and (2.17) are also valid, respectively.*

Proof. In Theorem 2.2, choose positive real numbers such that $\sum_{\rho=1}^{n} p_\rho = 1$ $\sum_{\rho=1}^{m} q_\rho = 1$, we get required result. $\square$

Next we have generalized form (for real weights) of Levinson's type inequality for $2n$-points given in [19] (see also [18]).

Theorem 2.6. *Let $\mathbf{f} : I = [\zeta_1, \zeta_2] \to \mathbb{R}$ be such that $\mathbf{f}^{(k-1)}$ is absolutely continuous, $(p_1, \dots, p_n)$ be positive n-tuple such that $\sum_{\rho=1}^{n} p_\rho = 1$. Also, let $x_\rho, y_\rho \in I$ such that $x_\rho + y_\rho = 2\check{c}$, $x_\rho + x_{n-\rho+1} \leqslant 2\check{c}$ and $\frac{p_\rho x_\rho + p_{n-\rho+1} x_{n-\rho+1}}{p_\rho + p_{n-\rho+1}} \leqslant \check{c}$ for $\rho = 1, \dots, n$. Then for the functional $J(\cdot)$ defined in (1.5), we have the following.*

(i) *If $\mathbf{f}$ is k-convex function, then (2.15) holds. Moreover if $\mathbf{f}^{(j)}(\zeta_1) \geqslant 0$ for $j = 3, \dots, k-1$, the R.H.S. of (2.15) will be non negative.*

(ii) *If* k *is even* **f** *is* k-*convex function, then* (2.17) *holds. Moreover if* $\mathbf{f}^{(j)}(\zeta_2) \geqslant 0$ *for* $j = 4, \ldots, k-2$ *and* $\mathbf{f}^{(j)}(\zeta_2) \leqslant 0$ *for* $j = 3, \ldots, k-1$ *the R.H.S. of* (2.17) *will be non negative.*

(iii) *If* k *is odd* **f** *is* k−*convex function, then* (2.17) *holds in reverse direction. Moreover if* $\mathbf{f}^{(j)}(\zeta_2) \leqslant 0$ *for* $j = 4, \ldots, k-2$ *and* $\mathbf{f}^{(j)}(\zeta_2) \geqslant 0$ *for* $j = 3, \ldots, k-1$ *the R.H.S. of* (2.17) *will be non positive.*

Proof. In Theorem 2.4, replace the condition (1.3), for x_ρ and y_ρ with the condition given in statement to get required result. ☐

Remark 2.7. Identities (2.12) and (2.13) are also valid for the points x_ρ and y_ρ in Theorem 2.6 satisfying $\max\{x_1 \ldots x_n\} \leqslant \min\{y_1 \ldots y_n\}$ and condition (1.6).

3. New bounds for Levinson's type inequality

Consider the Čebyšev functional for two Lebesgue integrable functions $f_1, f_2 : [\zeta_1, \zeta_2] \to \mathbb{R}$,

$$\Theta(f_1, f_2) = \frac{1}{\zeta_2 - \zeta_1} \int_{\zeta_1}^{\zeta_2} f_1(x) f_2(x) dx \frac{1}{\zeta_2 - \zeta_1} \int_{\zeta_1}^{\zeta_2} f_1(x) dx . \frac{1}{\zeta_2 - \zeta_1} \int_{\zeta_1}^{\zeta_2} f_2(x) dx,$$

where the integrals are assumed to exist.

Theorem 3.1 ([9]). *Let* $f_1 : [\zeta_1, \zeta_2] \to \mathbb{R}$ *be a Lebesgue integrable function and* $f_2 : [\zeta_1, \zeta_2] \to \mathbb{R}$ *be an absolutely continuous function with* $(., -\zeta_1)(., -\zeta_2)[f_2']^2 \in L[\zeta_1, \zeta_2]$. *Then*

$$|\Theta(f_1, f_2)| \leqslant \frac{1}{\sqrt{2}} [\Theta(f_1, f_1)]^{\frac{1}{2}} \frac{1}{\sqrt{\zeta_2 - \zeta_1}} \left(\int_{\zeta_1}^{\zeta_2} (t - \zeta_1)(\zeta_2 - t)[f_2'(t)]^2 dt \right)^{\frac{1}{2}},$$

$\frac{1}{\sqrt{2}}$ *is the best possible.*

Theorem 3.2 ([9]). *Let* $f_1 : [\zeta_1, \zeta_2] \to \mathbb{R}$ *be an absolutely continuous with* $f_1' \in L_\infty[\zeta_1, \zeta_2]$ *and* $f_2 : [\zeta_1, \zeta_2] \to \mathbb{R}$ *is monotonic non-decreasing on* $[\zeta_1, \zeta_2]$. *Then*

$$|\Theta(f_1, f_2)| \leqslant \frac{1}{2(\zeta_2 - \zeta_1)} \|f'\|_\infty \int_{\zeta_1}^{\zeta_2} (t - \zeta_1)(\zeta_2 - t)[f_2'(t)]^2 df_2(t),$$

$\frac{1}{2}$ *is the best possible.*

To generalize results given in previous section we will consider Theorems 3.1 and 3.2.

Theorem 3.3. *Let* $\mathbf{f} : I = [\zeta_1, \zeta_2] \to \mathbb{R}$ *be such that* $\mathbf{f}^{(k)}(\cdot)$ *is absolutely continuous with* $(. - \zeta_1)(\zeta_2 - .)[\mathbf{f}^{(k+1)}]^2 \in L[\zeta_1, \zeta_2]$. *Also, let* $(p_1, \ldots, p_n) \in \mathbb{R}^n$, $(q_1, \ldots, q_m) \in \mathbb{R}^m$ *be such that* $\sum_{\rho=1}^{n} p_\rho = 1$, $\sum_{\rho=1}^{m} q_\rho = 1$, $x_\rho, y_\rho, \sum_{\rho=1}^{n} p_\rho x_\rho, \sum_{\rho=1}^{m} q_\rho y_\rho \in I$. *Then*

(i)

$$\breve{J}(\mathbf{f}(\cdot)) = \breve{J}(T_{k-1}(\mathbf{f}; \zeta_1, \cdot)) + \frac{\mathbf{f}^{(k-1)}(\zeta_2) - \mathbf{f}^{(k-1)}(\zeta_1)}{(\zeta_2 - \zeta_1)} \breve{J}(R_{k-1}(\mathbf{f}; \zeta_1, \cdot)) + \mathcal{R}_m(\zeta_1, \zeta_2; \mathbf{f}), \tag{3.1}$$

where $\breve{J}(\mathbf{f}(\cdot))$, $\breve{J}(T_{k-1}(\mathbf{f}; \zeta_1, \cdot))$ *and* $\breve{J}(R_{k-1}(\mathbf{f}; \zeta_1, \cdot))$ *are defined in* (2.1), (2.3), *and* (2.4), *respectively, where the remainder* $\mathcal{R}_k(\zeta_1, \zeta_2; \mathbf{f})$ *satisfies the bound*

$$|\mathcal{R}_k(\zeta_1, \zeta_2; \mathbf{f})| \leqslant \frac{(\zeta_2 - \zeta_1)}{\sqrt{2}} \left[\Theta(\breve{J}(R_{k-1}(\mathbf{f}; \zeta_1, \cdot)), \breve{J}(R_{k-1}(\mathbf{f}; \zeta_1, \cdot))) \right]^{\frac{1}{2}}$$

$$\times \frac{1}{\sqrt{\zeta_2 - \zeta_1}} \left(\int_{\zeta_1}^{\zeta_2} (t - \zeta_1)(\zeta_2 - t)[\mathbf{f}^{(k+1)}(t)]^2 dt \right)^{\frac{1}{2}}. \tag{3.2}$$

(ii)

$$\breve{J}(\mathbf{f}(\cdot)) = \breve{J}(T_{k-1}(\mathbf{f};\zeta_2,\cdot)) + \frac{\mathbf{f}^{(k-1)}(\zeta_2) - \mathbf{f}^{(k-1)}(\zeta_1)}{(\zeta_2 - \zeta_1)} \breve{J}(R_{k-1}(\mathbf{f};\zeta_2,\cdot)) + \mathcal{R}'_m(\zeta_1,\zeta_2;\mathbf{f}), \qquad (3.3)$$

where $\breve{J}(\mathbf{f}(\cdot))$, $\breve{J}(T_{k-1}(\mathbf{f};\zeta_2,\cdot))$ *and* $\breve{J}(R_{k-1}(\mathbf{f};\zeta_2,\cdot))$ *are defined in* (2.1), (2.6), *and* (2.7), *respectively, where the remainder* $\mathcal{R}_k(\zeta_1,\zeta_2;\mathbf{f})$ *satisfies the bound*

$$|\mathcal{R}'_k(\zeta_1,\zeta_2;\mathbf{f})| \leqslant \frac{(\zeta_2 - \zeta_1)}{\sqrt{2}} \left[\Theta(\breve{J}(R_{k-1}(\mathbf{f};\zeta_1,\cdot)), \breve{J}(R_{k-1}(\mathbf{f};\zeta_1,\cdot))) \right]^{\frac{1}{2}}$$

$$\times \frac{1}{\sqrt{\zeta_2 - \zeta_1}} \left(\int_{\zeta_1}^{\zeta_2} (t - \zeta_1)(\zeta_2 - t)[\mathbf{f}^{(k+1)}(t)]^2 dt \right)^{\frac{1}{2}}.$$

Proof.

(i) Setting $f_1 \mapsto \breve{J}(R_{k-1}(\mathbf{f};\zeta_1,\cdot))$ and $f_2 \mapsto \mathbf{f}^{(k)}$ in Theorem 3.1, we get

$$\left| \frac{1}{\zeta_2 - \zeta_1} \int_{\zeta_1}^{\zeta_2} \breve{J}(R_{k-1}(\mathbf{f};\zeta_1,\cdot))\mathbf{f}^{(k)}(t)dt - \frac{1}{\zeta_2 - \zeta_1} \int_{\zeta_1}^{\zeta_2} \breve{J}(R_{k-1}(\mathbf{f};\zeta_1,\cdot))dt \frac{1}{\zeta_2 - \zeta_1} \int_{\zeta_1}^{\zeta_2} \mathbf{f}^{(k)}(s)dt \right|$$

$$\leqslant \frac{1}{\sqrt{2}} [\Theta(\breve{J}(R_{k-1}(\mathbf{f};\zeta_1,\cdot)), \breve{J}(R_{k-1}(\mathbf{f};\zeta_1,\cdot)))]^{\frac{1}{2}} \frac{1}{\sqrt{\zeta_2 - \zeta_1}} \left(\int_{\zeta_1}^{\zeta_2} (t - \zeta_1)(\zeta_2 - t)[\mathbf{f}^{(k+1)}(t)]^2 dt \right)^{\frac{1}{2}},$$

$$\left| \frac{1}{\zeta_2 - \zeta_1} \int_{\zeta_1}^{\zeta_2} \breve{J}(R_{k-1}(\mathbf{f};\zeta_1,\cdot))\mathbf{f}^{(k)}(s)dt - \frac{f^{(2)}(\zeta_2) - \mathbf{f}^{(2)}(\zeta_1)}{(\zeta_2 - \zeta_1)^2} \int_{\zeta_1}^{\zeta_2} \breve{J}(R_{k-1}(\mathbf{f};\zeta_1,\cdot))dt \right|$$

$$\leqslant \frac{1}{\sqrt{2}} \left[\Theta(\breve{J}(R_{k-1}(\mathbf{f};\zeta_1,\cdot)), \breve{J}(R_{k-1}(\mathbf{f};\zeta_1,\cdot))) \right]^{\frac{1}{2}} \frac{1}{\sqrt{\zeta_2 - \zeta_1}} \left(\int_{\zeta_1}^{\zeta_2} (t - \zeta_1)(\zeta_2 - t)[\mathbf{f}^{(k+1)}(s)]^2 dt \right)^{\frac{1}{2}}.$$

Multiplying $(\zeta_2 - \zeta_1)$ on both sides of above inequality and using the estimation (3.2), we get

$$\int_{\zeta_1}^{\zeta_2} \breve{J}(R_{k-1}(\mathbf{f};\zeta_1,\cdot))\mathbf{f}^{(k)}dt = \frac{f^{(2)}(\zeta_2) - \mathbf{f}^{(2)}(\zeta_1)}{(\zeta_2 - \zeta_1)} \int_{\zeta_1}^{\zeta_2} \breve{J}(R_{k-1}(\mathbf{f};\zeta_1,\cdot))dt + \mathcal{R}_k(\zeta_1,\zeta_1;\mathbf{f}).$$

Using identity (2.1), we get (3.1).

(ii) Similar to above part. □

By using Theorem 3.2, Grüss type inequalities can be obtained.

Theorem 3.4. *Let* $\mathbf{f} : I = [\zeta_1, \zeta_2] \to \mathbb{R}$ *be such that* $\mathbf{f}^{(k)}(\cdot)$ *is absolutely continuous and* $\mathbf{f}^{(k+1)}(\cdot) \geqslant 0$ *a.e. on* $[\zeta_1, \zeta_2]$. *Also, let* $(p_1,\ldots,p_n) \in \mathbb{R}^n$, $(q_1,\ldots,q_m) \in \mathbb{R}^m$ *be such that* $\sum_{\rho=1}^n p_\rho = 1$, $\sum_{\rho=1}^m q_\rho = 1$, $x_\rho, y_\rho, \sum_{\rho=1}^n p_\rho x_\rho, \sum_{\rho=1}^m q_\rho y_\rho \in I$. *Then*

(i) *Identity* (3.1) *holds, where the remainder satisfies the estimation*

$$|\mathcal{R}_k(\zeta_1,\zeta_2;\mathbf{f})| \leqslant (\zeta_2 - \zeta_1)\|(\breve{J}(R_{k-1}(\mathbf{f};\zeta_1,\cdot)))'\|_\infty \left[\frac{\mathbf{f}^{(k-1)}(\zeta_2) + \mathbf{f}^{(k-1)}(\zeta_1)}{2} - \frac{\mathbf{f}^{(k-2)}(\zeta_2) - \mathbf{f}^{(k-2)}(\zeta_1)}{\zeta_2 - \zeta_1} \right].$$

(ii) *Identity* (3.3) *holds, where the remainder satisfies the estimation*

$$|\mathcal{R}'_k(\zeta_1,\zeta_2;\mathbf{f})| \leqslant (\zeta_2 - \zeta_1)\|(\breve{J}(R_{k-1}(\mathbf{f};\zeta_1,\cdot)))'\|_\infty \left[\frac{\mathbf{f}^{(k-1)}(\zeta_2) + \mathbf{f}^{(k-1)}(\zeta_1)}{2} - \frac{\mathbf{f}^{(k-2)}(\zeta_2) - \mathbf{f}^{(k-2)}(\zeta_1)}{\zeta_2 - \zeta_1} \right].$$

Proof.

(i) Setting $f_1 \mapsto \check{J}(R_{k-1}(\mathbf{f}; \zeta_1,.))$ and $f_2 \mapsto \mathbf{f}^{(k)}$ in Theorem 3.2, we get

$$
\left| \frac{1}{\zeta_2 - \zeta_1} \int_{\zeta_1}^{\zeta_2} \check{J}(R_{k-1}(\mathbf{f}; \zeta_1, \cdot)) \mathbf{f}^{(k)}(t) dt - \frac{1}{\zeta_2 - \zeta_1} \int_{\zeta_1}^{\zeta_2} \check{J}(R_{k-1}(\mathbf{f}; \zeta_1, \cdot)) dt \frac{1}{\zeta_2 - \zeta_1} \int_{\zeta_1}^{\zeta_2} \mathbf{f}^{(k)}(t) dt \right|
$$
$$
\leqslant \frac{1}{2} \|(\check{J}(R_{k-1}(\mathbf{f}; \zeta_1, \cdot)))'\|_\infty \frac{1}{\zeta_2 - \zeta_1} \int_{\zeta_1}^{\zeta_2} (t - \zeta_1)(\zeta_2 - t)[\mathbf{f}^{(k+1)}(t)]^2 dt. \tag{3.4}
$$

Since,

$$
\int_{\zeta_1}^{\zeta_2} (t - \zeta_1)(\zeta_2 - t)[f^{(k+1)}(t)]^2 dt = \int_{\zeta_1}^{\zeta_2} [2t - \zeta_1 - \zeta_2] \mathbf{f}^k(t) dt
$$
$$
= (\zeta_2 - \zeta_1)[\mathbf{f}^{(k-1)}(\zeta_2) + \mathbf{f}^{(k-1)}(\zeta_1)] - 2(\mathbf{f}^{(k-1)}(\zeta_2) - \mathbf{f}^{(k-1)}(\zeta_1)), \tag{3.5}
$$

using (2.1), (3.4) and (3.5), we have (3.1).

(ii) Similar to above part. □

Theorem 3.5. *Let* $\mathbf{f} : I = [\zeta_1, \zeta_2] \to \mathbb{R}$ *be such that* $\mathbf{f}^{(k-1)}(\cdot)$ *is absolutely continuous. Also, let* $(p_1, \dots, p_n) \in \mathbb{R}^n$, $(q_1, \dots, q_m) \in \mathbb{R}^m$ *be such that* $\sum_{\rho=1}^n p_\rho = 1$, $\sum_{\rho=1}^m q_\rho = 1$, $x_\rho, y_\rho, \sum_{\rho=1}^n p_\rho x_\rho, \sum_{\rho=1}^m q_\rho y_\rho \in I$. *Also, let* (r, s) *be a pair of conjugate exponents that is* $1 \leqslant r, s, \leqslant \infty$, $\frac{1}{r} + \frac{1}{s} = 1$. *If* $|\mathbf{f}^{(k)}|^r : [\zeta_1, \zeta_2] \to \mathbb{R}$ *be Riemann integrable function, then*

(i)

$$
\left| \check{J}(\mathbf{f}(\cdot)) - \check{J}(T_{k-1}(\mathbf{f}; \zeta_1, \cdot)) \right| \leqslant \|\mathbf{f}^{(k)}\|_r \left(\int_{\zeta_1}^{\zeta_2} \left| \check{J}(R_{k-1}(\mathbf{f}; \zeta_1, \cdot)) dt \right|^s \right)^{\frac{1}{s}};
$$

(ii)

$$
\left| \check{J}(\mathbf{f}(\cdot)) - \check{J}'(T_{k-1}(\mathbf{f}; \zeta_2, \cdot)) \right| \leqslant \|\mathbf{f}^{(k)}\|_r \left(\int_{\zeta_1}^{\zeta_2} \left| \check{\mathfrak{J}}'(R_{k-1}(\mathbf{f}; \zeta_2, \cdot)) dt \right|^s \right)^{\frac{1}{s}}.
$$

Proof. For proof see Theorem 3.5 in [8]. □

Remark 3.6. Similar work can be done for Levinson's inequality (1.2) (one type of data points) for higher order-convex functions.

Remark 3.7. We can give related mean value theorems by using non-negative functionals (2.2) and (2.5) and also, we can construct the new families of k-exponentially convex functions and Cauchy means related to these functionals as given in Section 4 of [8].

4. Application to information theory

The idea of Shannon entropy is the central job of information speculation, now and again implied as measure of uncertainty. The entropy of a random variable is described with respect to probability distribution, and it can be shown that it is a decent measure of random. The assignment of Shannon entropy is to assess the typical least number of bits expected to encode a progression of pictures subject to the letters, including the size and the repetition of the symbols.

Divergences between probability distributions can be interpreted as measure of the distance between them. An assortment of sorts of divergences exist, for example the **f**-divergences (especially, Kullback-Leibler divergences, Hellinger distance and total variation distance), Rényi divergences, Jensen-Shannon divergences, etc (see [16, 25]). There are a lot of papers dealing with the subject of inequalities and entropies, see, e.g., [1–5, 12, 14, 24] and the references therein. Jensen's inequality deals with one kind of data points, Levinson's inequality deals two types of data points.

4.1. Csiszár divergence

Following definition is given by Csiszár in [10, 11].

Definition 4.1. Let $\mathbf{f} : \mathbb{R}^+ \to \mathbb{R}^+$ be a convex function. Also, let $\tilde{\mathbf{r}}, \tilde{\mathbf{k}} \in \mathbb{R}^n_+$ be such that $\sum_{\nu=1}^n r_\nu = 1$ and $\sum_{\nu=1}^n k_\nu = 1$. Then $\mathbf{f}$-divergence functional is defined by

$$I_{\mathbf{f}}(\tilde{\mathbf{r}}, \tilde{\mathbf{k}}) := \sum_{\nu=1}^n k_\nu f\left(\frac{r_\nu}{k_\nu}\right).$$

Define the following:

$$\mathbf{f}(0) := \lim_{x \to 0^+} \mathbf{f}(x); \quad 0\mathbf{f}\left(\frac{0}{0}\right) := 0; \quad 0\mathbf{f}\left(\frac{a}{0}\right) := \lim_{x \to 0^+} x\mathbf{f}\left(\frac{a}{x}\right), \quad a > 0.$$

Definition 4.2 ([13]). Let I be an interval contained in $\mathbb{R}$ and $\mathbf{f} : I \to \mathbb{R}$ be a function. Also, let $\tilde{\mathbf{r}} = (r_1, \ldots, r_n) \in \mathbb{R}^n$ and $\tilde{\mathbf{k}} = (k_1, \ldots, k_n) \in (0, \infty)^n$ be such that

$$\frac{r_\nu}{k_\nu} \in I, \quad \nu = 1, \ldots, n.$$

Then

$$\hat{I}_{\mathbf{f}}(\tilde{\mathbf{r}}, \tilde{\mathbf{k}}) := \sum_{\nu=1}^n k_\nu \mathbf{f}\left(\frac{r_\nu}{k_\nu}\right). \tag{4.1}$$

We apply Theorem 2.2 for k-convex functions to $\hat{I}_{\mathbf{f}}(\tilde{\mathbf{r}}, \tilde{\mathbf{k}})$.

Theorem 4.3. *Let* $\tilde{\mathbf{r}} = (r_1, \ldots, r_n) \in \mathbb{R}^n$, $\tilde{\mathbf{w}} = (w_1, \ldots, w_m) \in \mathbb{R}^m$, $\tilde{\mathbf{k}} = (k_1, \ldots, k_n) \in (0, \infty)^n$ *and* $\tilde{\mathbf{t}} = (t_1, \ldots, t_m) \in (0, \infty)^m$ *be such that*

$$\frac{r_\nu}{k_\nu} \in I, \quad \nu = 1, \ldots, n, \quad and \quad \frac{w_u}{t_u} \in I, \quad u = 1, \ldots, m.$$

Also, let $\mathbf{f} : I = [\zeta_1, \zeta_2] \to \mathbb{R}$ *be such that* $\mathbf{f}^{k-1}$ *is absolutely continuous and* $\mathbf{f}$ *is* k-*convex function, then*

(i)

$$J_{cis}(\mathbf{f}(\cdot)) \geqslant J(T_{k-1}(\mathbf{f}; \zeta_1, \cdot)), \tag{4.2}$$

where

$$J_{cis}(\mathbf{f}(\cdot)) = \frac{1}{\sum_{u=1}^m t_u} \hat{I}_{\mathbf{f}}(\tilde{\mathbf{w}}, \tilde{\mathbf{t}}) - \mathbf{f}\left(\sum_{u=1}^m \frac{w_u}{\sum_{u=1}^m t_u}\right) - \frac{1}{\sum_{\nu=1}^n k_\nu} \hat{I}_{\mathbf{f}}(\tilde{\mathbf{r}}, \tilde{\mathbf{k}}) + \mathbf{f}\left(\sum_{\nu=1}^n \frac{r_\nu}{\sum_{\nu=1}^n k_\nu}\right)$$

and

$$J(T_{k-1}(\mathbf{f}; \zeta_1, \cdot)) = \sum_{\rho=1}^m \frac{t_u}{\sum_{u=1}^m t_u} T_{k-1}\left(\mathbf{f}; \zeta_1, \frac{w_u}{t_u}\right) - T_{k-1}\left(\mathbf{f}; \zeta_1, \sum_{\rho=1}^m \frac{w_u}{\sum_{u=1}^m t_u}\right)$$

$$- \sum_{\rho=1}^n \frac{k_\nu}{\sum_{\nu=1}^n k_\nu} T_{k-1}\left(\mathbf{f}; \zeta_1, \frac{r_\nu}{k_\nu}\right) + T_{k-1}\left(\mathbf{f}; \zeta_1, \sum_{\rho=1}^n \frac{r_\nu}{\sum_{\nu=1}^n k_\nu}\right);$$

(ii)

$$J_{\mathrm{cis}}(\mathbf{f}(\cdot)) \geqslant J(T_{k-1}(\mathbf{f}; \zeta_2, \cdot)), \tag{4.3}$$

where

$$J(T_{k-1}(\mathbf{f}; \zeta_2, \cdot)) = \sum_{j=0}^{k-1} \frac{\mathbf{f}^j(\zeta_2)}{j!}(-1)^j \left[\sum_{\rho=1}^{m} \frac{t_u}{\sum_{u=1}^{m} t_u} \left(\zeta_2 - \frac{w_u}{t_u} \right)^j - \left(\zeta_2 - \sum_{\rho=1}^{m} \frac{w_u}{\sum_{u=1}^{m} t_u} \right)^j \right.$$
$$\left. - \left(\sum_{\rho=1}^{n} \frac{k_v}{\sum_{v=1}^{n} k_v}(\zeta_2 - \frac{r_v}{k_v})^j \right) + \left(\zeta_2 - \sum_{\rho=1}^{n} \frac{r_v}{\sum_{v=1}^{n} k_v} \right)^j \right].$$

Proof.

(i) Since $\mathbf{f}$ is k-convex and

$$\frac{d^3}{d(\cdot)^3}(\cdot - t)_+^{k-1} = \begin{cases} (k-1)(k-2)(k-3)(\cdot - t)^{k-4} \geqslant 0, & \text{for } t \leqslant \cdot, \\ 0, & t \leqslant \cdot, \end{cases}$$

hence $(\cdot - t)_+^{k-1}$ is 3-convex, therefore (2.8) holds. Using $p_\rho = \frac{k_v}{\sum_{v=1}^{n} k_v}$, $x_\rho = \frac{r_v}{k_v}$, $q_\rho = \frac{t_u}{\sum_{u=1}^{m} t_u}$, $y_\rho = \frac{w_u}{t_u}$ in Theorem 2.2 (i), (2.9) becomes (4.2), where $\hat{I}_{\mathbf{f}}(\tilde{\mathbf{r}}, \tilde{\mathbf{k}})$ is defined in (4.1) and

$$\hat{I}_{\mathbf{f}}(\tilde{\mathbf{w}}, \tilde{\mathbf{t}}) := \sum_{u=1}^{m} t_u \mathbf{f}\left(\frac{w_u}{t_u} \right).$$

(ii) By using the same substitutions as in Theorem 2.2 (ii) and following the similar steps as in (i), (2.11) becomes (4.3). $\qquad\square$

4.2. Shannon Entropy

Definition 4.4 ([13]). The Shannon entropy of positive probability distribution $\tilde{\mathbf{k}} = (k_1, \ldots, k_n)$ is defined by

$$S := -\sum_{v=1}^{n} k_v \log(k_v). \tag{4.4}$$

Corollary 4.5. *Let* $\tilde{\mathbf{k}} = (k_1, \ldots, k_n)$ *and* $\tilde{\mathbf{t}} = (t_1, \ldots, t_m)$ *be positive probability distributions. Also, let* $\tilde{\mathbf{r}} = (r_1, \ldots, r_n) \in (0, \infty)^n$ *and* $\tilde{\mathbf{w}} = (w_1, \ldots, w_m) \in (0, \infty)^m$. *If base of* $\log$ *is greater than 1 and* k=*odd* ($k = 3, 5, \ldots$), *then*

(i)

$$J_s(\cdot) \geqslant J(T_{k-1}(\log x; \zeta_1, \cdot)), \tag{4.5}$$

where

$$T_{k-1}(\log x; \zeta_1, \cdot) = \sum_{j=0}^{k-1} \frac{(-1)^{j-1}(j-1)!}{j!(\zeta_1)^j}(\cdot - c)^j$$

and

$$J_s(\cdot) = \sum_{u=1}^{m} t_u \log(w_u) + \tilde{S} - \log\left(\sum_{u=1}^{m} w_u \right) - \sum_{v=1}^{n} k_v \log(r_v) + S + \log\left(\sum_{v=1}^{n} r_v \right). \tag{4.6}$$

(ii)

$$J_s(\cdot) \geqslant J(T_{k-1}(\log x; \zeta_2, \cdot)), \tag{4.7}$$

where

$$J(T_{k-1}(\log x; \zeta_2, \cdot)) = \sum_{j=0}^{k-1} \frac{(-1)^j (j-1)!}{j!(\zeta_2)^j} \left(\sum_{\rho=1}^{m} \frac{t_u}{\sum_{u=1}^{m} t_u} \left(\zeta_2 - \frac{w_u}{t_u} \right)^j \right.$$
$$\left. - \left(\zeta_2 - \sum_{\rho=1}^{m} \frac{w_u}{\sum_{u=1}^{m} t_u} \right)^j - \left(\sum_{\rho=1}^{n} \frac{k_\nu}{\sum_{\nu=1}^{n} k_\nu} (\zeta_2 - \frac{r_\nu}{k_\nu})^j \right) + \left(\zeta_2 - \sum_{\rho=1}^{n} \frac{r_\nu}{\sum_{\nu=1}^{n} k_\nu} \right)^j \right);$$

(iii) *if base of* log *is less than 1 or* k=*even* (k = 4, 6, . . .), *then inequalities* (4.5) *and* (4.7) *are hold in reverse direction.*

Proof.

(i) The function $\mathbf{f} \to \log(x)$ is k-convex for $k = 3, 5, \ldots$ and base of log is greater than 1. Therefore using $\mathbf{f} = \log(x)$ in Theorem 4.3 (i), we get (4.5), where $\mathcal{S}$ is defined in (4.4) and

$$\tilde{\mathcal{S}} = - \sum_{u=1}^{m} t_u \log(t_u).$$

(ii) Using $\mathbf{f} = \log(x)$ in Theorem 4.3 (ii), we get (4.7).

(iii) Since the function $\mathbf{f} \to \log(x)$ is k-concave for $k = 4, 6, \ldots$, so by using Remark 2.3 (ii), (2.9) and (2.11) are hold in reverse direction. Therefore using $\mathbf{f} = \log(x)$ and $p_\rho = \frac{k_\nu}{\sum_{\nu=1}^{n} k_\nu}$, $x_\rho = \frac{r_\nu}{k_\nu}$, $q_\rho = \frac{t_u}{\sum_{u=1}^{m} t_u}$, $y_\rho = \frac{w_u}{t_u}$ in reversed inequalities of (2.9) and (2.11), we get (4.5) and (4.7) in reverse directions respectively. $\square$

Acknowledgment

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