J. Math. Computer Sci., 22 (2021), 38–48

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

A modified extra-gradient method for a family of strongly pseudomonotone equilibrium problems in real Hilbert spaces

Habib ur Rehman[a], Nuttapol Pakkaranang[a], Azhar Hussain[b], Nopparat Wairojjana[c,*]

[a]KMUTTFixed Point Research Laboratory, KMUTT-Fixed Point Theory and Applications Research Group, SCL 802 Fixed Point Laboratory, Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), 126 Pracha-Uthit Road, Bang Mod, Thrung Khru, Bangkok 10140, Thailand.

[b]Department of Mathematics, University of Sargodha, Sargodha-40100, Pakistan.

[c]Applied Mathematics Program, Faculty of Science and Technology, Valaya Alongkorn Rajabhat University under the Royal Patronage (VRU), 1 Moo 20 Phaholyothin Road, Klong Neung, Klong Luang, Pathumthani, 13180, Thailand.

Abstract

In this paper, we propose a modified extragradient method for solving a strongly pseudomonotone equilibrium problem in a real Hilbert space. A strong convergence theorem relative to our proposed method is proved and the proposed method has worked without having the information of a strongly pseudomonotone constant and the Lipschitz-type constants of a bifunction. We have carried out our numerical explanations to justify our well-established convergence results, and we can see that our proposed method has a substantial improvement over the time of execution and number iterations.

Keywords: Equilibrium problem, strongly pseudomonotone bifunction, strong convergence theorem, Lipschitz-type conditions, variational inequality problems.

2020 MSC: 47J25, 47H09, 47H06, 47J05.

1. Introduction

Let K to be a nonempty closed, convex subset of a Hilbert space $\mathbb{E}$ and $f : \mathbb{E} \times \mathbb{E} \to \mathbb{R}$ be a bifunction such that $f(x, x) = 0$ for all $x \in K$. The *equilibrium problem* for the bifunction f on K is defined as follows:

$$\text{Find } x^* \in K \text{ such that } f(x^*, y) \geqslant 0, \ \forall y \in K. \tag{EP}$$

Equilibrium problem (EP) was initially established in the unique format by Blum and Oettli [8] in 1994 and provided a comprehensive study on their theoretical properties. This study consists of considerable improvement in applied and pure science. It had been previously presented that the equilibrium problem

*Corresponding author

Email addresses: hrehman.hed@gmail.com.com (Habib ur Rehman), nuttapol.pak@mail.kmutt.ac.th (Nuttapol Pakkaranang), hafiziqbal30@yahoo.com (Azhar Hussain), nopparat@vru.ac.th (Nopparat Wairojjana)

doi: 10.22436/jmcs.022.01.04

Received: 2020-03-19 *Revised:* 2020-04-10 *Accepted:* 2020-04-27

theory has set up a unique approach to deal with many topics that are arisen from the social sciences, economics, finance, restoration of image, ecology, transport, networking, elasticity and optimization problems (see for details [3, 10, 11, 20, 28]). The equilibrium problem contains several mathematical problems as particular cases, i.e., minimization problems, variational inequality problems (VIP), the fixed point problems, the Nash equilibrium of non-cooperative games, complementarity problems, the problem of vector minimization and the saddle point problem [8, 13, 19, 21, 34].

On the other hand, iterative methods are significant and useful tools for studying the numerical solution of an equilibrium problem. A considerable number of methods was formed to deal with specific types of equilibrium problems for finite and infinite dimensional spaces (see [9, 12, 14, 17, 18, 24, 25, 27, 29, 32, 33]). More specifically, Hieu et al. in [15] described a sequence $\{x_n\}$ recursively as:

$$\begin{cases} x_0 \in K, \\ y_n = \arg\min\{\lambda_n f(x_n, y) + \frac{1}{2}\|x_n - y\|^2 : y \in K\}, \\ x_{n+1} = \arg\min\{\lambda_n f(y_n, y) + \frac{1}{2}\|x_n - y\|^2 : y \in K\}, \end{cases}$$

where $\{\lambda_n\}$ is a sequence of positive real numbers satisfy the following conditions, i.e.,

$$(\Psi_1) : \lim_{n\to\infty} \lambda_n = 0 \text{ and } (\Psi_2) : \sum_{n=0}^{\infty} \lambda_n = +\infty.$$

On the other hand, inertial-type methods are valuable and depending on the technique of the heavy-ball methods of the second-order time dynamic system. Polyak started by considering an inertial step as an acceleration process to deal with the problem of smooth convex minimization. Inertial-type methods are two-step iterative programs and the next iteration is determined by using the previous two iterations and may be used the accelerated step to boost up the iterative sequence (further details, see [1, 2, 6, 23, 30, 31, 35]).

In this paper, on the basis of the work of Hieu et al. [15], we propose a modified extragradient method for solving equilibrium problems involving bifunction f being strongly pseudomonotone. Our purpose method is carried out without any knowledge of the lipschitz-type and strongly pseudomonotone constants of the bifunction. This modification is based on the use of a step-size sequence that slowly converges to zero and is non-summable. Due to this factor and the strong pseudomonotonicity of the bifunction, the strong convergence of our method has been achieved. Despite that, it is not mandatory to have the information about these constants before, i.e., such constants should not be within the input parameters of the method. In the end, the numerical experiments are carried out and shown that proposed method is more efficient than the existing ones [15, 16] in term of number of iteration and execution time.

The paper is arranged according to the following. Section 2 provides definitions and essential lemmas which are used during this paper. Section 3 consists of our proposed method and corresponding strong convergence theorem. Section 4 sets out the numerical experimental work to indicate the numerical performance compared to existing methods.

2. Preliminaries

We take K convex and closed subset of a Hilbert space $\mathbb{E}$. The notion $\langle ., . \rangle$ and $\|.\|$ views for the inner product and norm on the Hilbert space, respectively. Moreover, $EP(f, K)$ stands for the solution set of an equilibrium problem over the set K and $VI(G, K)$ solution set of an variational inequality problem over the set K with x^* is any arbitrary member of $EP(f, K)$ or $VI(G, K)$.

Let $g : K \to \mathbb{R}$ is a convex function and *subdifferential of* g at $x \in K$ is defined as follows:

$$\partial g(x) = \{z \in \mathbb{E} : g(y) - g(x) \geqslant \langle z, y - x\rangle, \ \forall y \in K\}.$$

A *normal cone of* K at $x \in K$ is given as

$$N_K(x) = \{z \in \mathbb{E} : \langle z, y - x\rangle \leqslant 0, \ \forall y \in K\}.$$

Definition 2.1 ([7, 8])**.** $f : \mathbb{E} \times \mathbb{E} \to \mathbb{R}$ on K for $\gamma > 0$ is

(i) *strongly monotone* if
$$f(x,y) + f(y,x) \leqslant -\gamma\|x-y\|^2, \ \forall x,y \in K;$$

(ii) *monotone* if
$$f(x,y) + f(y,x) \leqslant 0, \ \forall x,y \in K;$$

(iii) *strongly pseudomonotone* if
$$f(x,y) \geqslant 0 \Longrightarrow f(y,x) \leqslant -\gamma\|x-y\|^2, \ \forall x,y \in K;$$

(iv) *pseudomonotone* if
$$f(x,y) \geqslant 0 \Longrightarrow f(y,x) \leqslant 0, \ \forall x,y \in K;$$

(v) satisfying the *Lipschitz-type condition* on K if there are $k_1, k_2 > 0$, such that
$$f(x,z) \leqslant f(x,y) + f(y,z) + k_1\|x-y\|^2 + k_2\|y-z\|^2, \ \forall x,y,z \in K.$$

Lemma 2.2 ([26])**.** *Let K be a nonempty, closed and convex subset of a real Hilbert space $\mathbb{E}$ and $g : K \to \mathbb{R}$ be a subdifferentiable, convex and lower semicontinuous function on K. Moreover, $x \in K$ is a minimizer of a function g if and only if $0 \in \partial g(x) + N_K(x)$, where $\partial g(x)$ and $N_K(x)$ stand for the subdifferential of g at x and the normal cone of K at x, respectively.*

Lemma 2.3 ([4])**.** *Assume a_n, b_n and c_n are sequences in $[0,+\infty)$ such that*
$$a_{n+1} \leqslant a_n + b_n(a_n - a_{n-1}) + c_n, \ \text{for all } n \geqslant 1, \ \text{with } \sum_{n=1}^{+\infty} c_n < +\infty,$$
and also with $b > 0$ such that $0 \leqslant b_n \leqslant b < 1$, for all $n \in \mathbb{N}$. Thus, the following relations are true.

(i) $\sum_{n=1}^{+\infty}[a_n - a_{n-1}]_+ < \infty$, *with* $[s]_+ := \max\{s, 0\}$;
(ii) $\lim_{n\to+\infty} a_n = a^* \in [0,\infty)$.

Lemma 2.4 ([5])**.** *For every $\alpha, \beta \in \mathbb{E}$ and $\mu \in \mathbb{R}$, the following item is true:*
$$\|\mu\alpha + (1-\mu)\beta\|^2 = \mu\|\alpha\|^2 + (1-\mu)\|\beta\|^2 - \mu(1-\mu)\|\alpha-\beta\|^2.$$

Lemma 2.5 ([22])**.** *Suppose $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences of nonnegative real numbers. If $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\sum_{n=1}^{\infty} \alpha_n\beta_n < \infty$, thus $\liminf_{n\to\infty} \beta_n = 0$.*

Assumption 1. Let $f : \mathbb{E} \times \mathbb{E} \to \mathbb{R}$ satisfying the following conditions:

(f1) $f(x,x) = 0$, for all $x \in K$ and f is strongly pseudomonotone on K;
(f2) f satisfy the Lipschitz-type conditions through two positive constants k_1 and k_2;
(f3) $f(x,.)$ is sub-differentiable and convex on K for each fixed $x \in K$.

3. An algorithm and its strong convergence analysis

We established an inertial method for dealing with strongly pseudomonotone equilibrium problem with a Lipschitz-type condition. However, it is not compulsory to have information about the Lipschitz-type constants k_1, k_2 and strongly pseudomonotone constant γ previously to generate the iterative sequence. The following is our method in detail.

Algorithm 3.1 (Modified extragradient method for strongly pseudomonotone equilibrium problems)**.**

Initialization: Choose $x_{-1}, x_0 \in \mathbb{E}$ and $0 \leqslant \theta_n < \sqrt{5} - 2$.

Iterative steps: Assume x_{n-1}, x_n are known for $n \geqslant 0$, and a sequence $\{\lambda_n\}$ satisfying the conditions:

$$(T_1): \lim_{n\to\infty} \lambda_n = 0 \text{ and } (T_2): \sum_{n=1}^{\infty} \lambda_n = +\infty.$$

Step 1: Compute

$$y_n = \arg\min_{y\in K}\{\lambda_n f(w_n, y) + \frac{1}{2}\|w_n - y\|^2\},$$

where $w_n = x_n + \theta_n(x_n - x_{n-1})$. If $y_n = w_n$ then stop and w_n is the solution of the equilibrium problem. Otherwise, go to **Step 2**.

Step 2: Compute

$$x_{n+1} = \arg\min_{y\in K}\{\lambda_n f(y_n, y) + \frac{1}{2}\|w_n - y\|^2\}.$$

Set $n := n+1$ and go back to **Step 1**.

Lemma 3.2. *From Algorithm 3.1, we have the following useful inequality.*

$$\lambda_n f(y_n, y) - \lambda_n f(y_n, x_{n+1}) \geqslant \langle w_n - x_{n+1}, y - x_{n+1}\rangle, \ \forall y \in K.$$

Proof. By Lemma 2.2, we can write

$$0 \in \partial_2\Big\{\lambda_n f(y_n, y) + \frac{1}{2}\|w_n - y\|^2\Big\}(x_{n+1}) + N_K(x_{n+1}).$$

Thus, for $\omega \in \partial_2 f(y_n, x_{n+1})$ and $\overline{\omega} \in N_K(x_{n+1})$ we have

$$\lambda_n \omega + x_{n+1} - w_n + \overline{\omega} = 0.$$

The above implies that

$$\langle w_n - x_{n+1}, y - x_{n+1}\rangle = \lambda_n\langle \omega, y - x_{n+1}\rangle + \langle \overline{\omega}, y - x_{n+1}\rangle, \ \forall y \in K.$$

Since $\overline{\omega} \in N_K(x_{n+1})$ then $\langle \overline{\omega}, y - x_{n+1}\rangle \leqslant 0$, for all $y \in K$. Thus, we obtain

$$\langle w_n - x_{n+1}, y - x_{n+1}\rangle \leqslant \lambda_n\langle \omega, y - x_{n+1}\rangle, \ \forall y \in K. \tag{3.1}$$

By $\omega \in \partial f(y_n, x_{n+1})$, we get

$$f(y_n, y) - f(y_n, x_{n+1}) \geqslant \langle \omega, y - x_{n+1}\rangle, \ \forall y \in K. \tag{3.2}$$

Combining (3.1) and (3.2) we get the required result

$$\lambda_n f(y_n, y) - \lambda_n f(y_n, x_{n+1}) \geqslant \langle w_n - x_{n+1}, y - x_{n+1}\rangle, \ \forall y \in K.$$

□

Lemma 3.3. *By Algorithm 3.1, we can also get the following inequality.*

$$\lambda_n f(w_n, y) - \lambda_n f(w_n, y_n) \geqslant \langle w_n - y_n, y - y_n\rangle, \ \forall y \in K.$$

Proof. It follows the same procedure as in Lemma 3.2. □

Lemma 3.4. *Let $f : K \to \mathbb{R}$ satisfies the Assumption 1 and the solution set $EP(f, K) \neq \emptyset$. Thus, for each $x^* \in EP(f, K)$, we have*

$$\|x_{n+1} - x^*\|^2 \leqslant \|w_n - x^*\|^2 - (1 - 2k_1\lambda_n)\|w_n - y_n\|^2 - (1 - 2k_2\lambda_n)\|y_n - x_{n+1}\|^2 - 2\gamma\lambda_n\|y_n - x^*\|^2.$$

Proof. By the Lemma 3.2 and replacing $y = x^*$, we have

$$\lambda_n f(y_n, x^*) - \lambda_n f(y_n, x_{n+1}) \geqslant \langle w_n - x_{n+1}, x^* - x_{n+1} \rangle.$$

Since $f(x^*, y_n) \geqslant 0$ then due to strong pseudomonotonicity implies that $f(y_n, x^*) \leqslant -\gamma \|y_n - x^*\|^2$, such that

$$\langle w_n - x_{n+1}, x_{n+1} - x^* \rangle \geqslant \lambda_n f(y_n, x_{n+1}) + \gamma \lambda_n \|y_n - x^*\|^2. \tag{3.3}$$

The Lipschitz-type continuity of a bifunction f leads to

$$f(w_n, x_{n+1}) \leqslant f(w_n, y_n) + f(y_n, x_{n+1}) + k_1 \|w_n - y_n\|^2 + k_2 \|y_n - x_{n+1}\|^2. \tag{3.4}$$

Combining the expression (3.3) and (3.4) we obtain

$$\begin{aligned} \langle w_n - x_{n+1}, x_{n+1} - x^* \rangle \geqslant \lambda_n \{ f(w_n, x_{n+1}) - f(w_n, y_n) \} - k_1 \lambda_n \|w_n - y_n\|^2 \\ - k_2 \lambda_n \|y_n - x_{n+1}\|^2 + \gamma \lambda_n \|y_n - x^*\|^2. \end{aligned} \tag{3.5}$$

Following Lemma 3.3 with $y = x_{n+1}$, we have

$$\lambda_n f(w_n, x_{n+1}) - \lambda_n f(w_n, y_n) \geqslant \langle w_n - y_n, x_{n+1} - y_n \rangle. \tag{3.6}$$

By the expression (3.5) and (3.6) we get

$$\begin{aligned} \langle w_n - x_{n+1}, x_{n+1} - x^* \rangle \geqslant \langle w_n - y_n, x_{n+1} - y_n \rangle - k_1 \lambda_n \|w_n - y_n\|^2 \\ - k_2 \lambda_n \|y_n - x_{n+1}\|^2 + \gamma \lambda_n \|y_n - x^*\|^2. \end{aligned} \tag{3.7}$$

Furthermore, we have the following facts:

$$\begin{aligned} -2\langle w_n - x_{n+1}, x_{n+1} - x^* \rangle &= -\|w_n - x^*\|^2 + \|x_{n+1} - w_n\|^2 + \|x_{n+1} - x^*\|^2, \\ 2\langle y_n - w_n, y_n - x_{n+1} \rangle &= \|w_n - y_n\|^2 + \|x_{n+1} - y_n\|^2 - \|w_n - x_{n+1}\|^2. \end{aligned}$$

From above two facts and (3.7) we get the desired result.

$$\|x_{n+1} - x^*\|^2 \leqslant \|w_n - x^*\|^2 - (1 - 2k_1\lambda_n)\|w_n - y_n\|^2 - (1 - 2k_2\lambda_n)\|y_n - x_{n+1}\|^2 - 2\gamma\lambda_n\|y_n - x^*\|^2. \quad \square$$

Theorem 3.5. *The sequences $\{x_n\}$, $\{y_n\}$, and $\{w_n\}$ generated by Algorithm 3.1 converge strongly to $x^* \in EP(f, K)$, where $0 \leqslant \theta_n \leqslant \theta < \sqrt{5} - 2$.*

Proof. Due to $\lambda_n \to 0$ there is an $N_0 \in \mathbb{N}$ such that for each $n \geqslant N_0$, we have

$$0 < \lambda_n \leqslant \frac{\frac{1}{2} - 2\theta - \frac{1}{2}\theta^2 - \tau}{\max\{k_1, k_2\}(1-\theta)^2} \quad \text{for some} \quad 0 < \tau < \frac{1}{2} - 2\theta - \frac{1}{2}\theta^2.$$

Thus, Lemma 3.4 for $n \geqslant N_0$, provides that

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leqslant \|w_n - x^*\|^2 - (1 - \beta\lambda_n)\left[\|w_n - y_n\|^2 + \|x_{n+1} - y_n\|^2\right] \\ &\leqslant \|w_n - x^*\|^2 - \frac{(1 - \beta\lambda_n)}{2}\|x_{n+1} - w_n\|^2, \end{aligned} \tag{3.8}$$

where $\beta = \max\{2k_1, 2k_2\}$. By Lemma 2.4, we obtain

$$\|w_n - x^*\|^2 = (1 + \theta_n)\|x_n - x^*\|^2 - \theta_n\|x_{n-1} - x^*\|^2 + \theta_n(1 + \theta_n)\|x_n - x_{n-1}\|^2, \tag{3.9}$$

and

$$\|x_{n+1} - w_n\|^2 = \|x_{n+1} - x_n - \theta_n(x_n - x_{n-1})\|^2$$

$$
\begin{aligned}
&= \|x_{n+1} - x_n\|^2 + \theta_n^2 \|x_n - x_{n-1}\|^2 - 2\theta_n \langle x_{n+1} - x_n, x_n - x_{n-1}\rangle \qquad (3.10)\\
&\geqslant \|x_{n+1} - x_n\|^2 + \theta_n^2 \|x_n - x_{n-1}\|^2 - 2\theta_n \|x_{n+1} - x_n\|\|x_n - x_{n-1}\|\\
&\geqslant \|x_{n+1} - x_n\|^2 + \theta_n^2 \|x_n - x_{n-1}\|^2 - \theta_n \|x_{n+1} - x_n\|^2 - \theta_n \|x_n - x_{n-1}\|^2\\
&= (1-\theta_n)\|x_{n+1} - x_n\|^2 + (\theta_n^2 - \theta_n)\|x_n - x_{n-1}\|^2. \qquad (3.11)
\end{aligned}
$$

Combining the expressions (3.8), (3.9), and (3.11) we obtain

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leqslant (1+\theta_n)\|x_n - x^*\|^2 - \theta_n\|x_{n-1} - x^*\|^2 + \theta_n(1+\theta_n)\|x_n - x_{n-1}\|^2\\
&\quad - \rho_n(1-\theta_n)\|x_{n+1} - x_n\|^2 - \rho_n(\theta_n^2 - \theta_n)\|x_n - x_{n-1}\|^2 \qquad (3.12)\\
&= (1+\theta_n)\|x_n - x^*\|^2 - \theta_n\|x_{n-1} - x^*\|^2 - \rho_n(1-\theta_n)\|x_{n+1} - x_n\|^2\\
&\quad + \Big[\theta_n(1+\theta_n) - \rho_n(\theta_n^2 - \theta_n)\Big]\|x_n - x_{n-1}\|^2\\
&= (1+\theta_n)\|x_n - x^*\|^2 - \theta_n\|x_{n-1} - x^*\|^2 - q_n\|x_{n+1} - x_n\|^2 + r_n\|x_n - x_{n-1}\|^2, \qquad (3.13)
\end{aligned}
$$

where $\rho_n := \frac{(1-\beta\lambda_n)}{2} \geqslant 0$ and $q_n := \rho_n(1-\theta_n) \geqslant 0$ with

$$
r_n := \theta_n(1+\theta_n) - \rho_n(\theta_n^2 - \theta_n) = \theta_n(1+\theta_n) + \rho_n\theta_n(1-\theta_n) \geqslant 0, \text{ for all } n \geqslant N_0.
$$

Next, we assume that

$$
\Pi_n = \|x_n - x^*\|^2 - \theta_n\|x_{n-1} - x^*\|^2 + r_n\|x_n - x_{n-1}\|^2.
$$

By using the expression (3.13) we can evaluate the following for $n \geqslant N_0$, such that

$$
\begin{aligned}
\Pi_{n+1} - \Pi_n &= \|x_{n+1} - x^*\|^2 - \theta_{n+1}\|x_n - x^*\|^2 + r_{n+1}\|x_{n+1} - x_n\|^2\\
&\quad - \|x_n - x^*\|^2 + \theta_n\|x_{n-1} - x^*\|^2 - r_n\|x_n - x_{n-1}\|^2\\
&\leqslant \|x_{n+1} - x^*\|^2 - (1+\theta_n)\|x_n - x^*\|^2 + \theta_n\|x_{n-1} - x^*\|^2 \qquad (3.14)\\
&\quad + r_{n+1}\|x_{n+1} - x_n\|^2 - r_n\|x_n - x_{n-1}\|^2\\
&\leqslant -(q_n - r_{n+1})\|x_{n+1} - x_n\|^2
\end{aligned}
$$

and

$$
\begin{aligned}
q_n - r_{n+1} &= \rho_n(1-\theta_n) - \theta_{n+1}(1+\theta_{n+1}) + \rho_{n+1}(\theta_{n+1}^2 - \theta_{n+1})\\
&\geqslant \rho_{n+1}(1-\theta_{n+1}) - \theta_{n+1}(1+\theta_{n+1}) + \rho_{n+1}(\theta_{n+1}^2 - \theta_{n+1})\\
&\geqslant \rho_{n+1}(1-\theta)^2 - \theta - \theta^2 \qquad (3.15)\\
&= \Big(\frac{1}{2} - \frac{\beta}{2}\lambda_{n+1}\Big)(1-\theta)^2 - \theta - \theta^2\\
&= \Big(\frac{1}{2} - 2\theta - \frac{1}{2}\theta^2\Big) - \frac{\beta}{2}\lambda_{n+1}(1-\theta)^2 \geqslant \tau.
\end{aligned}
$$

From the expression (3.14) and (3.15) we obtain

$$
\Pi_{n+1} - \Pi_n \leqslant -\tau\|x_{n+1} - x_n\|^2 \leqslant 0. \qquad (3.16)
$$

Hence the sequence $\{\Pi_n\}$ is non-increasing for $n \geqslant N_0$. The definition of $\{\Pi_n\}$ for $n \geqslant N_0$, implies that

$$
\begin{aligned}
\|x_n - x^*\|^2 &\leqslant \Pi_n + \theta_n\|x_{n-1} - x^*\|^2\\
&\leqslant \Pi_{N_0} + \theta\|x_{n-1} - x^*\|^2\\
&\leqslant \cdots \leqslant \Pi_{n_0}(\theta^{n-N_0} + \cdots + 1) + \theta^{n-N_0}\|x_{N_0} - x^*\|^2
\end{aligned}
$$

$$\leqslant \frac{\Pi_{N_0}}{1-\theta} + \theta^{n-N_0}\|x_{N_0} - x^*\|^2. \tag{3.17}$$

By the definition of $\{\Pi_{n+1}\}$ for $n \geqslant N_0$, with expression (3.17) implies that

$$-\Pi_{n+1} \leqslant \theta_{n+1}\|x_n - x^*\|^2 \leqslant \theta\|x_n - x^*\|^2 \leqslant \theta\frac{\Pi_{N_0}}{1-\theta} + \theta^{n-N_0+1}\|x_{N_0} - x^*\|^2. \tag{3.18}$$

It follows from expressions (3.16) and (3.18) such that

$$\tau\sum_{n=N_0}^{k}\|x_{n+1} - x_n\|^2 \leqslant \Pi_{N_0} - \Pi_{k+1} \leqslant \Pi_{N_0} + \theta\frac{\Pi_{N_0}}{1-\theta} + \theta^{k-N_0+1}\|x_{N_0} - x^*\|^2 \leqslant \frac{\Pi_{N_0}}{1-\theta} + \|x_{N_0} - x^*\|^2, \tag{3.19}$$

letting $k \to \infty$ in expression (3.19) implies that

$$\sum_{n=1}^{\infty}\|x_{n+1} - x_n\|^2 < +\infty \Longrightarrow \lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.20}$$

Thus, the expression (3.10) with (3.20) implies that

$$\|x_{n+1} - w_n\| \to 0 \text{ as } n \to \infty. \tag{3.21}$$

By expression (3.12), (3.20) with Lemma 2.3, implies that

$$\lim_{n\to\infty}\|x_n - x^*\|^2 = l \text{ and } \lim_{n\to\infty}\|w_n - x^*\|^2 = l. \tag{3.22}$$

To show $\lim_{n\to\infty}\|y_n - x^*\|^2 = l$, next, we use Lemma 3.4, for $n \geqslant N_0$ with relations (3.21) and (3.22) such that

$$\begin{aligned}(1-2k_1\lambda_n)\|w_n - y_n\|^2 &\leqslant \|w_n - x^*\|^2 - \|x_{n+1} - x^*\|^2\\ &= (\|w_n - x^*\| + \|x_{n+1} - x^*\|)(\|w_n - x^*\| - \|x_{n+1} - x^*\|)\\ &\leqslant (\|w_n - x^*\| + \|x_{n+1} - x^*\|)\|x_{n+1} - w_n\| \longrightarrow 0, \text{ as } n \to \infty.\end{aligned}$$

This implies that, the sequences $\{x_n\}$, $\{w_n\}$, and $\{y_n\}$ are bounded for each $x^* \in EP(f, K)$, and the $\lim_{n\to\infty}\|x_n - x^*\|$ exists. We prove $\{x_n\}$ strongly converges to x^*. By Lemma 3.4, with (3.9) for $n \geqslant N_0$, we have

$$\begin{aligned}2\gamma\lambda_n\|y_n - x^*\|^2 &\leqslant -\|x_{n+1} - x^*\|^2 + (1+\theta_n)\|x_n - x^*\|^2 - \theta_n\|x_{n-1} - x^*\|^2 + \theta_n(1+\theta_n)\|x_n - x_{n-1}\|^2\\ &\leqslant (\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2) + \theta(1+\theta)\|x_n - x_{n-1}\|^2\\ &\quad + (\theta_n\|x_n - x^*\|^2 - \theta_{n-1}\|x_{n-1} - x^*\|^2).\end{aligned}$$

Now summing up above expression for $k > N_0$, we obtain

$$\begin{aligned}\sum_{n=N_0}^{k} 2\gamma\lambda_n\|y_n - x^*\|^2 &\leqslant (\|x_{N_0} - x^*\|^2 - \|x_{k+1} - x^*\|^2) + \theta(1+\theta)\sum_{n=N_0}^{k}\|x_n - x_{n-1}\|^2\\ &\quad + (\theta_k\|x_k - x^*\|^2 - \theta_{N_0-1}\|x_{N_0-1} - x^*\|^2)\\ &\leqslant \|x_{N_0} - x^*\|^2 + \theta\|x_k - x^*\|^2 + \theta(1+\theta)\sum_{n=N_0}^{k}\|x_n - x_{n-1}\|^2 = M,\end{aligned}$$

for $M \geqslant 0$. It gives that

$$\sum_n 2\gamma\lambda_n\|y_n - x^*\|^2 < +\infty.$$

The above expression with Lemma 2.5, implies that

$$\liminf_{n\to\infty} \|y_n - x^*\| = 0.$$

Thus, there is a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $\lim_{k\to\infty} \|y_{n_k} - x^*\| = 0$. Since $\lim_{n\to\infty} \|x_n - y_n\| = 0$, we obtain $\lim_{k\to\infty} \|x_{n_k} - x^*\| = 0$. We have $\lim_{n\to\infty} \|x_n - x^*\| = l$. Therefore, $\lim_{n\to\infty} \|x_n - x^*\| = 0$, i.e., $x_n \to x^*$, $y_n \to x^*$ and $w_n \to x^*$ as $n \to \infty$. □

Note: If we assume that the bifunction $f(x, y) := \langle G(x), y - x\rangle$ for all $x, y \in K$, then the equilibrium problem converts into the variational inequality problem with $L = 2k_1 = 2k_2$. We deduce the results for a strongly pseudomonotone and Lipschitz continuous operator.

Corollary 3.6. *Let $G : K \to \mathbb{E}$ is strongly pseudomonotone with L-Lipschitz continuous on K for some positive constant $L > 0$ and solution set $VI(G, K) \neq \emptyset$. Let $\{w_n\}$, $\{x_n\}$, $\{y_n\}$ are sequences generated as follows.*

(i.) *Given $x_{n-1}, x_n \in K$ with $w_n = x_n + \theta_n(x_n - x_{n-1})$ for each $n \geqslant 0$, and $\theta_n \in [0, \sqrt{5} - 2)$, compute*

$$\begin{cases} y_n = P_K\big(w_n - \lambda_n G(w_n)\big), \\ x_{n+1} = P_K\big(w_n - \lambda_n G(y_n)\big), \end{cases}$$

where stepsize sequence λ_n satisfies the conditions

$$(\Psi_1) : \lim_{n\to\infty} \lambda_n = 0 \text{ and } (\Psi_2) : \sum_{n=1}^{\infty} \lambda_n = +\infty.$$

Then, the sequences $\{w_n\}$, $\{x_n\}$, and $\{y_n\}$ strongly converge to x^* of $VI(G, K)$.

4. Computational experiment

We will show some numerical experiments to explain the efficiency of our proposed method. The MATLAB codes run in MATLAB version 9.5 (R2018b) on a PC Intel(R) Core(TM)i5-6200 CPU @ 2.30GHz 2.40GHz, RAM 8.00 GB. We use $x_{-1} = x_0 = y_0 = (1, 1, 1, 1, 1)^T$, and y-axes show D_n while the x-axis points out to the number of iterations or the time elapsed (in seconds).

4.1. Nash-Cournot oligopolistic equilibrium model

Consider that there will be n firms which generate the same commodity. Let x sets for a vector in which each item x_i holds for the volume of the commodity generated by a firm i. We take the cost P as a decreasing affine function that depends upon on the subject matter of $S = \sum_{i=1}^{m} x_i$, i.e., $P_i(S) = \phi_i - \psi_i S$, where $\phi_i > 0$, $\psi_i > 0$. The profit function for each firm i is defined by $F_i(x) = P_i(S)x_i - t_i(x_i)$, where $t_i(x_i)$ is the tax value and cost for developing x_i. Consider that $K_i = [x_i^{\min}, x_i^{\max}]$ is the set of operations connects to each firm i, and the strategy work out for the whole design take the form as $K := K_1 \times K_2 \times \cdots \times K_n$. In fact, each firm try to arrive at its peak revenue by adopting the respective stage of production on the assumption that the production of the other firms is an input parameter. A broadly utilized technique to the model is based on the popular Nash equilibrium concept. We would like to point out that point $x^* \in K = K_1 \times K_2 \times \cdots \times K_n$ is the point of equilibrium of the model if $F_i(x^*) \geqslant F_i(x^*[x_i])$, $\forall x_i \in K_i$, $\forall i = 1, 2, \cdots, n$, with the vector $x^*[x_i]$ represent the vector get from x^* by taking x_i^* with x_i. Certainly, we have $f(x, y) := \varphi(x, y) - \varphi(x, x)$ with $\varphi(x, y) := -\sum_{i=1}^{n} F_i(x[y_i])$, and the problem of finding the Nash equilibrium point of the model may be as follows:

$$\text{find } x^* \in K : f(x^*, y) \geqslant 0, \ \forall y \in K.$$

It follows from [27], that the bifunction f could be taken in the following form

$$f(x, y) = \langle Ax + By + c, y - x\rangle,$$

where $c \in \mathbb{R}^5$ and A, B are

$$A = \begin{pmatrix} 3.1 & 2 & 0 & 0 & 0 \\ 2 & 3.6 & 0 & 0 & 0 \\ 0 & 0 & 3.5 & 2 & 0 \\ 0 & 0 & 2 & 3.3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1.6 & 1 & 0 & 0 & 0 \\ 1 & 1.6 & 0 & 0 & 0 \\ 0 & 0 & 1.5 & 1 & 0 \\ 0 & 0 & 1 & 1.5 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}, \quad c = \begin{pmatrix} 1 \\ -2 \\ -1 \\ 2 \\ -1 \end{pmatrix}.$$

B is symmetric positive semidefinite and $B - A$ is symmetric negative definite with Lipschitz constants $k_1 = k_2 = \frac{1}{2}\|A - B\|$ (for more details see [27]). The feasible set $K \subset \mathbb{R}^5$ is closed and convex and writen as

$$K := \{x \in \mathbb{R}^5 : -5 \leqslant x_i \leqslant 5\}.$$

The numerical results regarding model 4.1 are shown in Figures 1-6 and Table 1.

Table 1: The experimental results for Figures 1-6.

			Hieu-Algo1 [15]		Hieu-Algo2 [16]		Rehman-Algo1 3.1	
n	TOL	λ_n	iter.	time	iter.	time	iter.	time
5	10^{-6}	$\frac{1}{n+1}$	222	2.5697	179	1.9488	102	0.9632
5	10^{-6}	$\frac{1}{\sqrt{n+1}}$	31	0.2880	29	0.2712	18	0.1919
5	10^{-6}	$\frac{1}{\log(n+2)}$	19	0.1909	24	0.2338	13	0.1279
5	10^{-6}	$\frac{1}{(n+1)\log(n+3)}$	319	4.1638	385	5.1718	123	1.1854
5	10^{-6}	$\frac{\log(n+3)}{n+1}$	139	1.2722	135	1.3614	77	0.7494
5	10^{-6}	$\frac{1}{\log\log(n+20)}$	28	0.2564	83	0.7783	18	0.1797

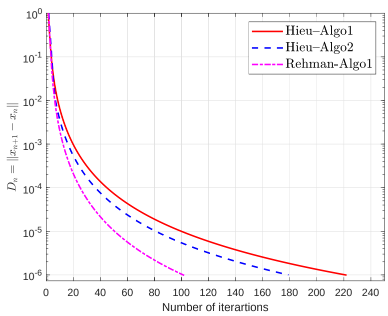

Figure 1: Equilibrium model 4.1 when $\lambda_n = \frac{1}{n+1}$.

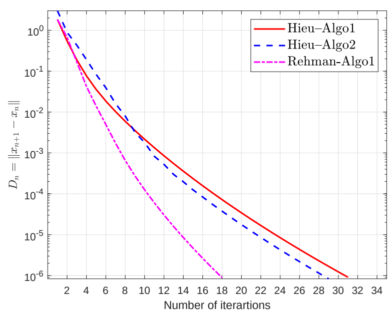

Figure 2: Equilibrium model 4.1 when $\lambda_n = \frac{1}{\sqrt{n+1}}$.

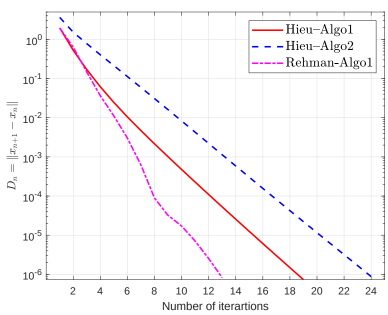

Figure 3: Equilibrium model 4.1 when $\lambda_n = \frac{1}{\log(n+2)}$.

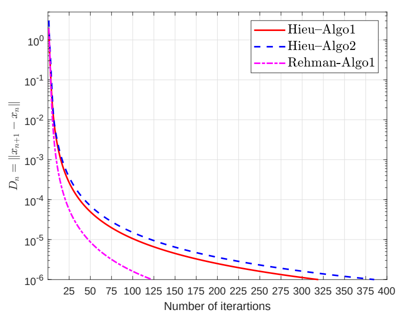

Figure 4: Equilibrium model 4.1 when $\lambda_n = \frac{1}{(n+1)\log(n+3)}$.

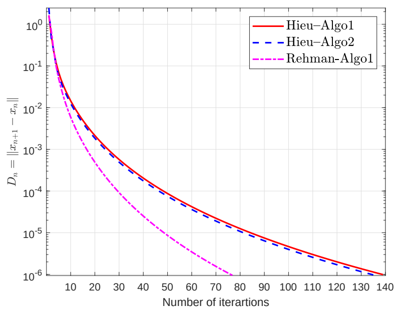

Figure 5: Equilibrium model 4.1 when $\lambda_n = \frac{\log(n+3)}{n+1}$.

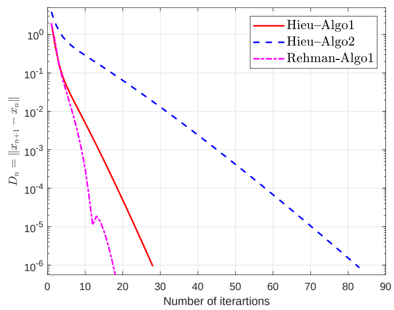

Figure 6: Equilibrium model 4.1 when $\lambda_n = \frac{1}{\log\log(n+20)}$.

5. Conclusion

In this study, we have established a new method by incorporating an inertial term with an extragradient method for solving a family of strongly pseudomonotone equilibrium problems. The prospective method requires a sequence of diminishing and non-summable stepsizes and the proposed method could be carried out without previous knowledge of the modulus of strong pseudomonotonicity and the Lipschitz-type constant of a cost bifunction. Two numerical experiments were presented to demonstrate the computational performance of the method in comparison to alternative existing methods. Such numerical results have confirmed that the method with inertial effects contributes to perform better than without inertial effects.

Acknowledgment

Habib ur Rehman has received support from "Petchra Pra Jom Klao Ph.D. Research Scholarship from King Mongkut's University of Technology Thonburi (Grant No.39/2560)". Nuttapol Pakkaranang thanks the "Royal Golden Jubilee Ph.D. (RGJ-PHD) Program (Grant No. PHD/0205/2561)". Nopparat Wairojjana was partially supported by Valaya Alongkorn Rajabhat University under the Royal Patronage, Thailand.

References

[1] J. Abubakar, P. Kumam, H. ur Rehman, A. H. Ibrahim, *Inertial iterative schemes with variable step sizes for variational inequality problem involving pseudomonotone operator*, Mathematics, **8** (2020), 25 pages. 1

[2] J. Abubakar, K. Sombut, H. ur Rehman, A. H. Ibrahim, *An accelerated subgradient extragradient algorithm for strongly pseudomonotone variational inequality problems*, Thai J. Math., **18** (2020), 166–187. 1

[3] M. Adeel, K. A. Khan, D. Pečarić, J. Pečarić, *Generalization of the Levinson inequality with applications to information theorey*, J. Inequal. Appl., **2019** (2019), 19 pages. 1

[4] F. Alvarez, H. Attouch, *An inertial proximal method for maximal monotone operators via discretization of a nonlinear oscillator with damping*, Set-Valued Anal., **9** (2001), 3–11. 2.3

[5] H. H. Bauschke, P. L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*, Springer, New York, (2017). 2.4

[6] A. Beck, M. Teboulle, *A fast iterative shrinkage-thresholding algorithm for linear inverse problems*, SIAM J. Imaging Sci., **2** (2009), 183–202. 1

[7] M. Bianchi, S. Schaible, *Generalized monotone bifunctions and equilibrium problems*, J. Optim. Theory Appl., **90** (1996), 31–43. 2.1

[8] E. Blum, *From optimization and variational inequalities to equilibrium problems*, Math. Student, **63** (1994), 123–145. 1, 2.1

[9] P. L. Combettes, S. A. Hirstoaga, *Equilibrium programming in hilbert spaces*, J. Nonlinear Convex Anal., **6** (2005), 117–136. 1

[10] S. Dafermos, *Traffic equilibrium and variational inequalities*, Transportation Sci., **14** (1980), 42–54. 1

[11] M. C. Ferris, J. S. Pang, *Engineering and economic applications of complementarity problems*, SIAM Rev., **39** (1997), 669–713. 1

[12] S. D. Flam, A. S. Antipin, *Equilibrium programming using proximal-like algorithms*, Math. Programming, **78** (1996), 29–41. 1

[13] F. Giannessi, A. Maugeri, P. M. Pardalos, *Equilibrium problems: nonsmooth optimization and variational inequality models*, Springer, New York, (2001). 1
[14] D. V. Hieu, *Parallel extragradient-proximal methods for split equilibrium problems*, Math. Model. Anal., **21** (2016), 478–501. 1
[15] D. V. Hieu, *New extragradient method for a class of equilibrium problems in Hilbert spaces*, Appl. Anal., **97** (2017), 811–824. 1, 1
[16] D. V. Hieu, *Convergence analysis of a new algorithm for strongly pseudomontone equilibrium problems*, Numer. Algorithms, **77** (2018), 983–1001. 1, 1
[17] A. N. Iusem, W. Sosa, *Iterative algorithms for equilibrium problems*, Optimization, **52** (2003), 301–316. 1
[18] I. V. Konnov, *Application of the proximal point method to nonmonotone equilibrium problems*, J. Optim. Theory Appl., **119** (2003), 317–333. 1
[19] I. Konnov, *Equilibrium models and variational inequalities*, Elsevier B. V., Amsterdam, (2007). 1
[20] A. Krylatov, V. Zakharov, T. Tuovinen, *Optimization Models and Methods for Equilibrium Traffic Assignment*, Springer, Cham, (2020). 1
[21] X. Li, A. Hussain, M. Adeel, E. Savas, *Fixed point theorems for Z_θ-contraction and applications to nonlinear integral equations*, IEEE Access, **7** (2019), 120023–120029. 1
[22] E. Ofoedu, *Strong convergence theorem for uniformly l-lipschitzian asymptotically pseudocontractive mapping in real Banach space*, J. Math. Anal. Appl., **321** (2006), 722–728. 2.5
[23] B.T. Polyak, *Some methods of speeding up the convergence of iteration methods*, U.S.S.R. Comput. Math. Math. Phys., **4** (1964), 1–17. 1
[24] T. D. Quoc, P. N. Anh, L. D. Muu, *Dual extragradient algorithms extended to equilibrium problems*, J. Global Optim., **52** (2011), 139–159. 1
[25] S. Takahashi, W. Takahashi, *Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces*, J. Math. Anal. Appl., **331** (2007), 506–515. 1
[26] J. V. Tiel, *Convex analysis: an introductory text*, Wiley, New York, (1984). 2.2
[27] D. Q. Tran, M. L. Dung, V. H. Nguyen, *Extragradient algorithms extended to equilibrium problems*, Optimization, **57** (2008), 749–776. 1, 4.1
[28] H. ur Rehman, D. Gopal, P. Kumam, *Generalizations of darbo's fixed point theorem for new condensing operators with application to a functional integral equation*, Demonstr. Math., **52** (2019), 166–182. 1
[29] H. ur Rehman, P. Kumam, A. B. Abubakar, Y. J. Cho, *The extragradient algorithm with inertial effects extended to equilibrium problems*, Comput. Appl. Math., **39** (2020), 26 pages. 1
[30] H. ur Rehman, P. Kumam, I. K. Argyros, N. A. Alreshidi, W. Kumam, W. Jirakitpuwapat, *A self-adaptive extragradient methods for a family of pseudomonotone equilibrium programming with application in different classes of variational inequality problems*, Symmetry, **12** (2020), 27 pages. 1
[31] H. ur Rehman, P. Kumam, I. K. Argyros, W. Deebani, W. Kumam, *Inertial extra-gradient method for solving a family of strongly pseudomonotone equilibrium problems in real hilbert spaces with application in variational inequality problem*, Symmetry, **12** (2020), 24 pages. 1
[32] H. ur Rehman, P. Kumam, Y. J. Cho, Y. I. Suleiman, W. Kumam, *Modified popov's explicit iterative algorithms for solving pseudomonotone equilibrium problems*, Opti. Methods Soft., **2020** (2020), 1–32. 1
[33] H. ur Rehman, P. Kumam, Y. J. Cho, P. Yordsorn, *Weak convergence of explicit extragradient algorithms for solving equilibirum problems*, J. Inequal. Appl., **1** (2019), 25 pages. 1
[34] H. ur Rehman, P. Kumam, S. Dhompongsa, *Existence of tripled fixed points and solution of functional integral equations through a measure of noncompactness*, Carpathian J. Math., **35** (2019), 193–208. 1
[35] H. ur Rehman, P. Kumam, W. Kumam, M. Shutaywi, W. Jirakitpuwapat, *The inertial sub-gradient extra-gradient method for a class of pseudo-monotone equilibrium problems*, Symmetry, **12** (2020), 25 pages. 1