J. Math. Computer Sci., 22 (2021), 16–37

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

# Properties and applications of beta Erlang-truncated exponential distribution

M. Shrahili[a], I. Elbatal[b,c], Isyaku Muhammad[d], Mustapha Muhammad[e,*]

[a]Department of Statistics and Operations Research, College of Science, King Saud University, P. O. Box 2455, Riyadh 11451, Saudi Arabia.

[b] Department of Mathematics and Statistics, College of Science, Imam Mohammad Ibn Saud Islamic University (IMSIU), Saudi Arabia.

[c]Faculty of Graduate Studies for Statistical Research, Cairo University, Egypt.

[d]Department of Mechanical Engineering, School of Technology, Kano State Polytechnic, Nigeria.

[e]Department of Mathematical Sciences, Faculty of Physical Sciences, Bayero University Kano (BUK), Nigeria.

## Abstract

In this article, we proposed a new four-parameter distribution called beta Erlang truncated exponential distribution (BETE). Some important mathematical and statistical properties of the proposed distribution are examined. The stochastic ordering result for the BETE was also discussed. Moreover, the $r^{th}$ moment, moment generating function, incomplete moments, mean deviations, Bonferroni and Lorenz curves, moments of residual life, Shannon and Renyi entropies, and Kullback–Leibler divergence measure are derived. The maximum-likelihood estimate for the unknown parameters of the BETE was established and assessed by the simulation studies. The maximum likelihood estimation of the stress-strength parameter is discussed and its asymptotic distribution is obtained. The effectiveness and usefulness of the BETE are demonstrated by the use of three real data set, in which the BETE provide a better fit than some other existing distributions and demonstrated its capability in the stress-strength reliability analysis.

**Keywords:** Erlang-truncated exponential distribution, beta-G distribution, moments, entropy, maximum likelihood estimation, stress-strength parameter estimation.

**2020 MSC:** 62E05, 62F10.



## 1. Introduction

The method of extending family of distributions for added flexibility and potentiality is a familiar technique in the literature. In random phenomena, modeling and analyzing lifetime data are very essential in the fields of sciences and applied sciences such medicine, engineering, finance, economics, biomedical sciences, public health, among others. Several lifetime distributions have been used to analyze such kinds

*Corresponding author
Email addresses: msharahili@ksu.edu.sa (M. Shrahili), iielbatal@imamu.edu.sa, i_elbatal@staff.cu.edu.eg (I. Elbatal), isyakuedu@yahoo.com (Isyaku Muhammad), mmmahmoud12@sci.just.edu.jo, mmuhammad.mth@buk.edu.ng (Mustapha Muhammad)

doi: 10.22436/jmcs.022.01.03

*Received:* 2019-02-03 *Revised:* 2020-02-26 *Accepted:* 2020-04-08

of data in practice, but the quality of the procedure used in statistical study depends on the assumed probability model.

However, there are still various vital problems where there is need of higher accuracy in their statistical studies. In such cases, the known classical probability distribution may not provide a satisfactory result due to its incapability of flexibility. Consequently, significant effort has been spent in the development of new classes of flexible probability distributions along with relevant statistical methodologies in recent years, for example, the gamma-G [44], transmuted Weibull-G [6], Poisson odd-generalized exponential-G [26], new Weibull-X [4], an extended alpha power transformed [3], Topp Leone-G Poisson [1], Type II Power Topp-Leone-G [8], among others. One of the generalized family of probability distributions that received a considerable attention by authors over the years is the beta generalized family of distributions defined by [21] as follows.

Let $G(x;\phi)$ be an arbitrary cumulative distribution function (cdf) of an absolutely continuous random variable X, where $\phi$ is a parameter vector, then the cdf of the beta generalized family of distributions from X is

$$F(x;a,b,\phi) = I_{G(x;\phi)}(a,b) = \frac{B_{G(x;\phi)}(a,b)}{B(a,b)} = \frac{1}{B(a,b)}\int_0^{G(x;\phi)} w^{a-1}(1-w)^{b-1}dw, \tag{1.1}$$

where $a > 0$ and $b > 0$ are two additional parameters whose purpose is to improve the skewness and to vary tail weights. $B_y(a,b) = \int_0^y w^{a-1}(1-w)^{b-1}dw$ is the incomplete beta function, $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ is the complete beta function, $\Gamma(.)$ a gamma function, and $I_y(a,b) = \frac{B_y(a,b)}{B(a,b)}$. Notice that, if K is a beta random variable with parameters a and b, then $X = G^{-1}(K)$ is a random variable with cdf given in (1.1). Moreover, if $b = 1$ we obtain the class of exponentiated-G distributions as a special case of the beta generalized family of distributions. It is among the advantages of this generalized class the ability of fitting skewed data that other existing models may fail.

Several generalized distributions have been proposed due to (1.1) which include: the beta normal by [21], beta Gumbel distribution [28], beta Frechet [10], beta exponential [29], the beta-Weibull [19], beta Pareto [5], beta-Rayleigh [31], Beta modified Weibull [42], beta generalized half-normal [37], beta generalized Pareto [24], beta Birnbaum–Saunders [16], beta Laplace [17], beta-half-Cauchy [18], beta Burr XII [36], beta generalized linear exponential [41], beta generalized exponential [11], beta power exponential [2], and beta-Gompertz by [23], Beta exponentiated Nadarajah-Haghighi [39], among others.

Here, we propose a new four-parameter lifetime model called the beta Erlang-truncated exponential distribution. The new distribution generalizes some existing lifetime distributions, and can be considered as an alternative to some lifetime distributions.

The arrangement of the article is as follows. In Section 2, the new model and some mathematical and statistical properties are computed and studied. In Section 3, we discuss the estimation of the parameter by maximum likelihood procedure and statistical testing of hypotheses about the unknown parameters of the proposed model. The assessment of the behavior of the MLEs by simulation studies is provided. The stress-strength analysis and the construction of asymptotic confidence interval using the asymptotic distribution is discussed in Section 4. Applications of the BETE to real data is provided in Section 5. Conclusions are presented in Section 6.

## 2. The BETE and properties

[20] introduced a two parameter distribtion called the Erlang-truncated exponential (ETE) with the cumulative distribution function and density function given by

$$G(x;\alpha,\theta) = 1 - e^{-\alpha(1-e^{-\theta})x}, \quad x > 0, \tag{2.1}$$

and

$$g(x;\alpha,\theta) = \alpha(1-e^{-\theta})e^{-\alpha(1-e^{-\theta})x}, \quad x > 0,$$

respectively, where $\theta$, $\alpha > 0$. The proposed BETE distribution can be derived by plugging-in (2.1) to equation (1.1). Therefore, the cdf of BETE is given by

$$F(x) = \frac{1}{B(a,b)} \int_0^{1-e^{-\alpha(1-e^{-\theta})x}} w^{a-1}(1-w)^{b-1}dw, \quad \alpha,\ \theta,\ a,\ b,\ x > 0. \tag{2.2}$$

In other form, the cdf of the BETE can be expressed in terms of the hyper-geometric function as

$$F(x) = \frac{\left(1-e^{-\alpha(1-e^{-\theta})x}\right)^a}{a\,B(a,b)} {}_2F_1(a, 1-b; a+1; (1-e^{-\alpha(1-e^{-\theta})x})), \quad \alpha,\ \theta,\ a,\ b, x > 0,$$

where ${}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k k!} z^k$ and $(a)_k = a(a+1)\cdots(a+k-1)$.

The corresponding probability density function (pdf) $f(x)$ and the hazard rate function (hrf) $h(x)$ are

$$f(x) = \frac{\alpha(1-e^{-\theta})e^{-\alpha b(1-e^{-\theta})x}\left(1-e^{-\alpha(1-e^{-\theta})x}\right)^{a-1}}{B(a,b)}, \quad \alpha,\ \theta,\ a,\ b,\ x > 0, \tag{2.3}$$

and

$$h(x) = \frac{\alpha(1-e^{-\theta})e^{-\alpha b(1-e^{-\theta})x}\left(1-e^{-\alpha(1-e^{-\theta})x}\right)^{a-1}}{B(a,b)(1-I_{\{G(x;\alpha,\theta)\}}(a,b))}, \quad \alpha,\ \theta,\ a,\ b,\ x > 0, \tag{2.4}$$

respectively.

We denoted by $X \backsim \mathrm{BETE}(\varphi)$ the random variable $X$ with the pdf in (2.3), where $\varphi = (\alpha, \theta, a, b)^T$ is a parameter vector.

Notice that, if $a = b = 1$ we have Erlang truncated exponential distribution [20], and when $b = 1$ we have the extended Erlang truncated exponential (EETE) distribution [35].

**Theorem 2.1.** *Let $f(x)$ be the pdf in (2.3), where $\varphi = (\alpha,\theta,a,b)^T$ and $(\alpha,\theta,a,b)^T \in \mathbb{R}_+^4$, then,* (i) *$f(x)$ is decreasing corresponding to the parameter vector $\varphi$ and $a \in (0,1]$;* (ii) *$f(x)$ is unimodal with mode at $x_0 = -\log\left(\frac{b}{a+b-1}\right)/(\alpha(1-e^{-\theta}))$, corresponding to the parameter vector $\varphi$ and $a > 1$.*

*Proof.* Let $v(x) = \frac{d}{dx}\log f(x) = \alpha(1-e^{-\theta})\left[\frac{(a+b-1)e^{-\alpha(1-e^{-\theta})x}-b}{1-e^{-\alpha(1-e^{-\theta})x}}\right]$. Observe that for $a \leqslant 1$, $v(x) < 0$ thus (2.3) is decreasing monotonically. Define $\tau(x) = (a+b-1)e^{-\alpha(1-e^{-\theta})x} - b$, then $\tau(x_0) = 0$ and $x_0 = -\log\left(\frac{b}{a+b-1}\right)/(\alpha(1-e^{-\theta}))$, $a > 1$, hence $v(x_0) = 0$. Therefore, for $0 < x < x_0$, $v(x_0) > 0$, and for $x_0 < x < \infty$, $v(x_0) < 0$, thus (2.3) is a unimodal function with mode at $x_0 = -\log\left(\frac{b}{a+b-1}\right)/(\alpha(1-e^{-\theta}))$. □

It is also found out that (i) when $x \to 0$ and $a < 1$ then $f(x) \sim 0$; (ii) when $x \to 0$ and $a > 1$, then $f(x) \sim \infty$; (iii), if $x \to 0$ and $a = 1$, $f(x) \sim \alpha(1-e^{-\theta})/B(a,b)$; (iv) if $x \to \infty$, then $f(x) \sim 0$. Figure 1 (i) provides some possible shapes of the pdf given by equation (2.3).

**Theorem 2.2.** *Let $h(x)$ be the hrf in (2.4), where $\varphi = (\alpha,\theta,a,b)^T$ and $(\alpha,\theta,a,b)^T \in \mathbb{R}_+^4$, then,* (i) *$h(x)$ is decreasing corresponding to the parameter vector $\varphi$ and $a \in (0,1]$;* (ii) *$h(x)$ is increasing corresponding to the parameter vector $\varphi$ and $a > 1$.*

*Proof.* We follow the theorem provided in [22]. Define

$$\eta(x) = \frac{f'(x)}{f(x)} = \alpha b(1-e^{-\theta}) - (a-1)\alpha(1-e^{-\theta})e^{-\alpha b(1-e^{-\theta})x}\left(1-e^{-\alpha(1-e^{-\theta})x}\right)^{-1},$$

and

$$\eta'(x) = (a-1)\alpha^2(1-e^{-\theta})^2 e^{-\alpha b(1-e^{-\theta})x}\left(1-e^{-\alpha(1-e^{-\theta})x}\right)^{-1}$$

$$\times \left[ b + e^{-\alpha(1-e^{-\theta})x} \left(1 - e^{-\alpha(1-e^{-\theta})x}\right)^{-1} \right].$$

Clearly, when $a < 1, \eta'(x) < 0$ equation (2.4) is monotonically decreasing and when $a > 1, \eta'(x) > 0$ and equation (2.4) is increasing function. □

Notice that, the shapes behavior of BETE($\varphi$) density and hazard depend primarily on the parameters $a$. Figure 1 (ii) provides some possible shapes of the hazard rate function given by equation (2.4).

Further, we can define the quantile of BETE as follows. Let $X \sim BTET(\varphi)$, and let $K \sim B(a, b)$ with parameter $a, b > 0$, then, the quantile function of X can be obtained by inverting (2.2), i.e., the solution of $K = 1 - e^{-\alpha(1-e^{-\theta})x}$ as

$$\xi(k) = -(\ln(1-k))/(\alpha(1-e^{-\theta})). \tag{2.5}$$

We can use equation (2.5) for simulating random observations that follow BETE. The median of $X \sim BETE$ can be obtained from (2.5) as

$$\text{Med}(X) = -(\ln(1 - \text{Med}(k)))/(\alpha(1-e^{-\theta})). \tag{2.6}$$

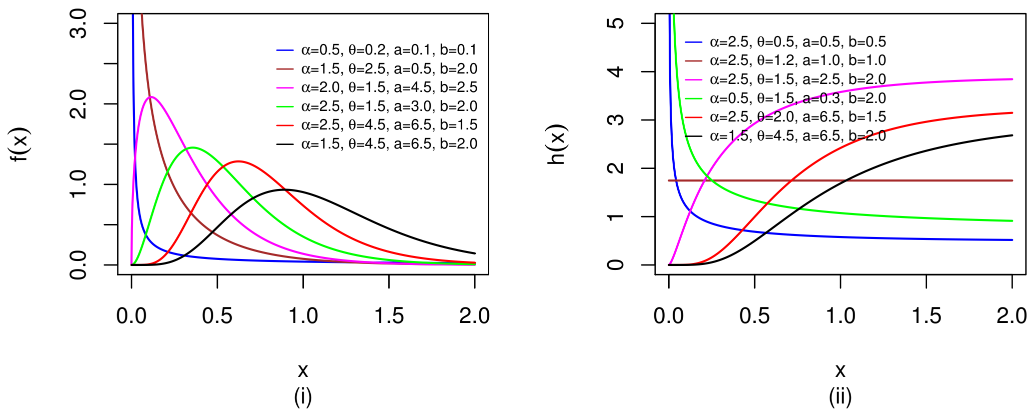


Figure 1: Plots of some possible shapes of the pdf and hrf of the BETE for some value of parameters.

Now, we derive some useful representations of the pdf of BETE distribution which make it easier in the computation of most of the properties of BETE. The mathematical relations given below are useful throughout this work. If $b$ is a positive real non-integer and $|z| < 1$ then

$$(1-z)^{b-1} = \sum_{i=0}^{\infty} \binom{b-1}{i} (-z)^i.$$

Thus, one can re-express equations (2.3) as

$$f(x, \varphi) = \sum_{j=0}^{\infty} (-1)^j \binom{a-1}{j} \frac{\alpha(1-e^{-\theta})}{B(a,b)} e^{-\alpha(b+j)(1-e^{-\theta})x} = \omega_j \, e^{-\alpha(b+j)(1-e^{-\theta})x}, \tag{2.7}$$

where $\omega_j = \sum_{j=0}^{\infty} (-1)^j \binom{a-1}{j} \frac{\alpha(1-e^{-\theta})}{B(a,b)}$.

### 2.1. *Stochastic ordering*

The most common technique of comparing two random variables is through their associated means and variances. In statistical studies, one can express many important studies about the underlying distributions regarding their survival functions, hazard rate functions, mean residual functions, and other useful functions of probability distributions. The stochastic ordering of random variables is an important tool for comparative studies of random variables based on such functions. The stochastic ordering of lifetime random variables has many applications in both theoretic and applied mathematics. For example, applications of stochastic ordering can be found in [9, 38, 43], among others. Recently, the stochastic ordering for the skew-symmetric-Laplace distributions was analyzed by [30, 32] studied the stochastic orders of the Marshall-Olkin extended distribution.

A random variable $X_1$ is said to be stochastically smaller than $X_2$ denoted by $X_1 \prec_s X_2$ if $F_{X_1}(t) \geqslant F_{X_2}(t)$ for all $t > 0$. Another two important criterion are the hazard rate ordering denoted by $X_1 \prec_{hr} X_2$ if $h_{X_1}(t) \geqslant h_{X_2}(t)$ and likelihood ratio ordering denoted by $X_1 \prec_{lr} X_2$ if $f_{X_1}(t)/f_{X_2}(t)$ is decreasing. [40] proved that

$$X_1 \prec_{lr} X_2 \Rightarrow X_1 \prec_{hr} X_2 \Rightarrow X_1 \prec_s X_2.$$

Another important stochastic measure between two random variables are the up likelihood ratio order if $f_X(x+t)/f_Y(t)$ is decreasing, denoted by $X \prec_{lr\uparrow} Y$ and down likelihood ratio order if $f_X(x)/f_Y(x+t)$ is decreasing, denoted by $X \prec_{lr\downarrow} Y$, $t > 0$. To read more about stochastic ordering one can see [13, 14].

Let $X_1$ and $X_2$ be a random variables with $\mathrm{BETE}_{X_1}(\alpha_1, \theta_1, a_1, b_1)$ and $\mathrm{BETE}_{X_2}(\alpha_2, \theta_2, a_2, b_2)$, respectively, then we consider that

$$\begin{aligned}\frac{d}{dx}\left(\log\left(\frac{f_1(t)}{f_2(t)}\right)\right) &= -\alpha_1 b_1(1-e^{-\theta_1}) + \alpha_2 b_2(1-e^{-\theta_2}) + \frac{(a_1-1)\alpha_1(1-e^{-\theta_1})e^{-\alpha_1(1-e^{-\theta_1})t}}{1-e^{-\alpha_1(1-e^{-\theta_1})t}} \\ &\quad - \frac{(a_2-1)\alpha_2(1-e^{-\theta_2})e^{-\alpha_2(1-e^{-\theta_1})t}}{1-e^{-\alpha_2(1-e^{-\theta_1})t}}.\end{aligned} \tag{2.8}$$

Observe that, (i) equation (2.8) is negative when $\alpha_1 \geqslant \alpha_2$, $\theta_1 \geqslant \theta_2$, $b_1 \geqslant b_2$, $a_1 \leqslant 1$ and $a_2 \geqslant 1$; (ii) equation (2.8) is negative when $\alpha_1 = \alpha_2$, $\theta_1 \geqslant \theta_2$, $b_1 \geqslant b_2$, and $a_1 < a_2$; (iii) equation (2.8) is negative when $\alpha_1 = \alpha_2$, $\theta_1 = \theta_2$, $b_1 \geqslant b_2$, and $a_2 > a_1 > 1$.

Here, we let $X \sim \mathrm{EETE}_X(\alpha, \theta, a)$ and $Y \sim \mathrm{BETE}_Y(\alpha, \theta, a, b)$, then we have the following

$$\frac{d}{dx}\left(\log\frac{f_X(x+t)}{f_Y(x)}\right) = \alpha(b-1)(1-e^{-\theta}) + (a-1)\left[\frac{1}{e^{\alpha(1-e^{-\theta})(x+t)}-1} - \frac{1}{e^{\alpha(1-e^{-\theta})x}-1}\right], \tag{2.9}$$

$$\frac{d}{dx}\left(\log\frac{f_X(x)}{f_Y(x+t)}\right) = \alpha(b-1)(1-e^{-\theta}) + (a-1)\left[\frac{1}{e^{\alpha(1-e^{-\theta})x}-1} - \frac{1}{e^{\alpha(1-e^{-\theta})(x+t)}-1}\right]. \tag{2.10}$$

Thus, for $a > 1$, $b < 1$ (2.9) is negative and for $a < 1$, $b < 1$ (2.10) is negative. Hence, we establish the following

**Theorem 2.3.** *Let $X_1$ and $X_2$ be two random variables with $\mathrm{BETE}_{X_1}(\alpha_1, \theta_1, a_1, b_1)$ and $\mathrm{BETE}_{X_2}(\alpha_2, \theta_2, a_2, b_2)$, respectively, then, we have:*

(i) *if $\alpha_1 \geqslant \alpha_2$, $\theta_1 \geqslant \theta_2$, $b_1 \geqslant b_2$, $a_1 \leqslant 1$ and $a_2 \geqslant 1$, then, $X_1 \prec_{lr} X_2, X_1 \prec_{hr} X_2$ and $X_1 \prec_s X_2$;*

(ii) *if $\alpha_1 = \alpha_2$, $\theta_1 \geqslant \theta_2$, $b_1 \geqslant b_2$, and $a_1 < a_2$, then, $X_1 \prec_{lr} X_2, X_1 \prec_{hr} X_2$ and $X_1 \prec_s X_2$;*

(iii) *if $\alpha_1 = \alpha_2$, $\theta_1 = \theta_2$, $b_1 \geqslant b_2$, and $a_2 > a_1 > 1$, then, $X_1 \prec_{lr} X_2, X_1 \prec_{hr} X_2$ and $X_1 \prec_s X_2$.*

**Proposition 2.4.** *Let $X \sim \mathrm{GETE}_X(\alpha, \theta, a)$ and $Y \sim \mathrm{BETE}_Y(\alpha, \theta, a, b)$, if $a > 1$, $b < 1$ then $X \prec_{lr\uparrow} Y$ and if $a < 1$, $b < 1$ then $X \prec_{lr\downarrow} Y$.*

*2.2. Moments*

In this subsection, some important statistical properties of the BETE are obtained such as the $r^{th}$ moments, moment generating function, characteristic function, skewness and kurtosis.

**Theorem 2.5.** *If* $X$ *has the* $\text{BETE}(\varphi)$, *then the* $r^{th}$ *moment of* $X$ *is given by*

$$\mu_r^{'} = \omega_j \left[ \frac{\Gamma(r+1)}{[\alpha(b+j)(1-e^{-\theta})]^{r+1}} \right]. \tag{2.11}$$

*Proof.* By using (2.7) we have

$$\mu_r^{'} = \int_0^\infty x^r f(x)dx = \omega_j \int_0^\infty x^r e^{-\alpha(b+j)(1-e^{-\theta})x} dx = \omega_j \left[ \frac{\Gamma(r+1)}{[\alpha(b+j)(1-e^{-\theta})]^{r+1}} \right],$$

where $\omega_j = \sum_{j=0}^{\infty} (-1)^j \binom{a-1}{j} \frac{\alpha(1-e^{-\theta})}{B(a,b)}$. □

**Theorem 2.6.** *If* $X$ *has the* $\text{BETE}(\varphi)$, *then the moment generating function (mgf) of* $X$ *is given by*

$$M_X(t) = \frac{\omega_j}{[\alpha(b+j)(1-e^{-\theta}) - t]}.$$

*Proof.* Taking $E[e^{tx}]$ and follow similar to (2.11). □

**Proposition 2.7.** *Let* $X \sim \text{BETE}(\varphi)$, *then the moment generating function (mgf) and characteristic function of* $X$ *can be presented as*

$$M_X(t) = \frac{B(b - \frac{t}{\alpha(1-e^{-\theta})}, a)}{B(a,b)} \quad \text{and} \quad \psi_X(t) = \frac{B(b - \frac{it}{\alpha(1-e^{-\theta})}, a)}{B(a,b)},$$

*respectively, where* $i = \sqrt{-1}$.

*Proof.* Using

$$M_X(t) = \int_0^\infty e^{tx} f(x)dx = \frac{\alpha(1-e^{-\theta})}{B(a,b)} \int_0^\infty e^{(t-\alpha b(1-e^{-\theta}))x} \left(1 - e^{-\alpha(1-e^{-\theta})x}\right)^{a-1} dx,$$

letting $u = e^{-\alpha(1-e^{-\theta})x}$ gives the result and $\psi_X(t)$ follows similar by takig $E[e^{itx}]$. □

Further, the central moments $\mu_r^{'}$ in (2.11) can be used to obtain the higher order moments by substituting $r = 1, 2, 3, \ldots$.

**Corollary 2.8.** *Let* $X \sim \text{BETE}(\varphi)$, *then the variance* ($\sigma^2$), *coefficient of variation (CV), skewness* ($\gamma^3$) *and kurtosis* ($\gamma^4$) *could be obtain from*

$$\sigma^2 = \mu_2^{'} - \mu_1^{'2}, \quad CV = \sqrt{\frac{\mu_2^{'}}{\mu_1^{'2}} - 1}, \quad \gamma^3 = \frac{\mu_3^{'} - 3\mu_2^{'}\mu_1^{'} + 2\mu_1^{'3}}{(\mu_2^{'} - \mu_1^{'2})^{3/2}}, \quad \text{and} \quad \gamma^4 = \frac{\mu_4^{'} - 4\mu_3^{'}\mu_1^{'} + 6\mu_2^{'}\mu_1^{'2} + 3\mu_1^{'4}}{(\mu_2^{'} - \mu_1^{'2})^2}.$$

Table 1 provides some numerical values of the first six moments, $\sigma^2$, CV, $\gamma^3$ and $\gamma^4$ for some parameter values. Observe that, as the parameters $\alpha, \theta, a$ and $b$ increase the first six moments, variance, CV and the skewness decreases, while the kurtosis increases. Figure 2 shows that the skewness of the BETE is decreasing in both $a$ and $b$ while the kurtosis is decreasing in $a$ and increasing in $b$ for the fixed value of $\alpha = 1.5$ and $\theta = 2.5$.

Table 1: First six moments, variance ($\sigma^2$), coefficient of variation (CV), skewness $\gamma^3$, and kurtosis $\gamma^4$ of BETE for some parameter values.

| $\alpha$ | $\theta$ | $a$ | $b$ | $\mu_1$ | $\mu_2$ | $\mu_3$ | $\mu_4$ | $\mu_5$ | $\mu_6$ | $\sigma^2$ | CV | $\gamma^3$ | $\gamma^4$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.5 | 0.6 | 0.5 | 0.6 | 4.9508 | 67.7955 | 1469.86 | 43121.76 | 1588843 | 70364563 | 43.2854 | 1.3289 | 2.4778 | 9.5164 |
| 0.8 | 0.9 | 0.6 | 0.7 | 2.2135 | 12.441 | 109.866 | 1311.9 | 19677.01 | 354832.4 | 7.5418 | 1.2407 | 2.3629 | 10.133 |
| 1.0 | 1.3 | 1.2 | 1.5 | 1.0451 | 2.0136 | 5.6714 | 21.059 | 97.187 | 536.59 | 0.9214 | 0.9184 | 1.8556 | 15.9684 |
| 1.5 | 1. | 1.5 | 2.0 | 0.5683 | 0.5425 | 0.7303 | 1.2726 | 2.7274 | 6.9480 | 0.2195 | 0.8244 | 1.6771 | 36.8479 |
| 3.0 | 2.0 | 2.0 | 4.0 | 0.1735 | 0.0453 | 0.01586 | 0.0070 | 0.0037 | 0.0023 | 0.0152 | 0.7115 | 1.4399 | 197.613 |
| 3.2 | 2.3 | 2.5 | 4.2 | 0.1786 | 0.0449 | 0.0146 | 0.0058 | 0.0028 | 0.0015 | 0.0130 | 0.6382 | 1.2987 | 275.82 |
| 3.9 | 2.6 | 2.9 | 4.8 | 0.1424 | 0.0274 | 0.0067 | 0.0020 | 0.0007 | 0.00027 | 0.0071 | 0.5928 | 1.2077 | 449.255 |
| 4.0 | 3.0 | 3.1 | 5.0 | 1.3750 | 0.02514 | 0.0058 | 0.0016 | 0.0005 | 0.00019 | 0.0062 | 0.5737 | 1.1698 | 522.208 |
| 4.5 | 3.5 | 3.9 | 5.5 | 0.1319 | 0.02196 | 0.0044 | 0.0011 | 0.00029 | 8.840e-5 | 0.0046 | 0.5128 | 1.0513 | 812.702 |
| 5.0 | 10 | 9.0 | 15 | 0.0966 | 0.0104 | 0.0012 | 0.00016 | 2.24e-5 | 3.392e-6 | 0.0011 | 0.3365 | 0.6859 | 5344.36 |

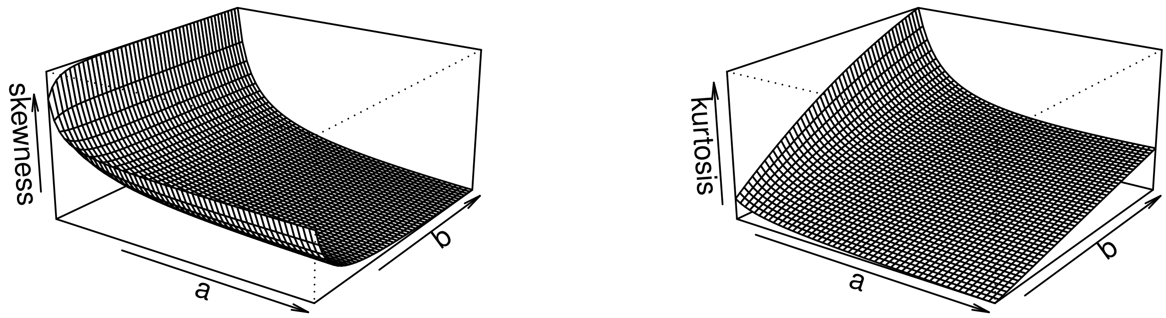


Figure 2: Plots of the skewness ($\gamma^3$) and kurtosis ($\gamma^4$) of the BETE distribution for $\alpha = 1.5$ and $\theta = 2.5$.

### 2.3. Mean deviation

The mean deviation around the mean and the mean deviation around the median can be defined as $\delta_1(x) = \int_0^\infty \mid x-\mu_1 \mid f(x)dx = 2\left[\mu_1 F(\mu_1) - J(\mu_1)\right]$ and $\delta_2(x) = \int_0^\infty \mid x - M \mid f(x)dx = \mu_1 - 2J(M)$, respectively. The value of $\mu_1$ can be obtained from (2.11) when $r = 1$ and $M = \mathrm{Med}(X)$ from (2.6). To compute $\delta_1(x)$ and $\delta_2(x)$ it is enough to derive $J(.)$ as

$$J(d) = \int_0^d x\, f(x)dx = \omega_j \int_0^d x e^{-\alpha(b+j)(1-e^{-\theta})x} dx = \frac{\omega_j\, \gamma(2, \alpha(b+j)(1-e^{-\theta})d)}{[\alpha(b+j)(1-e^{-\theta})]^2}. \tag{2.12}$$

### 2.4. Bonferroni and Lorenz curves

In this subsection, we introduced the Bonferroni curve defined by $B(p) = \frac{1}{\mu p}\int_0^q x\, f(x)dx$ and Lorenz curve defined by $L(p) = \frac{1}{\mu}\int_0^q x\, f(x)dx$. These curves has many applications not only in economics to study income and poverty, but also in other fields like reliability, medicine and insurance. By considering

(2.12), the Bonferroni and Lorenz curves of $X \sim \text{BETE}(\varphi)$ are computed as

$$B(p) = \frac{1}{\mu p}\int_0^q x\,f(x)dx = \frac{J(q)}{\mu p} = \frac{\omega_j\,\gamma(2, \alpha(b+j)(1-e^{-\theta})q)}{\mu\,[\alpha(b+j)(1-e^{-\theta})]^2},$$

and

$$L(p) = \frac{1}{\mu}\int_0^q x\,f(x)dx = \frac{J(q)}{\mu} = \frac{\omega_j\,\gamma(2, \alpha(b+j)(1-e^{-\theta})q)}{\mu\,[\alpha(b+j)(1-e^{-\theta})]^2},$$

respectively, where $q = F^{-1}(p)$ is the quantile value at $p$ which can be computed directly from (2.5).

### 2.5. *Entropy and Kullback-Leibler divergence*

The entropy of a random variable $X$ is a measure of variation of the uncertainty. We derived the two most commonly considered entropy measures that is Renyi and Shannon entropies. The Renyi entropy of a random variable $X$ with probability density function $f(x)$ is defined as follows: $I_{R(\rho)} = (1-\rho)^{-1}\ln\int_0^\infty f^\rho(x)dx$, for $\rho > 0$ and $\rho \neq 1$. The following theorem gives the Renyi entropy of the BETE.

**Theorem 2.9.** *Let the random variable $X \sim \text{BETE}$, let $\rho > 0$ and $\rho \neq 1$, then the Renyi entropy of $X$ is given by*

$$I_{R(\rho)} = (1-\rho)^{-1}\ln\left(\frac{(a-1)}{(a+b-1)}\right) - \ln\alpha - \ln(1-e^{-\theta}) + \frac{1}{(1-\rho)}\ln\left(\frac{B(\rho(a-1), \rho b)}{B^\rho(a,b)}\right).$$

*Proof.* We compute the $\int_0^\infty f^\rho(x)dx$ as

$$\int_0^\infty f^\rho(x)dx = \int_0^\infty \alpha^\rho(1-e^{-\theta})^\rho e^{-\alpha b\rho(1-e^{-\theta})x}\left(1-e^{-\alpha(1-e^{-\theta})x}\right)^{\rho(a-1)} B^{-\rho}(a,b)dx,$$

letting $u = 1-e^{-\alpha(1-e^{-\theta})x}$, we get

$$\begin{aligned}\int_0^\infty f^\rho(x)dx &= \frac{\alpha^{\rho-1}(1-e^{-\theta})^{\rho-1}}{B^\rho(a,b)}\int_0^1 (1-u)^{b\rho-1}u^{\rho(a-1)}du = \alpha^{\rho-1}(1-e^{-\theta})^{\rho-1}\frac{B(\rho(a-1)+1, b\rho)}{B^\rho(a,b)}\\ &= \frac{(a-1)\alpha^{\rho-1}(1-e^{-\theta})^{\rho-1}B(\rho(a-1), b\rho)}{(a+b-1)B^\rho(a,b)},\end{aligned}$$

therefore,

$$I_{R(\rho)} = (1-\rho)^{-1}\ln\left(\frac{(a-1)}{(a+b-1)}\right) - \ln\alpha - \ln(1-e^{-\theta}) + \frac{1}{(1-\rho)}\ln\left(\frac{B(\rho(a-1), \rho b)}{B^\rho(a,b)}\right). \qquad \square$$

Table 2 provides some numerical values of the Renyi entropy for some parameter values and $\rho$. Observe that the Renyi entropy is decreasing when $a, b, \alpha$ and $\rho$ are increasing. It is seen that this entropy may be negative in some cases.

The Shannon entropy of $X$ is defined by $E[-\log f(x)]$ it is also a particular case of the Renyi entropy when $\rho \to 1$. The Shannon entropy of BETE is obtained by applying lemma 2.10.

**Lemma 2.10.** *Let $X \sim \text{BETE}$, then,* $E\left[\log(1-e^{-\alpha(1-e^{-\theta})x})\right] = \frac{\alpha(1-e^{-\theta})}{B(a,b)}\frac{\partial}{\partial t}B(b, a+t)|_{t=0}$.

**Theorem 2.11.** *Let the random variable $X \sim \text{BETE}$, then the Shannon entropy of $X$ is given by*

$$E[-\log f(x)] = \log\left(\frac{B(a,b)}{(\alpha(1-e^{-\theta}))}\right) + \alpha b(1-e^{-\theta})\mu_1^{/} - \frac{(a-1)\alpha(1-e^{-\theta})}{B(a,b)}\frac{\partial}{\partial t}B(a+t, b)|_{t=0},$$

*where $\mu_1^{/}$ is the mean of $X$ and can be computed from (2.11), when $r = 1$.*

*Proof.*

$$E[-\log f(x)] = \log\left(\frac{B(a,b)}{(\alpha(1-e^{-\theta}))}\right) + \alpha b(1-e^{-\theta})E[X] - (a-1)E\left[\log(1-e^{-\alpha(1-e^{-\theta})x})\right], \tag{2.13}$$

the result is obtained by Lemma 2.10 and (2.11) when $r = 1$. □

The Shannon entropy of X can be represented in another form if we consider the last term of (2.13) as (2.14) which is obtained by using log expansion and exponential expansion in $\log(1-e^{-\alpha(1-e^{-\theta})x})$ as.

$$\begin{aligned} E(\log(1-e^{-\alpha(1-e^{-\theta})x})) &= \sum_{k=1}^{\infty}\frac{(-1)^{2k+1}}{k}\int_0^{\infty} e^{-k\alpha(1-e^{-\theta})x} f(x)dx \\ &= \sum_{k=1}^{\infty}\sum_{r=0}^{\infty}\frac{(-1)^{2k+r+1}\alpha^r k^r (1-e^{-\theta})^r}{kr!}\int_0^{\infty} x^r f(x)dx \\ &= \sum_{k=1}^{\infty}\sum_{r=0}^{\infty}\frac{(-1)^{2k+r+1}\alpha^r k^r (1-e^{-\theta})^r}{k\, r!}\mu_r^{/}. \end{aligned} \tag{2.14}$$

Table 3 provides some numerical values of the Shannon entropy for some parameter values, showing that for some fixed values of $\theta, b$ and $\alpha$ the Shannon entropy is increasing when $a$ is increasing. It is seen that the Shannon entropy may be negative in some cases.

Now, we shall compute the Kullback-Leibler (KL) divergence for the BETE distributions. The KL-divergence is a fundamental equation of information theory that measures the proximity of two probability distributions, i.e., it measures the distance between two density functions. It is also called the information divergence and relative entropy. For random variables $X_1 \sim \mathrm{BETE}(\varphi_1)$ and $X_2 \sim \mathrm{BETE}(\varphi_2)$, let $f_1(x) = \frac{\alpha_1(1-e^{-\theta_1})e^{-\alpha_1 b_1(1-e^{-\theta_1})x}\left(1-e^{-\alpha_1(1-e^{-\theta_1})x}\right)^{a_1-1}}{B(a_1,b_1)}$, and $f_2(x) = \frac{\alpha_2(1-e^{-\theta_2})e^{-\alpha_2 b_2(1-e^{-\theta_2})x}\left(1-e^{-\alpha_2(1-e^{-\theta_2})x}\right)^{a_2-1}}{B(a_2,b_2)}$, be the density of $X_1$ and $X_2$, then the KL divergence measure of $f_1$ and $f_2$ is defined as $KL(f_1\|f_2) = \int_0^{\infty} f_1 \log\left(\frac{f_1}{f_2}\right)dx$.

**Proposition 2.12.** *Let $X_1 \sim \mathrm{BETE}(\varphi_1)$ and $X_2 \sim \mathrm{BETE}(\varphi_2)$, let $f_1$ and $f_2$ be the density of $X_1$ and $X_2$, then the Kullback-Leibler (KL) divergence measure of $f_1$ and $f_2$ is*

$$\begin{aligned} KL(f_1\|f_2) = \log\left(\frac{\alpha_1(1-e^{-\theta_1})}{\alpha_2(1-e^{-\theta_2})}\right) - \left(\alpha_2 b_2(1-e^{-\theta_2}) - \alpha_1 b_1(1-e^{-\theta_1})\right)\mu_{1_{f1}} \\ + \frac{(a_1-1)\alpha_1(1-e^{-\theta_1})}{B(a_1,b_1)}\frac{\partial}{\partial t}B(b_1,a_1+t)|_{t=0} - (a_2-1)\sum_{k=1}^{\infty}\sum_{r=0}^{\infty}\frac{(-1)^{2k+r+1}\alpha_2^r k^r(1-e^{-\theta_2})^r}{k\, r!}\mu_{r_{f_1}}^{/}, \end{aligned}$$

*where $\mu_{1_{f1}}^{/}$ is the first moment of X from $f_1$ and $\mu_{r_{f_1}}^{/}$ is the $r^{th}$ moment of X from $f_1$ which can be determine from* (2.11).

*Proof.* We first simplify

$$\begin{aligned} f_1\log\left(\frac{f_1}{f_2}\right) = f_1[\log\left(\frac{\alpha_1(1-e^{-\theta_1})}{\alpha_2(1-e^{-\theta_2})}\right) - (\alpha_2 b_2(1-e^{-\theta_2}) - \alpha_1 b_1(1-e^{-\theta_1}))x \\ + (a_1-1)\log\left(1-e^{-\alpha_1(1-e^{-\theta_1})x}\right) - (a_2-1)\log\left(1-e^{-\alpha_2(1-e^{-\theta_2})x}\right)], \end{aligned}$$

therefore,

$$\begin{aligned} \int_0^{\infty} f_1\log\left(\frac{f_1}{f_2}\right)dx = \log\left(\frac{\alpha_1(1-e^{-\theta_1})}{\alpha_2(1-e^{-\theta_2})}\right) - (\alpha_2 b_2(1-e^{-\theta_2}) - \alpha_1 b_1(1-e^{-\theta_1}))E_{f_1}[x] \\ + (a_1-1)E_{f_1}\left[\log\left(1-e^{-\alpha_1(1-e^{-\theta_1})x}\right)\right] - (a_2-1)E_{f_1}\left[\log\left(1-e^{-\alpha_2(1-e^{-\theta_2})x}\right)\right]. \end{aligned} \tag{2.15}$$

Thus, by considering the computations in Lemma 2.10 for the third term in (2.15) and follow that of the equation (2.14) for the last term in (2.15) we have

$$KL(f_1||f_2) = \log\left(\frac{\alpha_1(1-e^{-\theta_1})}{\alpha_2(1-e^{-\theta_2})}\right) - \left(\alpha_2 b_2(1-e^{-\theta_2}) - \alpha_1 b_1(1-e^{-\theta_1})\right)\mu_{1_{f1}}$$
$$+ \frac{(a_1-1)\alpha_1(1-e^{-\theta_1})}{B(a_1,b_1)}\frac{\partial}{\partial t}B(b_1,a_1+t)|_{t=0} - (a_2-1)\sum_{k=1}^{\infty}\sum_{r=0}^{\infty}\frac{(-1)^{2k+r+1}\alpha_2^r k^r(1-e^{-\theta_2})^r}{k\, r!}\mu'_{r_{f_1}},$$

where $E_{f_1}[t] = \int_0^\infty t\, f_1(x)dx$. □

Table 2: Renyi entropy for some parameter values and $\rho$.

| $\alpha$ | $\theta$ | $a$ | $b$ | $\rho$ | $I_{R(\rho)}$ | $\alpha$ | $\theta$ | $a$ | $b$ | $\rho$ | $I_{R(\rho)}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.2 | 0.5 | 0.1 | 0.2 | 0.1 | 6.5945 | 1.2 | 1.8 | 1.7 | 1.5 | 0.8 | 0.9777 |
| 0.3 | 0.4 | 0.15 | 0.3 | 0.2 | 5.2917 | 1.8 | 2.1 | 1.9 | 2.0 | 1.1 | 0.1977 |
| 0.5 | 0.7 | 0.2 | 0.5 | 0.3 | 3.3949 | 2.5 | 2.5 | 3.1 | 2.5 | 1.5 | -0.2889 |
| 0.8 | 0.6 | 0.5 | 0.9 | 0.4 | 2.3971 | 3.5 | 4.2 | 5.0 | 7.0 | 2.0 | -1.4132 |
| 0.9 | 0.8 | 0.7 | 1.0 | 0.5 | 1.9337 | 5.0 | 8.0 | 9.0 | 12.0 | 5.0 | -2.1738 |

Table 3: Shannon Entropy for some parameter values.

| Parameters | $\alpha = 0.2,\ \theta = 0.5,\ b = 0.4$ | | $\alpha = 1.5,\ \theta = 2.1,\ b = 3.5$ | |
|---|---|---|---|---|
| | $a$ | Shannon Entropy | $a$ | Shannon Entropy |
| | 0.2 | 1.8743 | 0.3 | -2.4837 |
| | 0.4 | 3.6642 | 0.5 | -1.3677 |
| | 0.6 | 4.1457 | 0.9 | -0.6236 |
| | 0.7 | 4.2658 | 1.1 | -0.4479 |
| | 0.9 | 4.4117 | 3.6 | 0.1749 |
| | 1.3 | 4.5478 | 4.0 | 0.2081 |
| | 1.9 | 4.6297 | 4.5 | 0.2421 |

### 2.6. *Order statistics*

Order statistics are one of the fundamental tools for modeling random phenomena that arise commonly in reliability analysis, life testing, quality control, and survival analysis etc. Let $X_1, X_2, \cdots, X_n$ be the order statistics obtained from BETE, then, the probability density function of the $i^{th}$-order statistics represented by $f_{i:n}(x)$ is given as follows. For $f(x)$ in (2.3) and $F(x)$ in (2.2), we have

$$f_{i:n}(x) = \sum_{l=0}^{n-i}\frac{(-1)^l\binom{n-i}{l}f(x)}{B(i,n-i+1)}F^{i+l-1}(x), \tag{2.16}$$
$$F^{i+l-1}(x) = \left[\sum_{k=0}^{\infty}\frac{(1-b)_k}{k!(a+k)B(a,b)}\left(1-e^{-\alpha(1-e^{-\theta})x}\right)^{a+k}\right]^{i+l-1}.$$

Recall that, $\left(\sum_{k=0}^{\infty}a_k\right)^m = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\cdots\sum_{k_m=0}^{\infty}a_{k_1}a_{k_2}\cdots a_{k_m}$, therefore,

$$F^{i+l-1}(x) = \sum_{k_1,k_2,\cdots,k_{i+l-1}=0}^{\infty} C(k_w)(1-e^{-\alpha(1-e^{-\theta})x})^{\sum_{w=1}^{i+l-1}k_w + a(i+l-1)}, \tag{2.17}$$

where $C(k_w) = \prod_{w=1}^{i+l-1} c_{k_w}$ and $c_k = \frac{(1-b)_k}{(a+k)k!B(a,b)}$. By substituting (2.17) in (2.16) we get

$$f_{i:n}(x) = \sum_{l=0}^{n-i} \sum_{k_1,k_2,\cdots,k_{i+l-1}=0}^{\infty} \gamma_{k_w,l}(a,b)\, f_{k_w,l}(x;\alpha,\theta,a(i+l)+\sum_{w=1}^{i+l-1} k_w, b),$$

where $\gamma_{k_w,l}(a,b) = \frac{(-1)^l \binom{n-i}{l} C(k_w)\, B(a(i+l)+\sum_{w=1}^{i+l-1} k_w, b)}{B(i,i+l-1)B(a,b)}$ and

$$f_{k_w,l}(x;\alpha,\theta,a(i+l)+\sum_{w=1}^{i+l-1} k_w, b) = \frac{\alpha(1-e^{-\theta})e^{-\alpha b(1-e^{-\theta})x}}{B(a(i+l)+\sum_{w=1}^{i+l-1} k_w, b)}(1-e^{-\alpha(1-e^{-\theta})x})^{a(i+l)+\sum_{w=1}^{i+l-1} k_w - 1}$$

is the density function of the BETE($\varphi$) with $\varphi = (\alpha,\theta,a(i+l)+\sum_{w=1}^{i+l-1} k_w, b)$. Therefore, the pdf of the $i^{th}$-order statistics can be expressed as a mixture of BETE distribution, hence, (2.11) can be used to obtain the $r^{th}$ moment for the $i^{th}$-order statistics.

### 2.7. *Moments of residual life and reversed residual life*

Suppose that a component survives up to time $t \geqslant 0$, the residual life is the period beyond $t$ until the time of failure and defined by the conditional random variable $X - t|X > t$. The reversed residual life can be defined as the conditional random variable $t - X|X \leqslant t$ which denotes the time elapsed from the failure of a component given that its life is less than or equal to $t$.

**Proposition 2.13.** *Let* $X \sim$ BETE($\varphi$), *let* $r, t > 0$, *let* $\mu_r(t) = E((X-t)^r \mid X > t)$ *denote the* $r^{th}$ *moment of the residual life of* $X$ *and* $m_r(t) = E((t-X)^r \mid X \leqslant t)$ *denote the* $r^{th}$ *moment of the reversed residual life of* $X$*, then,*

$$\mu_r(t) = \frac{\omega_j}{\bar{F}(t)} \sum_{m=0}^{r} (-t)^m \binom{r}{m} \left[\frac{\Gamma(r-m+1, \alpha(b+j)(1-e^{-\theta})t)}{[\alpha(b+j)(1-e^{-\theta})]^{r-m+1}}\right],$$

*and*

$$m_r(t) = \frac{\omega_j}{F(t)} \sum_{m=0}^{r} (-1)^{r-m}(t)^m \binom{r}{m} \left[\frac{\gamma(r-m+1, \alpha(b+j)(1-e^{-\theta})t)}{[\alpha(b+j)(1-e^{-\theta})]^{r-m+1}}\right],$$

*where* $\bar{F}(t) = 1 - F(t)$, $\Gamma(s,t) = \int_t^{\infty} x^{s-1}e^{-x}dx$ *is the upper incomplete gamma function and* $\gamma(s,t) = \int_0^t x^{s-1}e^{-x}dx$ *is the lower incomplete gamma function.*

*Proof.* Applying the binomial expansion of $(X-t)^r$ and $(t-X)^r$ then follow similar in (2.11) by using (2.7) for the expansion of $f(x)$. $\mu_r(t) = E((X-t)^r \mid X > t) = \frac{1}{\bar{F}(t)} \int_t^{\infty} (x-t)^r f(x)dx$, and $m_r(t) = E((t-X)^r \mid X \leqslant t) = \frac{1}{F(t)} \int_0^t (t-x)^r f(x)dx$, respectively. □

## 3. Maximum likelihood estimation

In this section, we determine the maximum likelihood estimates (MLEs) of the parameters of the BETE distribution from complete data. Let $X_1, X_2, \ldots, X_n$ be a random sample of size $n$ from BETE($\varphi$) where $\varphi = (\alpha,\theta,a,b)^T$ is the parameter vector. The log likelihood function for the vector of parameters $\varphi = (\alpha,\theta,a,b)$ can be written as

$$\log L = n\log(\alpha) + n\log(1-e^{-\theta}) - \alpha b(1-e^{-\theta})\sum_{i=1}^{n} x_i + (a-1)\sum_{i=1}^{n} \log\left(1-e^{-\alpha(1-e^{-\theta})x_i}\right) - n\log B(a,b).$$

The associated score function is given by

$$U_n(\varphi) = \left[\frac{\partial L}{\partial \alpha}, \frac{\partial L}{\partial \theta}, \frac{\partial L}{\partial a}, \frac{\partial L}{\partial b}\right]^T.$$

The components of the score vector are given by

$$\begin{aligned}
\frac{\partial \log\ L}{\partial \alpha} &= \frac{n}{\alpha} - b(1-e^{-\theta})\sum_{i=1}^{n} x_i + (a-1)\sum_{i=1}^{n} \frac{(1-e^{-\theta})x_i e^{-\alpha(1-e^{-\theta})x_i}}{\left(1-e^{-\alpha(1-e^{-\theta})x_i}\right)},\\
\frac{\partial \log\ L}{\partial \theta} &= \frac{ne^{-\theta}}{(1-e^{-\theta})} - \alpha b e^{-\theta}\sum_{i=1}^{n} x_i + (a-1)\sum_{i=1}^{n} \frac{\alpha e^{-\theta} x_i e^{-\alpha(1-e^{-\theta})x_i}}{1-e^{-\alpha(1-e^{-\theta})x_i}},\\
\frac{\partial \log\ L}{\partial a} &= \sum_{i=1}^{n} \log\left(1-e^{-\alpha(1-e^{-\theta})x_i}\right) + n\psi(a+b) - n\psi(a),\\
\frac{\partial \log\ L}{\partial b} &= n\psi(a+b) - n\psi(b) - \alpha(1-e^{-\theta})\sum_{i=1}^{n} x_i.
\end{aligned}$$

The maximum likelihood estimation (MLE) of $\varphi$, say $\widehat{\varphi}$, is obtained by solving the nonlinear system $U_n(\varphi) = 0$. These equations cannot be solved analytically, but Mathematical software can be used to solve them numerically via iterative methods. The `nlminb, nlm, maxBFGS or optimx in R-software` can be used. For interval estimation and hypothesis tests on the model parameters, we require the information matrix. The $4 \times 4$ observed information matrix is given by $I_n(\varphi) = -\left(\frac{\partial^2(\log L)}{\partial\varphi\partial\varphi^T}\right)$. Applying the usual large sample approximation, MLE of $\varphi$, i.e. $\widehat{\varphi}$ can be treated as being approximately $N_4(\varphi, J_n(\varphi)^{-1})$, where $J_n(\varphi) = E[I_n(\varphi)]$. Under conditions that are fulfilled for parameters in the interior of the parameter space but not on the boundary, the asymptotic distribution of $\sqrt{n}(\widehat{\varphi} - \varphi)$ is $N_4(0, J(\varphi)^{-1})$ , where $J(\varphi) = \lim_{n\to\infty} n^{-1} I_n(\varphi)$ is the unit information matrix. The estimated asymptotic multivariate normal $N_4(\varphi, I_n(\widehat{\varphi})^{-1})$ distribution of $\widehat{\varphi}$ can be used to construct approximate confidence intervals for the parameters. An $100(1-\xi)$ asymptotic confidence interval for each parameter $\varphi_r$ is given by $ACI_r = \left(\widehat{\varphi}_r - z_{\frac{\xi}{2}}\sqrt{\widehat{I_{rr}}}, \widehat{\varphi}_r + z_{\frac{\xi}{2}}\sqrt{\widehat{I_{rr}}}\right)$ where $\widehat{I_{rr}}$ is the $(r, r)$ diagonal element of $I_n(\widehat{\varphi})^{-1}$ for $r = 1, 2, 3, 4$, and $z_{\frac{\xi}{2}}$ is the quantile $1 - \frac{\xi}{2}$ of the standard normal distribution. The elements of $I_n(\varphi)$ are given in Appendix A.

Further, BETE can be compared with some of its sub model by conducting a likelihood ratio test (LR). Let consider $\hat{\varphi}$ and $\check{\varphi}$ be the unrestricted and restricted MLEs of $\varphi$ respectively, then the LR testing between the null hypothesis $H_0 : \varphi_1 = \varphi_1^0$ versus alternative hypothesis $H_1 : \varphi_1 \neq \varphi_1^0$ is given by $w = -2(L(\check{\varphi}) - L(\hat{\varphi}))$, where $\check{\varphi}$ is under $H_0$ and $\hat{\varphi}$ under the full model, i.e., BETE. The LR test under $H_0$ is asymptotically distributed as $\chi^2_d$ with degree of freedom $d$, where $d$ is the difference in parameter dimension between the unrestricted model and the restricted model. The LR test rejects $H_0$ at level $\xi$ whenever $w > \chi^2_{d,1-\xi}$, where $\chi^2_{d,1-\xi}$ is the $1-\xi$ quantile of Chi-square distribution with degree of freedom $d$.

### *3.1. Simulation*

In this subsection, we assess the performance of the maximum likelihood estimates by conducting simulation studies. We generate 10,000 samples from the BETE each of sample size $n = 30, 50, 100, 200$, and 300. The actual values, estimated values and standard deviations for various values of parameters are listed in Table 4. The result of the simulation shows that the maximum likelihood method performed consistently and the standard deviations of the MLEs decrease as the sample size increases.

Table 4: MLEs, estimated values and standard deviations for various values of parameter.

| Sample size | Actual values | | | | Estimated values | | | | Standard deviations | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | $\alpha$ | $\theta$ | $a$ | $b$ | $\hat{\alpha}$ | $\hat{\theta}$ | $\hat{a}$ | $\hat{b}$ | $sd(\alpha)$ | $sd(\theta)$ | $sd(a)$ | $sd(b)$ |
| 30 | 0.3 | 0.6 | 2.5 | 1.5 | 0.5644 | 0.9503 | 2.9326 | 0.5853 | 0.1271 | 0.8414 | 1.7860 | 0.1506 |
| | 0.5 | 0.5 | 0.5 | 0.5 | 0.6327 | 1.0894 | 0.5559 | 0.5776 | 0.4014 | 1.6830 | 0.1461 | 0.3549 |
| | 1.8 | 1.5 | 4.0 | 2.0 | 1.9143 | 3.6626 | 4.9250 | 2.1371 | 0.6818 | 3.1671 | 1.8691 | 1.1249 |
| | 0.4 | 0.6 | 1.0 | 1.0 | 0.6335 | 1.0960 | 1.07020 | 0.6248 | 0.2484 | 1.4539 | 0.3178 | 0.2528 |
| | 1.9 | 2.0 | 4.0 | 0.8 | 1.4471 | 2.8677 | 4.2289 | 1.5262 | 0.4350 | 2.7613 | 1.6459 | 0.6213 |
| | 1.0 | 1.0 | 0.5 | 0.5 | 1.0891 | 1.6279 | 0.5842 | 1.0302 | 1.0452 | 2.2349 | 0.1489 | 1.0742 |
| | 1.0 | 1.0 | 1.0 | 1.0 | 1.1919 | 2.1213 | 1.1470 | 1.2280 | 0.7160 | 2.5451 | 0.3216 | 0.7757 |
| | 1.2 | 2.0 | 0.9 | 1.0 | 1.6641 | 2.4606 | 1.0484 | 1.7398 | 1.47580 | 2.7933 | 0.2835 | 1.5541 |
| 50 | 0.3 | 0.6 | 2.5 | 1.5 | 0.5412 | 0.8316 | 2.4817 | 0.5732 | 0.0909 | 0.4097 | 0.9532 | 0.1076 |
| | 0.5 | 0.5 | 0.5 | 0.5 | 0.5634 | 0.7687 | 0.5310 | 0.5404 | 0.2375 | 0.9961 | 0.0988 | 0.2101 |
| | 1.8 | 1.5 | 4.0 | 2.0 | 1.8378 | 3.4066 | 4.5268 | 1.9692 | 0.4711 | 2.9214 | 1.1851 | 0.6844 |
| | 0.4 | 0.6 | 1.0 | 1.0 | 0.6021 | 0.8691 | 1.0091 | 0.5985 | 0.1722 | 0.8888 | 0.2129 | 0.1770 |
| | 1.9 | 2.0 | 4.0 | 0.8 | 1.3881 | 2.5289 | 3.8349 | 1.4471 | 0.3073 | 2.2266 | 1.0150 | 0.4001 |
| | 1.0 | 1.0 | 0.5 | 0.5 | 0.9260 | 1.2531 | 0.5580 | 0.8893 | 0.5613 | 1.6953 | 0.1039 | 0.4741 |
| | 1.0 | 1.0 | 1.0 | 1.0 | 1.0930 | 1.0786 | 1.0786 | 1.1208 | 0.4281 | 1.1445 | 0.2108 | 0.4743 |
| | 1.2 | 2.0 | 0.9 | 1.0 | 1.4677 | 2.0872 | 0.9898 | 1.4704 | 0.8595 | 2.3647 | 0.1913 | 0.8686 |
| 100 | 0.3 | 0.6 | 2.5 | 1.5 | 0.5221 | 0.7828 | 2.2329 | 0.5683 | 0.0608 | 0.1989 | 0.5229 | 0.0730 |
| | 0.5 | 0.5 | 0.5 | 0.5 | 0.5332 | 0.5806 | 0.5143 | 0.5167 | 0.1512 | 0.3689 | 0.0645 | 0.1285 |
| | 1.8 | 1.5 | 4.0 | 2.0 | 1.7807 | 2.9891 | 4.2473 | 1.8684 | 0.3021 | 2.3318 | 0.7248 | 0.4033 |
| | 0.4 | 0.6 | 1.0 | 1.0 | 0.5798 | 0.7245 | 0.9637 | 0.5806 | 0.1130 | 0.3659 | 0.1332 | 0.1132 |
| | 1.9 | 2.0 | 4.0 | 0.8 | 1.3458 | 2.1242 | 3.5994 | 1.3988 | 0.2001 | 1.6742 | 0.6413 | 0.2499 |
| | 1.0 | 1.0 | 0.5 | 0.5 | 0.8400 | 0.9005 | 0.5387 | 0.8227 | 0.3066 | 0.8468 | 0.0678 | 0.2623 |
| | 1.0 | 1.0 | 1.0 | 1.0 | 1.0501 | 1.3181 | 1.0393 | 1.0476 | 0.2760 | 1.2611 | 0.1385 | 0.2773 |
| | 1.2 | 2.0 | 0.9 | 1.0 | 1.3427 | 1.6607 | 0.9478 | 1.3378 | 0.4937 | 1.6874 | 0.1217 | 0.4752 |
| 200 | 0.3 | 0.6 | 2.5 | 1.5 | 0.5167 | 0.7624 | 2.1429 | 0.5630 | 0.0413 | 0.1241 | 0.3183 | 0.0501 |
| | 0.5 | 0.5 | 0.5 | 0.5 | 0.5163 | 0.5342 | 0.5077 | 0.5062 | 0.0959 | 0.1616 | 0.0446 | 0.0842 |
| | 1.8 | 1.5 | 4.0 | 2.0 | 1.7572 | 2.5890 | 4.1267 | 1.8283 | 0.2076 | 1.7302 | 0.4865 | 0.2694 |
| | 0.4 | 0.6 | 1.0 | 1.0 | 0.5739 | 0.6802 | 0.9473 | 0.5696 | 0.0786 | 0.1619 | 0.0903 | 0.0772 |
| | 1.9 | 2.0 | 4.0 | 0.8 | 1.3313 | 1.8219 | 3.4963 | 1.3768 | 0.1392 | 1.1497 | 0.4314 | 0.1714 |
| | 1.0 | 1.0 | 0.5 | 0.5 | 0.8017 | 0.7651 | 0.5309 | 0.7991 | 0.1865 | 0.3382 | 0.0468 | 0.1663 |
| | 1.0 | 1.0 | 1.0 | 1.0 | 1.0226 | 1.1043 | 1.0164 | 1.0246 | 0.1862 | 0.5278 | 0.0934 | 0.1870 |
| | 1.2 | 2.0 | 0.9 | 1.0 | 1.286 | 1.3440 | 0.9335 | 1.2890 | 0.3105 | 0.9285 | 0.0830 | 0.3097 |
| 300 | 0.3 | 0.6 | 2.5 | 1.5 | 0.5144 | 0.7556 | 2.1150 | 0.5625 | 0.0335 | 0.0966 | 0.2554 | 0.0407 |
| | 0.5 | 0.5 | 0.5 | 0.5 | 0.5112 | 0.5209 | 0.5049 | 0.5039 | 0.0775 | 0.1215 | 0.0355 | 0.0678 |
| | 1.8 | 1.5 | 4.0 | 2.0 | 1.7467 | 2.3949 | 4.0922 | 1.8226 | 0.1667 | 1.3344 | 0.3881 | 0.2139 |
| | 0.4 | 0.6 | 1.0 | 1.0 | 0.5692 | 0.6692 | 0.9400 | 0.5687 | 0.0630 | 0.1257 | 0.0728 | 0.0625 |
| | 1.9 | 2.0 | 4.0 | 0.8 | 1.3280 | 1.7072 | 3.4586 | 1.3676 | 0.1136 | 0.5472 | 0.3446 | 0.1389 |
| | 1.0 | 1.0 | 0.5 | 0.5 | 0.7903 | 0.7405 | 0.5282 | 0.7904 | 0.1457 | 0.2072 | 0.0374 | 0.1314 |
| | 1.0 | 1.0 | 1.0 | 1.0 | 1.0150 | 1.0577 | 1.0132 | 1.0161 | 0.1472 | 0.3091 | 0.0760 | 0.1494 |
| | 1.2 | 2.0 | 0.9 | 1.0 | 1.2680 | 1.2780 | 0.9285 | 1.2724 | 0.2419 | 3.4216 | 0.0688 | 0.2402 |

## 4. The stress-strength analysis

Suppose that, the independent random variables $X_1 \sim \mathrm{BETE}(\alpha_1, \theta_1, a_1, b_1)$ and $X_2 \sim \mathrm{BETE}(\alpha_2, \theta_2, a_2, b_2)$. In reliability studies, the stress-strength model describes the life of a component which has a random strength $X_1$ that is subjected to a random stress $X_2$. If $X_1 > X_2$ the component will function satisfactorily and when $X_2 > X_1$ the component will fail because the stress applied exceed the strength. The reliability of a component is defined by $R=P(X_2 < X_1) = \int_0^\infty f_1(x)F_2(x)dx$. This has many applications in the diverse fields of engineering in the study of fatigue failure of component or structures etc. The reliability function $R$ of the BETE is given as follows.

**Theorem 4.1.** *Let $X_1 \sim \mathrm{BETE}(\alpha_1, \theta_1, a_1, b_1)$ and $X_2 \sim \mathrm{BETE}(\alpha_2, \theta_2, a_2, b_2)$ be independent random variables, then the reliability $R = P(X_2 < X_1)$ is given by*

$$R = \int_0^\infty f_1 F_2 dx = \sum_{n=0}^\infty \sum_{k=0}^\infty \Psi_{n,k}(a_2, b_2) \frac{B\left(a_1, b_1 + \frac{\alpha_2 k(1-e^{-\theta_2})}{\alpha_1(1-e^{-\theta_1})}\right)}{B(a_1, b_1)B(a_2, b_2)},$$

*where $\Psi_{n,k}(a_2, b_2) = \frac{(1-b_2)_n \binom{a_2+n}{k}(-1)^k}{n!(a_2+n)}$ and $(1-b_2)_n = \frac{\Gamma(1-b_2+n)}{\Gamma(1-b_2)}$ is a Pochhammer symbol.*

*Proof.* For real value of $a_2$ and $b_2$, we can express $F(x; \alpha_2, \theta_2, a_2, b_2) = I_{(1-e^{-\alpha_2(1-e^{-\theta_2})x})}(a_1, b_2)$ as (see. http://mathworld.wolfram.com/IncompleteBetaFunction.html)

$$F(x; \alpha_2, \theta_2, a_2, b_2) = \sum_{n=0}^\infty \frac{(1-b_2)_n}{n!(a_2+n)B(a_2, b_2)} \left(1 - e^{-\alpha_2(1-e^{-\theta_2})x}\right)^{a_2+n},$$

therefore,

$$f_1 F_2 = \frac{\alpha_1(1-e^{-\theta_1})}{B(a_1, b_1)B(a_2, b_2)} \sum_{n=0}^\infty \sum_{k=0}^\infty \frac{(1-b_2)_n \binom{a_2+n}{k}(-1)^k}{n!(a_2+n)} \times \left(1 - e^{-\alpha_1(1-e^{-\theta_1})x}\right)^{a_1-1} e^{-(\alpha_1 b_1(1-e^{-\theta_1})+\alpha_2 k(1-e^{-\theta_2}))x}.$$

By taking the integral and letting $u = 1 - e^{-\alpha_1(1-e^{-\theta_1})x}$, we have

$$\int_0^\infty f_1 F_2 dx = \sum_{n=0}^\infty \sum_{k=0}^\infty \frac{(1-b_2)_n \binom{a_2+n}{k}(-1)^k}{n!(a_2+n)B(a_1, b_1)B(a_2, b_2)} \int_0^1 u^{a_1-1}(1-u)^{b_1+\frac{\alpha_2 k(1-e^{-\theta_2})}{\alpha_1(1-e^{-\theta_1})}-1} du.$$

Thus,

$$R = \int_0^\infty f_1 F_2 dx = \sum_{n=0}^\infty \sum_{k=0}^\infty \Psi_{n,k}(a_2, b_2) \frac{B\left(a_1, b_1 + \frac{\alpha_2 k(1-e^{-\theta_2})}{\alpha_1(1-e^{-\theta_1})}\right)}{B(a_1, b_1)B(a_2, b_2)},$$

where $\Psi_{n,k}(a_2, b_2) = \frac{(1-b_2)_n \binom{a_2+n}{k}(-1)^k}{n!(a_2+n)}$.

□

For convenience, we will consider the estimation of $R$ for an independent $X_1$, $X_2$ with different shape parameters $a_1, a_2$ and common parameters $\alpha, \theta$ and $b$.

#### 4.1. *Estimation of R with common parameters* $\alpha$, $\theta$ *and* $b$

In this subsection, we compute R with common parameters $\alpha$, $\theta$ and $b$. The MLEs of R and the asymptotic confidence interval of R are discussed.

**Proposition 4.2.** *Let* $X_1 \sim \text{BETE}(\alpha, \theta, a_1, b)$ *and* $X_2 \sim \text{BETE}(\alpha, \theta, a_2, b)$ *be independent random variables, then the reliability* $R = P(X_2 < X_1)$ *is given as*

$$R = \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \Psi^{*}_{l,k}(a_2, b) \frac{B(a_1, b+k))}{B(a_1, b)B(a_2, b)},$$

*where* $\Psi^{*}_{l,k}(a_2, b) = \frac{(1-b)_l \binom{a_2+l}{k} (-1)^k}{l!(a_2+l)}$ *and* $(1-b)_l = \frac{\Gamma(1-b+l)}{\Gamma(1-b)}$ *is a Pochhammer symbol.*

Notice that R is independent of $\alpha$ and $\theta$.

##### 4.1.1. *MLE and asymptotic confidence interval of R*

Let $X_1$, $X_2$, $X_3, \ldots, X_n$ be an independent random sample of size $n$ from the BETE population with parameters $\alpha, \theta, a_1$, and $b$, let $Y_1$, $Y_2$, $Y_3, \ldots, Y_m$ be an independent random sample of size $m$ from the BETE population with parameters $\alpha, \theta, a_2$, and $b$, so the two samples has common parameters $\alpha, \theta$, and $b$. We wish to estimate the parameters $a_1$, $a_2$, $b$, $\alpha$, and $\theta$ by method of maximum likelihood.

The log-likelihood ($\log \ell$) function of the observed samples is given by

$$\begin{aligned}\log \ell = {} & (n+m)\log(\alpha) + (n+m)\log(1-e^{-\theta}) - \alpha b(1-e^{-\theta})\sum_{i=1}^{n} x_i \\ & - \alpha b(1-e^{-\theta})\sum_{j=1}^{m} y_j + (a_1-1)\sum_{i=1}^{n}\log\left(1-e^{-\alpha(1-e^{-\theta})x_i}\right) \\ & + (a_2-1)\sum_{j=1}^{m}\log\left(1-e^{-\alpha(1-e^{-\theta})y_j}\right) - n\log B(a_1,b) - m\log B(a_2,b).\end{aligned}$$

The MLEs of $\Theta = (a_1,\ a_2,\ b,\ \alpha,\ \theta)^T$, say $\hat{\Theta} = (\hat{a_1},\ \hat{a_2},\ \hat{b},\ \hat{\alpha},\ \hat{\theta},\ )^T$ can be obtained by the solution of the non-linear equations (4.1), (4.2), (4.3), (4.4), and (4.5).

$$\frac{\partial \log \ell}{\partial a_1} = \sum_{i=1}^{n}\log\left(1-e^{-\alpha(1-e^{-\theta})x_i}\right) + n\psi(a_1+b) - n\psi(a_1), \tag{4.1}$$

$$\frac{\partial \log \ell}{\partial a_2} = \sum_{i=1}^{m}\log\left(1-e^{-\alpha(1-e^{-\theta})y_i}\right) + m\psi(a_2+b) - m\psi(a_2), \tag{4.2}$$

$$\frac{\partial \log \ell}{\partial b} = n\psi(a_1+b) - n\psi(b) + m\psi(a_2+b) - m\psi(b) - \alpha(1-e^{-\theta})\sum_{i=1}^{n} x_i - \alpha(1-e^{-\theta})\sum_{j=1}^{m} y_j, \tag{4.3}$$

$$\begin{aligned}\frac{\partial \log \ell}{\partial \alpha} = {} & \frac{n+m}{\alpha} - b(1-e^{-\theta})\sum_{i=1}^{n} x_i - b(1-e^{-\theta})\sum_{j=1}^{m} y_j \\ & + (a_1-1)\sum_{i=1}^{n}\frac{(1-e^{-\theta})x_i e^{-\alpha(1-e^{-\theta})x_i}}{\left(1-e^{-\alpha(1-e^{-\theta})x_i}\right)} + (a_2-1)\sum_{j=1}^{m}\frac{(1-e^{-\theta})y_j e^{-\alpha(1-e^{-\theta})y_j}}{\left(1-e^{-\alpha(1-e^{-\theta})y_j}\right)},\end{aligned} \tag{4.4}$$

$$\frac{\partial \log \ell}{\partial \theta} = \frac{ne^{-\theta}}{(1-e^{-\theta})} + \frac{me^{-\theta}}{(1-e^{-\theta})} - \alpha b e^{-\theta}\sum_{i=1}^{n} x_i - \alpha b e^{-\theta}\sum_{j=1}^{m} x_j$$

$$+(a_1-1)\sum_{i=1}^{n}\frac{\alpha e^{-\theta}x_i e^{-\alpha(1-e^{-\theta})x_i}}{1-e^{-\alpha(1-e^{-\theta})x_i}}+(a_2-1)\sum_{j=1}^{m}\frac{\alpha e^{-\theta}y_j e^{-\alpha(1-e^{-\theta})y_j}}{1-e^{-\alpha(1-e^{-\theta})y_j}}. \tag{4.5}$$

Here, we discussed the asymptotic distribution of the $\hat{\Theta}=(\hat{a_1},\ \hat{a_2},\ \hat{b},\ \ \hat{\alpha},\ \hat{\theta})^T$ and the asymptotic distribution of $\hat{R}$. Based on the asymptotic distribution of $\hat{R}$, the asymptotic confidence interval of $R$ is derived. We denote the expected Fisher information matrix by $I(\Theta)=-E\left(\frac{\partial^2(\log \ell)}{\partial\Theta\partial\Theta^T}\right)$. The elements of $\left(\frac{\partial^2(\log \ell)}{\partial\Theta\partial\Theta^T}\right)$ can be deduced from appendix A. Let $I(\Theta)$ be represented by

$$I(\Theta)=-\begin{bmatrix} I_{a_1a_1} & I_{a_1a_2} & I_{a_1b} & I_{a_1\alpha} & I_{a_1\theta}\\ I_{a_2a_1} & I_{a_2a_2} & I_{a_2b} & I_{a_2\alpha} & I_{a_2\theta}\\ I_{ba_1} & I_{ba_2} & I_{bb} & I_{b\alpha} & I_{b\theta}\\ I_{\alpha a_1} & I_{\alpha a_2} & I_{\alpha b} & I_{\alpha\alpha} & I_{\alpha\theta}\\ I_{\theta a_1} & I_{\theta a_2} & I_{\theta b} & I_{\theta\alpha} & I_{bb}\end{bmatrix}.$$

The element of $I(\Theta)$ are given in appendix B.

**Theorem 4.3.** *As* $n\to\infty$, $m\to\infty$ *and* $n/m\to p$, *then,* $[\sqrt{n}(\hat{a_1}-a_1),\sqrt{m}(\hat{a_2}-a_2),\sqrt{n}(\hat{b}-b),\sqrt{n}(\hat{\alpha}-\alpha),\sqrt{n}(\hat{\theta}-\theta)]\to N_5(0,A^{-1}(\Theta))$, *where*

$$A(\Theta)=\begin{bmatrix} \lim_{m,n\to\infty}\frac{I_{a_1a_1}}{n} & 0 & \lim_{m,n\to\infty}\frac{I_{a_1b}}{n} & \lim_{m,n\to\infty}\frac{I_{a_1\alpha}}{n} & \lim_{m,n\to\infty}\frac{I_{a_1\theta}}{n}\\ 0 & \lim_{m,n\to\infty}\frac{I_{a_2a_2}}{m} & \lim_{m,n\to\infty}\frac{I_{a_2b}}{\sqrt{mn}} & \lim_{m,n\to\infty}\frac{I_{a_2\alpha}}{\sqrt{mn}} & \lim_{m,n\to\infty}\frac{I_{a_2\theta}}{\sqrt{mn}}\\ \lim_{m,n\to\infty}\frac{I_{ba_1}}{n} & \lim_{m,n\to\infty}\frac{I_{ba_2}}{\sqrt{mn}} & \lim_{m,n\to\infty}\frac{I_{bb}}{n} & \lim_{m,n\to\infty}\frac{I_{b\alpha}}{n} & \lim_{m,n\to\infty}\frac{I_{b\theta}}{n}\\ \lim_{m,n\to\infty}\frac{I_{\alpha a_1}}{n} & \lim_{m,n\to\infty}\frac{I_{\alpha a_2}}{\sqrt{mn}} & \lim_{m,n\to\infty}\frac{I_{\alpha b}}{n} & \lim_{m,n\to\infty}\frac{I_{\alpha\alpha}}{n} & \lim_{m,n\to\infty}\frac{I_{\alpha\theta}}{n}\\ \lim_{m,n\to\infty}\frac{I_{\theta a_1}}{n} & \lim_{m,n\to\infty}\frac{I_{\theta a_2}}{\sqrt{mn}} & \lim_{m,n\to\infty}\frac{I_{\theta b}}{n} & \lim_{m,n\to\infty}\frac{I_{\theta\alpha}}{n} & \lim_{m,n\to\infty}\frac{I_{bb}}{n}\end{bmatrix}.$$

*Proof.* The proof follows from the asymptotic normality of maximum likelihood estimation. □

The asymptotic variances and covariances of the estimators $\hat{a_1}$, $\hat{a_2}$, $\hat{\theta}$, $\hat{\alpha}$ and $\hat{b}$ is required to compute the variance of the estimator of $\hat{R}$ (see. [15]). The variance covariance matrix is given by $I^{-1}$

$$I^{-1}=\begin{bmatrix} Var(\hat{a_1}) & Cov(\hat{a_1},\hat{a_2}) & Cov(\hat{a_1},\hat{b}) & Cov(\hat{a_1},\hat{\alpha}) & Cov(\hat{a_1},\hat{\theta})\\ Cov(\hat{a_2},\hat{a_1}) & Var(\hat{a_2}) & Cov(\hat{a_2},\hat{b}) & Cov(\hat{a_2},\hat{\alpha}) & Cov(\hat{a_2},\hat{\theta})\\ Cov(\hat{b},\hat{a_1}) & Cov(\hat{b},\hat{a_2}) & Var(\hat{b}) & Cov(\hat{b},\hat{\alpha}) & Cov(\hat{b},\hat{\theta})\\ Cov(\hat{\alpha},\hat{a_1}) & Cov(\hat{\alpha},\hat{a_2}) & Cov(\hat{\alpha},\hat{b}) & Var(\hat{\alpha}) & Cov(\hat{\alpha},\hat{\theta})\\ Cov(\hat{\theta},\hat{a_1}) & Cov(\hat{\theta},\hat{a_2}) & Cov(\hat{\theta},\hat{b}) & Cov(\hat{\theta},\hat{\alpha}) & Var(\hat{\theta})\end{bmatrix}.$$

In order to construct the confidence interval of $\hat{R}$, we first determine the variance of $\hat{R}$. The asymptotic variance of $\hat{R}$ is given by

$$\begin{aligned}Var(\hat{R})=&\left(\frac{\partial R}{\partial a_1}\right)^2 Var(a_1)+\left(\frac{\partial R}{\partial a_2}\right)^2 Var(a_2)+\left(\frac{\partial R}{\partial b}\right)^2 Var(b)\\ &+2\left(\frac{\partial R}{\partial a_1}\frac{\partial R}{\partial a_2}\right)Cov(a1,a_2)+2\left(\frac{\partial R}{\partial a_1}\frac{\partial R}{\partial b}\right)Cov(a_1,b)+2\left(\frac{\partial R}{\partial a_2}\frac{\partial R}{\partial b}\right)Cov(a_2,b).\end{aligned}$$

Now, let us derive the expressions for the $\frac{\partial R}{\partial a_1}$, $\frac{\partial R}{\partial a_2}$ and $\frac{\partial R}{\partial b}$, which can be computed numerically using mathematical packages.

$$\frac{\partial R}{\partial a_1}=\frac{\alpha(1-e^{-\theta})}{B(a_1,b)}\int_0^\infty e^{-\alpha b(1-e^{-\theta})x}\left(1-e^{-\alpha(1-e^{-\theta})x}\right)^{a_1-1}I_{\{G(x;\alpha,\theta)\}}(a_2,b)$$

$$
\begin{aligned}
&\times\left(\log\left(1-e^{-\alpha(1-e^{-\theta})x}\right)-(\psi(a_1+b)-\psi(a_1))\right)dx,\\
\frac{\partial R}{\partial a_2}&=\frac{\alpha(1-e^{-\theta})}{B(a_1,b)}\int_0^\infty e^{-\alpha b(1-e^{-\theta})x}\left(1-e^{-\alpha(1-e^{-\theta})x}\right)^{a_1-1}\\
&\times\left(\left[\log(1-e^{-\alpha(1-e^{-\theta})x})-\psi(a_2)+\psi(a_2+b)\right]I_{\{G(x;\alpha,\theta)\}}(a_2,b)\right.\\
&\left.-\frac{\Gamma(a_2)\Gamma(a_2+b)}{\Gamma(b)}\left(1-e^{-\alpha(1-e^{-\theta})x}\right)^{a_2}\frac{{}_3F_2(a_2,a_2,1-b;a_2+1,a_2+1;1-e^{-\alpha(1-e^{-\theta})x})}{(\Gamma(a_2+1))^2}\right)dx,\\
\frac{\partial R}{\partial b}&=\frac{\alpha(1-e^{-\theta})}{B(a_1,b)}\int_0^\infty e^{-\alpha b(1-e^{-\theta})x}\left(1-e^{-\alpha(1-e^{-\theta})x}\right)^{a_1-1}\\
&\times\left[-\alpha(1-e^{-\theta})xI_{\{G(x;\alpha,\theta)\}}(a_2,b)+\frac{\partial I_{\{G(x;\alpha,\theta)\}}(a_2,b)}{\partial b}-I_{\{G(x;\alpha,\theta)\}}(a_2,b)[\psi(b)-\psi(a_2+b)]\right]dx,
\end{aligned}
$$

where

$$
\begin{aligned}
\frac{\partial I_{\{G(x;\alpha,\theta)\}}(a_2,b)}{\partial b}&=\frac{\Gamma(b)\Gamma(a_2+b)}{\Gamma(a_2)\,e^{\alpha b(1-e^{-\theta})x}}\,{}_3\bar{F}_2(b,b,1-a_2;b+1,b+1;e^{-\alpha(1-e^{-\theta})x})\\
&+I_{\{1-G(x;\alpha,\theta)\}}(b,a_2)\left(\psi(b)-\psi(a_2+b)-\log(e^{-\alpha(1-e^{-\theta})x})\right),
\end{aligned}
$$

and ${}_3\bar{F}_2(\ldots;..;z), |z|<1$ is the regularized hypergeometric function, see (http://mathworld.wolfram.com/RegularizedHypergeometricFunction.html). The hypergeometric function in Subsection 5.2 is computed using the package `hypergeo in R` with 2000 iteration. and $I_{\{.\}}(.,.)$ is incomplete beta function.

Once an estimated $\text{Var}(\hat{R})$ is obtained using the MLEs $\hat{\Theta}$, we can compute the 95% asymptotic confidence interval of R defined by $\hat{R}\pm 1.96\sqrt{\hat{\text{Var}}(\hat{R})}$.

## 5. Applications

In this section, we provide two applications of the BETE distribution. Firstly, we fitting BETE to a real data and compares the fit with some other existing distributions. Secondly, an application of the BETE in stress-strength analysis to a real data is considered.

### 5.1. Data analysis I

In this subsection, we perform an application to real data and demonstrate the superiority performance of BETE distribution as compared to some of its sub-models and other well-known lifetime distributions. In comparison, the model selection criteria known as the Akaike Information Citerion (AIC) and Bayesian Information Criterion (BIC) are used to compare the BETE and other models. Moreover the goodness of fit statistics known as the Kolmogorov Smirnov (KS), Anderson-Darling ($A^*$) and Cramer-von Mises ($W^*$) are considered. The KS, $A^*$ and the $W^*$ statistics are widely used to determine how closely a specific distribution whose associated cumulative distribution function fits the empirical distribution associated with a given data. The $A^*$ and $W^*$ statistics are defined by

$$
\begin{aligned}
A^*&=-(\frac{9}{4n^2}+\frac{3}{4n}+1)(n+\frac{1}{n}\sum_{i=1}^{n}(2i-1)\log(z_i(1-z_{n-i+1}))),\\
W^*&=(\frac{1}{2n}+1)(\sum_{i=1}^{n}(z_i-\frac{2i-1}{2n})^2),
\end{aligned}
$$

respectively, where $z_i=F(x_i)$ and $x_i$ the ordered observations. The upper tail percentiles of the asymptotic distributions of $A^*$ and $W^*$ were tabulated in [33]. Further, we conducted the likelihood ratio (LR) between the BETE and its sub model.

The data used consists of 101 observations of the fatigue life of 6061-T6 aluminum coupons cut parallel to the direction of rolling and oscillated at 18 cycles per second. It is provided by [12]. The data are: 70, 90, 96, 97, 99, 100, 103, 104, 104, 105, 107, 108, 108, 108, 109,109, 112, 112, 113, 114, 114, 114, 116, 119, 120, 120, 120, 121, 121, 123, 124, 124, 124, 124, 124, 128, 128, 129, 139, 130, 130, 130, 131, 131, 131, 131, 131, 132, 132, 132, 133, 134, 134, 134, 134, 134, 136, 136, 137, 138, 138, 138, 139, 139, 141, 141, 142, 142, 142, 142, 142, 142, 144, 144, 145, 146, 148, 148, 149, 151, 151, 152, 155, 156, 157, 157, 157, 157, 158, 159, 162, 163, 163, 164, 166, 166, 168, 170, 174, 196, 212.

The competing distributions include the Beta Frechet (BF) by [10], Marshall-Olkin generalized Erlang-truncated exponential (MOGETE) [34], Extended Erlang-truncated exponential (EETE) [35], and generalized half logistic Poisson (GHLP) [25]. The numerical values of the estimators and the test statistics of each model are presented in the Table 5.

Next, we consider the LR test between the null hypothesis $H_0$ : EETE versus alternative hypothesis $H_1$ : BETE, i.e., $H_0 : b = 1$ vs $H_1 : b \neq 1$. The value of the LR test is 12.57 with $p-\text{value} = 0.00039$. Therefore, the null hypothesis $H_0$ is rejected in favor of the alternative hypothesis $H_1$ : BETE model.

Clearly, we can see from Table 5, the results indicated that BETE has the smallest value of all the measures, thus, BETE fit the data better than the other existing models. In support of that, Figure 3 shows the plot of the (i) histogram and the fitted BETE, (ii) empirical and fitted BETE cdfs of the given data set, while (iii) is the quantile-quantile plot of the BETE for the data set.

Table 5: MLEs, L, AIC, BIC, $A^*$, $W^*$, KS and p-value of the competing distributions for the given data set.

| Model | $\hat{\alpha}$ | $\hat{\theta}$ | $\hat{\lambda}$ | $\hat{\beta}$ | $\hat{\gamma}$ | $\hat{a}$ | $\hat{b}$ | L | AIC | BIC | $A^*$ | $W^*$ | KS | $p-value$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| BETE | 0.0801 | 0.1307 | - | - | - | 40.3836 | 15.3598 | -456.53 | 921.06 | 931.52 | 0.4202 | 0.0712 | 0.0778 | 0.5735 |
| EETE | 257.0 | - | 3.2840 | 0.0469 | - | - | - | -462.82 | 931.63 | 939.48 | 1.5050 | 0.2393 | 0.1048 | 0.2174 |
| BFr | - | - | 0.795 | - | 112.53 | 78.23 | 108.70 | -458.07 | 924.15 | 934.61 | 4.2035 | 0.5576 | 0.1244 | 0.0879 |
| GHLP | $1.353\times10^{-5}$ | - | 6773.4630 | 278.803 | - | - | - | -462.79 | 931.59 | 939.43 | 1.1710 | 0.2106 | 0.1096 | 0.1764 |
| MOGETE | 9.0700 | $6.402\times10^{-3}$ | 1989.0 | - | - | - | - | -463.02 | 932.04 | 939.89 | 2.5561 | 0.3818 | 0.1060 | 0.2063 |

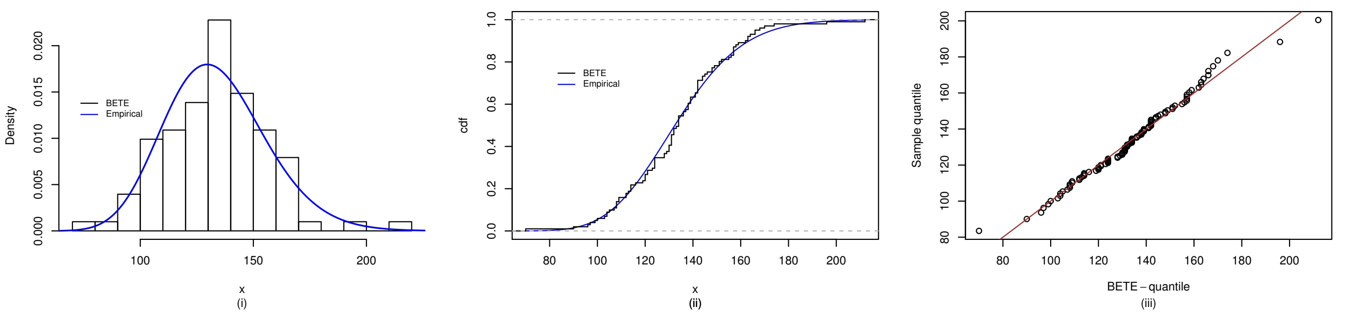


Figure 3: Plots of (i) histogram and estimated BETE pdf; (ii) empirical and estimated BETE cdf; (iii) quantile-quantile plot for the given data set.

## *5.2. Data analysis II*

In this subsection, we consider two real data sets for illustrative purposes. The two data sets were originally provided by [7], is the failure stresses (in GPa) of single carbon fibers of lengths 20 mm and 50 mm, respectively, also analyzed by [27]. The data sets are considered as $X(n = 69)$: 1.312, 1.314, 1.479, 1.552, 1.700, 1.803, 1.861, 1.865, 1.944, 1.958, 1.966, 1.997, 2.006, 2.021, 2.027, 2.055, 2.063, 2.098, 2.14, 2.179, 2.224, 2.240, 2.253, 2.270, 2.272, 2.274, 2.301, 2.301, 2.359, 2.382, 2.382, 2.426, 2.434, 2.435, 2.478, 2.490, 2.511, 2.514, 2.535, 2.554, 2.566, 2.57, 2.586, 2.629, 2.633, 2.642, 2.648, 2.684, 2.697, 2.726, 2.770, 2.773, 2.800, 2.809, 2.818, 2.821, 2.848, 2.88, 2.954, 3.012, 3.067, 3.084, 3.090, 3.096, 3.128, 3.233, 3.433, 3.585, 3.585, and $Y(m = 65)$: 1.339, 1.434, 1.549, 1.574, 1.589, 1.613, 1.746, 1.753, 1.764, 1.807, 1.812, 1.84, 1.852, 1.852, 1.862, 1.864, 1.931, 1.952, 1.974, 2.019, 2.051, 2.055, 2.058, 2.088, 2.125, 2.162, 2.171, 2.172, 2.18, 2.194, 2.211, 2.27, 2.272, 2.28, 2.299, 2.308, 2.335, 2.349, 2.356, 2.386, 2.39, 2.41, 2.43, 2.431, 2.458, 2.471, 2.497, 2.514, 2.558, 2.577, 2.593, 2.601, 2.604, 2.62, 2.633, 2.67, 2.682, 2.699, 2.705, 2.735, 2.785, 3.02, 3.042, 3.116, 3.174.

We estimate the unknown parameters of the BETE for the two data set by considering the case when $\alpha, \theta$ and $b$ are common. The goodness of fit statistics known as Kolmogorov-Smirnov (K-S) test is used to demonstrate how BETE fitted the two data sets. The numerical values of the MLEs, log-likelihood, K-S and p-value for the two data sets are computed as $\hat{a}_1 = 26.9896, \hat{a}_2 = 24.290, \hat{b} = 36.821, \hat{\alpha} = 1.010, \hat{\theta} = 0.255, \ell(\Theta) = -85.9765$. The goodness of fit statistics for X are K-S= 0.0622, p-value=0.9638 while for Y are K-S= 0.0708, p-value=0.8773.

We provide supportive plots in Figure 4 which shows (i) the plot of empirical and estimated BETE cdf of X, (ii) the quantile-quantile plot of X, (iii) the plot of empirical and estimated BETE cdf of Y, (vi) the quantile-quantile plot of Y. Figure 5 provides the profile log-likelihood of $a_1, a_2, b, \alpha$ and $\theta$, indicating that the maximum unique.

Based on the estimation we obtain $R = 0.6150$, the 95% asymptotic confidence interval of R as $(0.5920, 0.6379)$ with confidence length 0.0459. The confidence interval based on the MLEs is good.

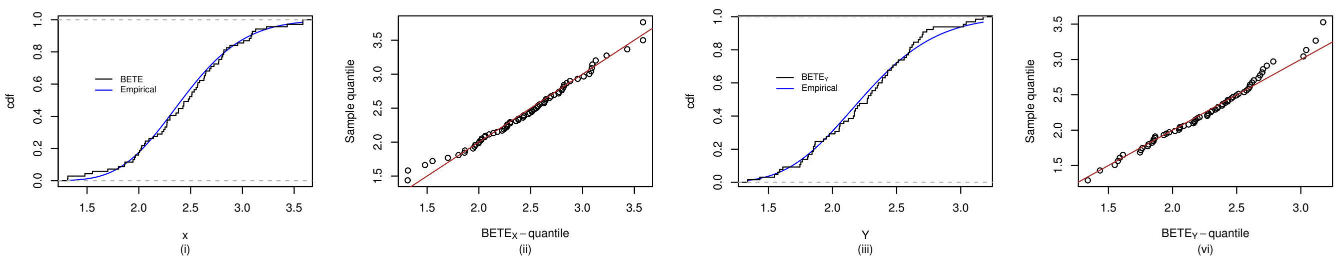


Figure 4: Plots of (i) empirical and estimated BETE cdf of X, (ii) quantile-quantile plot of X, (iii) empirical and estimated BETE cdf of Y, (vi) quantile-quantile plot of Y.

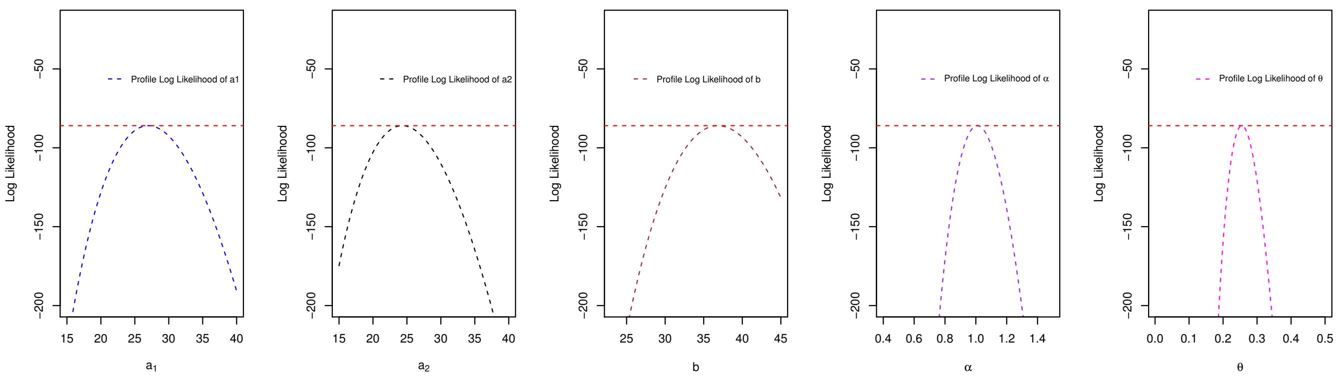


Figure 5: Plots of the profile log-likelihood of $a_1, a_2, b, \alpha$ and $\theta$.

## 6. Conclusion

We have proposed a new four-parameter distribution called the beta Erlang truncated exponential (BETE). Several mathematical and statistical properties of the BETE are investigated such as the stochastic ordering results, an explicit formula for $r^{th}$ moment, moment generating function, mean deviations, Bonferroni and Lorenz curves, moments of residual life, order statistics, entropies, and Kullback-Leibler divergence. We proposed maximum-likelihood as the means of parameter estimation of BETE and the information matrix is determined. The finite sample behavior of the MLEs was assessed by simulation studies. The stress-strength analysis is discussed. The maximum likelihood estimator and its asymptotic distribution of $P(X_2 < X_1)$ for independent random samples $X_1$ and $X_2$ of BETE for common scale parameters is analyzed. Two real applications of BETE are considered for illustration. The first application shows the superior performance in terms of fit of the BETE over several existing lifetime models. The second application of BETE in stress-strength analysis demonstrated that BETE can be a good choice in reliability analysis.

**Appendix A: Elements of the information matrix for MLEs**

The elements of the observed Fisher information matrix $I_n(\varphi)$ are given in the following. Let $p_i = e^{-\alpha(1-e^{-\theta})x_i}$, then, we have

$$\begin{aligned}
\frac{\partial^2 L}{\partial \alpha^2} &= -\frac{n}{\alpha^2} + (1-a)(1-e^{-\theta})^2 \sum_{i=1}^{n} \frac{x_i^2 p_i}{1-p_i} - (a-1)(1-e^{-\theta})^2 \sum_{i=1}^{n} \frac{x_i^2 p_i^2}{(1-p_i)^2},\\
\frac{\partial^2 L}{\partial a \partial \alpha} &= (1-e^{-\theta}) \sum_{i=1}^{n} \frac{x_i p_i}{1-p_i},\\
\frac{\partial^2 L}{\partial b \partial \alpha} &= -(1-e^{-\theta}) \sum_{i=1}^{n} x_i,\\
\frac{\partial^2 L}{\partial \theta \partial \alpha} &= -be^{-\theta} \sum_{i=1}^{n} x_i + (a-1)e^{-\theta} \sum_{i=1}^{n} \frac{x_i p_i}{1-p_i} - (a-1)\alpha(1-e^{-\theta})e^{-\theta} \sum_{i=1}^{n} \frac{x_i^2 p_i}{1-p_i}\\
&\quad - (a-1)\alpha(1-e^{-\theta})e^{-\theta} \sum_{i=1}^{n} \frac{x_i^2 p_i^2}{(1-p_i)^2},\\
\frac{\partial^2 L}{\partial \theta \partial a} &= \alpha e^{-\theta} \sum_{i=1}^{n} \frac{x_i p_i}{1-p_i},\\
\frac{\partial^2 L}{\partial \theta \partial b} &= -\alpha e^{-\theta} \sum_{i=1}^{n} x_i,\\
\frac{\partial^2 L}{\partial b \partial a} &= n\psi'(a+b),\\
\frac{\partial^2 L}{\partial a^2} &= n\psi'(a+b) - n\psi'(a),\\
\frac{\partial^2 L}{\partial b^2} &= n\psi'(a+b) - n\psi'(b),\\
\frac{\partial^2 L}{\partial \theta^2} &= -\frac{ne^{-\theta}}{(e^{\theta}-1)^2} + \alpha b e^{-\theta} \sum_{i=1}^{n} x_i + \alpha(1-a)e^{-\theta} \sum_{i=1}^{n} \frac{x_i p_i}{1-p_i}\\
&\quad - (a-1)\alpha^2 e^{-2\theta} \sum_{i=1}^{n} \frac{x_i^2 p_i}{1-p_i} + (a-1)\alpha^2 e^{-2\theta} \sum_{i=1}^{n} \frac{x_i^2 p_i^2}{(1-p_i)^2},
\end{aligned}$$

**Appendix B: Elements of the expected information matrix for MLEs of R**

The following lemma is used in the computation of some elements of I.

**Lemma 6.1.** *Let $\jmath_1, \jmath_2$ and $\jmath_3$ in $\mathbb{R}$, let*

$$J(\jmath_1, \jmath_2, \jmath_3) = \frac{\alpha(1-e^{-\theta})}{B(a,b)} \int_0^{\infty} x^{\jmath_1} e^{-\alpha[b+\jmath_2](1-e^{-\theta})x} \left(1 - e^{-\alpha(1-e^{-\theta})x}\right)^{a-\jmath_3-1} dx,$$

*then,*

$$J(\jmath_1,\jmath_2,\jmath_3)=\frac{\alpha(1-e^{-\theta})}{B(a,b)}\sum_{k=0}^{\infty}\frac{\binom{a-\jmath_3-1}{k}(-1)^k\Gamma(\jmath_1+1)}{[\alpha(1-e^{-\theta})(\jmath_2+b+k)]^{\jmath_1+1}}.$$

*Proof.* Follow in the same way in computation of (2.11). □

Notice that $J(1,0,0)=\mu$ is the mean of BETE. The elements of the expected Fisher information matrix for the MLEs of R, i.e., $I(\Theta)$ are given as follows.

- $I_{a_1a_1}=n\psi'(a_1+b)-n\psi'(a_1),\quad I_{a_1a_2}=I_{a_2a_1}=0,\quad I_{a_1b}=I_{ba_1}=n\psi'(a_1+b);$
- $I_{a_1\alpha}=I_{\alpha a_1}=n(1-e^{-\theta})J(1,1,1),\quad I_{a_1\theta}=I_{\theta a_1}=n\alpha e^{-\theta}J(1,1,1);$
- $I_{a_2a_2}=m\psi'(a_2+b)-m\psi'(a_2),\quad I_{a_2b}=I_{ba_2}=m\psi'(a_2+b);$
- $I_{a_2\alpha}=I_{\alpha a_2}=m(1-e^{-\theta})J(1,1,1),\quad I_{a_2\theta}=I_{\theta a_2}=m\alpha e^{-\theta}J(1,1,1);$
- $I_{bb}=n\psi'(a_1+b)-n\psi'(b)+m\psi'(a_2+b)-m\psi'(b);$
- $I_{b\alpha}=I_{\alpha b}=-n(1-e^{-\theta})\mu-m(1-e^{-\theta})\mu,\quad I_{b\theta}=I_{\theta b}=-n\alpha e^{-\theta}\mu-m\alpha e^{-\theta}\mu;$
- $I_{\alpha\alpha}=-\frac{n+m}{\alpha^2}-n(a_1-1)(1-e^{-\theta})^2J(2,2,2)+n(a_1-1)(1-e^{-\theta})^2J(2,1,1)-m(a_2-1)(1-e^{-\theta})^2$ $J(2,2,2)+m(a_2-1)(1-e^{-\theta})^2J(2,1,1);$
- $I_{\alpha\theta}=I_{\theta\alpha}=-nbe^{-\theta}\mu-mbe^{-\theta}\mu-n(a_1-1)\alpha(1-e^{-\theta})e^{-\theta}J(2,2,2)-n(a_1-1)\alpha(1-e^{-\theta})e^{-\theta}$ $J(2,1,1)-m(a_2-1)\alpha(1-e^{-\theta})e^{-\theta}J(2,2,2)-m(a_2-1)\alpha(1-e^{-\theta})e^{-\theta}J(2,1,1);$
- $I_{\theta\theta}=\frac{(n+m)e^{\theta}}{(e^{\theta}-1)^2}+n\alpha be^{-\theta}\mu+m\alpha be^{-\theta}\mu-n(a_1-1)\alpha e^{-\theta}J(1,1,1)-n(a_1-1)\alpha^2e^{-2\theta}J(2,1,1)-n(a_1-$ $1)\alpha^2e^{-2\theta}J(2,2,2)-m(a_2-1)\alpha e^{-\theta}J(1,1,1)-m(a_2-1)\alpha^2e^{-2\theta}J(2,1,1)-m(a_2-1)\alpha^2e^{-2\theta}J(2,2,2).$


## Acknowledgment

The authors extend their appreciation to the Deanship of Scientific Research at King Saud University for funding this work through research group No. (RG-1438-086)


## References


[1] T. H. M. Abouelmagda, M. S. Hameda, L. Handiqueb, H. Goualc, M. M. Alid, H. M. Yousofe, N. C. Korkmazf, *A new class of distributions based on the zero truncated Poisson distribution with properties and applications*, J. Nonlinear Sci. Appl., **12** (2019), 152–164. 1
[2] K. A. Adepoju, A. Chukwu, M. Wang, *The Beta Power Exponential Distribution*, J. Stat. Sci. Appl., **2** (2014), 37–46. 1
[3] Z. Ahmad, M. Elgarhy, N. Abbas, *A new extended alpha power transformed family of distributions: properties and applications*, J. Statist. Model. Theory Appl., **1** (2018), 13–28. 1
[4] Z. Ahmad, M. Elgarhy, G. G. Hamedani, *A new Weibull-X family of distributions: properties, characterizations and applications*, J. Stat. Distrib. Appl., **5** (2018), 18 pages. 1
[5] A. Akinsete, F. Famoye, C. Lee, *The beta-Pareto distribution*, Statistics, **42** (2008), 547–563. 1
[6] M. Alizadeh, M. Rasekhi, H. M. Yousof, G. G. Hamedani, *The transmuted Weibull-G family of distributions*, Hacet. J. Math. Stat., **47** (2018), 1671–1689. 1
[7] M. G. Bader, A. M. Priest, *Statistical aspects of fiber and bundle strength in hybrid composites*, Progress in Science and Engineering of Composites, **1982** (1982), 1129–1136. 5.2
[8] R. A. R. Bantan, F. Jamal, C. Chesneau, M. Elgarhy, *Type II Power Topp-Leone Generated Family of Distributions with Statistical Inference and Applications*, Symmetry, **12** (2020), 24 pages. 1
[9] R. E. Barlow, F. Proschan, *Statistical theory of Reliability and life testing*, Holt, Rinehart and Winston, New York-Montreal, (1975). 2.1
[10] W. Barreto-Souza, G. M. Cordeiro, A. B. Simas, *Some results for beta Frechet distribution*, Comm. Statist. Theory Methods, **40** (2011), 798–811. 1, 5.1

[11] W. Barreto-Souza, A. H. S. Santos, G. M. Cordeiro, *The beta generalized exponential distribution*, J. Stat. Comput. Simul., **80** (2010), 159–172. 1

[12] Z. W. Birnbaum, S. C. Saunders, *Estimation for a family of life distributions with applications to fatigue*, J. Appl. Probability, **6** (1969), 328–347. 5.1

[13] P. J. Boland, T. Hu, M. Shaked, J. G. Shanthikumar, *Stochastic ordering of order statistics II*, in: Modeling uncertainty, **2005** (2005), 607–623. 2.1

[14] P. J. Boland, M. Shaked, J. G. Shanthikumar, *Stochastic ordering of order statistics*, Handbook of Statistics, **1998** (1998), 89–103. 2.1

[15] G. Casella, R. L. Berger, *Statistical Inference*, Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove, (1990). 4.1.1

[16] G. M. Cordeiro, A. J. Lemonte, *The β-Birnbaum-Saunders distribution: an improved distribution for fatigue life modeling*, Comput. Statist. Data Anal., **55** (2011), 1445–1461. 1

[17] G. M. Cordeiro, A. J. Lemonte, *The beta Laplace distribution*, Statist. Probab. Lett., **81** (2011), 973–982. 1

[18] G. M. Cordeiro, A. J. Lemonte, *The beta-half-Cauchy distribution*, J. Probab. Stat., **2011** (2011), 18 pages. 1

[19] G. M. Cordeiro, A. B. Simas, B. D. Stosic, *Explicit expressions for moments of the beta Weibull distribution*, arXiv, **2008** (2008), 17 pages. 1

[20] A. R. El-Alosey, *Random sum of new type of mixture of distribution*, Int. J. Statist. Syst., **2** (2007), 49–57. 2, 2

[21] N. Eugene, C. Lee, F. Famoye *Beta-normal distribution and its applications*, Comm. Statist. Theory Methods, **31** (2002), 497–512. 1, 1

[22] R. E. Glaser, *Bathtub and related failure rate characterizations*, J. Amer. Statist. Assoc., **75** (1980), 667–672. 2

[23] A. A. Jafari, S. Tahmasebi, M. Alizadeh, *The beta-Gompertz distribution*, Rev. Colombiana Estadist., **37** (2014), 141–158. 1

[24] E. Mahmoudi, *The beta generalized Pareto distribution with application to lifetime data*, Math. Comput. Simulation, **81** (2011), 2414–2430. 1

[25] M. Muhammad, *Generalized Half Logistic Poisson Distributions*, Comm. Stat. Appl. Methods, **24** (2017), 353–365. 5.1

[26] M. Muhammad, *Poisson-odd generalized exponential family of distributions: theory and applications*, Hacet. J. Math. Stat., **47** (2018), 1652–1670. 1

[27] M. Muhammad, L. Liu, *A New Extension of the Generalized Half Logistic Distribution with Applications to Real Data*, Entropy, **21** (2019), 37 pages. 5.2

[28] S. Nadarajah, S. Kotz, *The beta Gumbel distribution*, Math. Probl. Eng., **2004** (2004), 323–332. 1

[29] S Nadarajah, S Kotz, *The beta exponential distribution*, Reliab. Eng. Syst. Safety, **91** (2006), 689–697. 1

[30] A. K. Nanda, S. Das, *Stochastic orders of the Marshall-Olkin extended distribution*, Statist. Probab. Lett., **82** (2012), 295–302. 2.1

[31] V. Nekoukhou, *The Beta-Rayleigh Distribution on the Lattice of Integers*, J. Statist. Res. Iran, **12** (2015), 205–224. 1

[32] V. Nekoukhou, M. H. Alamatsaz, *A family of skew-symmetric-Laplace distributions*, Statist. Papers, **53** (2012), 685–696. 2.1

[33] M. D. Nichols, W. J. Padgett, *A bootstrap control chart for Weibull percentiles*, Quality Reliab. Eng. Int., **22** (2006), 141–151. 5.1

[34] I. E. Okorie, A. C. Akpanta, J. Ohakwe, *Marshall–Olkin generalized Erlang–truncated exponential distribution: Properties and applications*, Cogent Math., **4** (2017), 19 pages. 5.1

[35] I. E. Okorie, A. C. Akpanta, J. Ohakwe, D. C. Chikezie, *The Extended Erlang-Truncated Exponential distribution: Properties and application to rainfall data*, Heliyon, **3** (2017), 12 pages. 2, 5.1

[36] P. F. Paranaíba, E. M. M. Ortega, G. M. Cordeiro, R. R. Pescim, *The beta Burr XII distribution with application to lifetime data*, Comput. Statist. Data Anal., **55** (2011), 1118–1136. 1

[37] R. R. Pescim, C. G. B. Demétrio, G. M. Cordeiro, E. M. Ortega, M. R. Urbano, *The beta generalized half-normal distribution*, Comput. Statist. Data Anal., **54** (2010), 945–957. 1

[38] S. M. Ross, *Stochastic processes*, Second ed., John Wiley & Sons, New York, (1996). 2.1

[39] A. Saboor, M. N. Khan, G. M. Cordeiro, I. Elbatal, R. R. Pescim, *The Beta exponentiated Nadarajah-Haghighi distribution: Theory, regression model and application*, Math. Slovaca, **69** (2019), 939–952. 1

[40] M. Shaked, J. G. Shanthikumar, J. B. Valdez-Torres, *Discrete probabilistic orderings in reliability theory*, Statist. Sinica, **4** (1994), 567–579. 2.1

[41] M. K. Shakhatreh, A. Yusuf, A. R. Mugdadi, *The beta generalized linear exponential distribution*, Statistics, **50** (2016), 1346–1362. 1

[42] G. O. Silva, E. M. M. Ortega, G. M. Cordeiro, *The beta modified Weibull distribution*, Lifetime Data Anal., **16** (2010), 409–430. 1

[43] D. Stoyan, *Comparison methods for queues and other stochastic models*, John Wiley & Sons, Chichester, (1983). 2.1

[44] K. Zografos, N. Balakrishnan, *On families of beta and generalized gamma-generated distributions and associated inference*, Stat. Methodol., **6** (2009), 344–362. 1