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Non-solvability of Balakrishnan-Taylor system with memory term in $\mathbb{R}^N$

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Abstract

This study establishes a novel nonexistence result for a strongly coupled viscoelastic system with Balakrishnan-Taylor damping and a nonlinear source in the whole space. Sufficient conditions ensuring the nonexistence of solutions are established using the test function method.

Keywords: Balakrishnan-Taylor system, non-solvability, memory term.

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1. Introduction

The question of non-solvability of evolution equations has been treated and discussed from different angles using different methods and techniques. The basic idea in most of these works is to compare solutions with sub-solutions that blow-up in finite time. Our concern in this paper is a strongly coupled system with Balakrishnan-Taylor damping and a power-type source acting as an external force on the whole $\mathbb{R}^N$ space with $N \geqslant 1$. Although, we study the special case where the kernels g and h decay polynomially, the results of the study remain valid for a range of other kernel types such as exponentially decaying functions. We consider the system described by

$$\begin{cases} u_{tt} - M(t)\Delta u + \int_0^t g(t-s)\Delta u ds = |v|^p, \text{ in } (0,+\infty)\times\mathbb{R}^N, \\ v_{tt} - M(t)\Delta v + \int_0^t h(t-s)\Delta v ds = |u|^q, \text{ in } (0,+\infty)\times\mathbb{R}^N, \end{cases} \tag{1.1}$$

where $p, q > 1$. We assume continuous and bounded initial data

$$\begin{cases} u(0,x) = u_0(x), u_t(0,x) = u_1(x), \text{ in } \mathbb{R}^N, \\ v(0,x) = v_0(x), v_t(0,x) = v_1(x), \text{ in } \mathbb{R}^N, \end{cases}$$

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and

$$M(t) = \xi_0 + \xi_1 \|\nabla u(t)\|_2^2 + \xi_2 \|\nabla v(t)\|_2^2 + \xi_3 (\nabla u(t), \nabla u_t(t)) + \xi_4 (\nabla v(t), \nabla v_t(t)),$$

where ξ_i, $i = 0, 1, 2, 3, 4$, are positive constants. The relaxation functions g and h describe the properties of two different viscoelastic materials.

Model (1.1) was initially proposed by Balakrishnan and Taylor in 1989 [1] and further examined by Bass and Zes [2] within a bounded domain Ω of $\mathbb{R}^N$ for the case of a single equation with Balakrishnan-Taylor damping ($\xi_3 > 0$) and $g = 0$. This class of nonlinear models was proposed for energy conservations systems to account for the experimental damping effects observed in the SCOLE configuration at NASA. It is related to the panel flutter equation and to the spillover problem. So far, the one-dimensional model has been studied by You [12], Clark [3] and Tatar and Zaraï [9, 10, 13]. Several results on the exponential decay and blow up in finite time have been achieved.

For coupled wave systems with Balakrishnan-Taylor dampings defined in a bounded domain, Mu and Ma [7] proved that the decay rate of the solution energy is similar to that of relaxation functions which is not necessarily of exponential or polynomial type under suitable assumptions on relaxation functions and source terms. For more results concerning a wave equation with Balakrishnan-Taylor damping, one can refer to [8, 11, 15].

In this paper, we are interested in establishing sufficient conditions for the non-solvability of (1.1). In order to achieve our goal, we make use of the test function method developed by Mitidieri and Pohozaev [6]. We present a proof by contradiction involving apriori estimates of the weak solutions of (1.1) and carefull choices of a special test function and a scaling argument. The main goal of the study is to find a range of values for p and q for which we have nonexistence under minimal assumptions on g and h. The results obtained in this paper extend previous results by Zaraï and Tatar [14].

The remaining parts of this paper are arranged as follows. The next section sets the necessary notation and defines the concept of a (weak) solution to our problem. Section 3 contains the main result concerning the nonexistence of solutions. In Section 4, we present some necessary conditions for the local and global existence of solutions.

2. Preliminaries

We start our paper by making some necessary definitions. Throughout the paper, we shall denote by Q_T the set $Q_T := (0, T) \times \mathbb{R}^N$ and by Q the set $Q_\infty := (0, \infty) \times \mathbb{R}^N$. The following definition explains what is meant by a weak solution of (1.1).

Definition 2.1. The pair (u, v) is said to be a local weak solution of (1.1) on $(0, T)$ if $u \in L^q_{loc}(Q_T)$, $v \in L^q_{loc}(Q_T)$, and the equalities

$$\begin{aligned}\int_{Q_T} |v|^p \varphi \, dx dt + \int_{\mathbb{R}^N} u_1(x) \varphi(0, x) \, dx \\ = \int_{Q_T} u \varphi_{tt} dx dt - \int_{Q_T} M(t) u \Delta\varphi dx dt + \int_{Q_T} u(s, x) \left(\int_s^T g(t-s) \Delta\varphi(t) dt \right) ds dx,\end{aligned} \tag{2.1}$$

and

$$\begin{aligned}\int_{Q_T} |u|^q \varphi \, dx dt + \int_{\mathbb{R}^N} v_1(x) \varphi(0, x) \, dx \\ = \int_{Q_T} v \varphi_{tt} dx dt - \int_{Q_T} M(t) v \Delta\varphi dx dt + \int_{Q_T} v(s, x) \left(\int_s^T h(t-s) \Delta\varphi(t) dt \right) ds dx,\end{aligned} \tag{2.2}$$

hold for any $\varphi \in C_0^2(Q_T)$ satisfying $\varphi \geqslant 0$ and

$$\varphi(T, x) = \varphi_t(T, x) = \varphi_t(0, x) = 0.$$

Note that by $\varphi \in C_0^2(Q_T)$ we refer to φ being a function in $C_{t,x}^{2,2}$ with compact support. We are now ready to state the hypothesis.

Hypothesis 2.2. Let g, h be some bounded C^1-functions from $\mathbb{R}^+$ to $\mathbb{R}^+$ satisfying

$$g(t), h(t) \leqslant \frac{K}{(1+t)^{\rho}}, \tag{2.3}$$

for $t \geqslant 0$ and the constants $K > 0$ and $\rho \in (2, \infty)$.

3. Nonexistence result

In this section, we extend the results of Zaraï and Tatar [14] regarding the nonexistence of solutions for the coupled system (1.1). We establish a range of values for parameters p and q over which no weak solutions can exist globally in time. The following theorem presents our result.

Theorem 3.1. *Suppose that*

$$\int_{\mathbb{R}^N} u_1(x)\,dx > 0, \quad \int_{\mathbb{R}^N} v_1(x)\,dx > 0,$$

and (2.3) *holds. Assume that* $N \geqslant 1$ *and*

$$1 < p, q < 1 + \min\left\{\frac{1}{N+\theta-1}, \frac{2\theta}{N-\theta}\right\}. \tag{3.1}$$

Then, Problem (1.1) *does not admit global nontrivial weak solutions in time.*

Proof. We aim to prove Theorem 3.1 by contradiction. Assume that a weak solution of (1.1) exists globally in time. We introduce the test functions

$$\varphi_i(t,x) := \phi\left(\frac{|x|}{R}\right)\mu\left(\frac{t}{R^{\theta_i}}\right), \ i = 1, 2,$$

with $\phi \in C_0^\infty(\mathbb{R}^N)$, $\phi \geqslant 0$, $\mu \in C^2(\mathbb{R}^+)$, and $\mu \geqslant 0$ such that

$$\phi(w), \mu(w) = \begin{cases} 1, & |w| \leqslant 1, \\ 0, & |w| > 2, \end{cases}$$

and μ satisfies $-C \leqslant \mu'(t) \leqslant 0$, $\mu'(2R^{\theta_i}) = 0$ for $R \gg 1$. The functions $\varphi_i(t,x)$ are supposed to have bounded second order partial derivatives. Moreover, we assume without loss of generality that

$$\int_{U_1} |\varphi_{1tt}|^{q'} (\varphi_2)^{1-q'} dxdt + \int_{U_2} M(t) |\Delta\varphi_1|^{q'} (\varphi_2)^{1-q'} dxdt < \infty, \tag{3.2}$$

and

$$\int_{V_1} |\varphi_{2tt}|^{p'} (\varphi_1)^{1-p'} dxdt + \int_{V_2} M(t) |\Delta\varphi_2|^{p'} (\varphi_1)^{1-p'} dxdt < \infty, \tag{3.3}$$

where $U_1 := \operatorname{supp}\varphi_{1tt} \cap \operatorname{supp}\varphi_2$, $U_2 := \operatorname{supp}\Delta\varphi_1 \cap \operatorname{supp}\varphi_2$, $V_1 := \operatorname{supp}\varphi_{2tt} \cap \operatorname{supp}\varphi_1$ and $V_2 := \operatorname{supp}\Delta\varphi_2 \cap \operatorname{supp}\varphi_1$. We denote by p' and q', respectively, the conjugate exponents of p and q. If these conditions are not satisfied for our functions $\varphi_i(t,x)$, $i = 1, 2$, then we pick $\varphi_i^\lambda(t,x)$, $i = 1, 2$, with some sufficiently large $\lambda > 0$.

Next, we estimate the different terms on the right hand side of (2.1) and (2.2) in terms of the expressions in the left hand sides. By multiplying and dividing by $\varphi_2^{1/p}$ and then applying Hölder's inequality, we see that

$$\begin{aligned} \int_{U_1} u\varphi_{1tt}\,dtdx &\leqslant \int_{U_1} u\varphi_2^{1/q}\varphi_2^{-1/q}\varphi_{1tt}\,dtdx \\ &\leqslant \left(\int_{U_1} |u|^q \varphi_2\,dtdx\right)^{1/q} \left(\int_{U_1} \varphi_2^{-q'/q} |\varphi_{1tt}|^{q'}\,dtdx\right)^{1/q'}. \end{aligned} \tag{3.4}$$

Similarly, we have

$$-\int_{U_2} M(t)u\Delta\varphi_1 dxdt \leqslant \left(\int_{U_2} |u|^q \varphi_2 dxdt\right)^{1/q} \left(\int_{U_2} |M(t)|^{q'} \varphi_2^{-q'/q} |\Delta\varphi_1|^{q'} dxdt\right)^{1/q'}, \tag{3.5}$$

and

$$\begin{aligned}\int_{U_2} u \left(\int_s^{+\infty} g(t-s)\Delta\varphi_1(t)dt\right) dsdx \leqslant & \left(\int_{U_2} |u|^q \varphi_2 dxdt\right)^{1/q} \\ & \times \left(\int_{U_2} \varphi_2^{-q'/q} \left|\int_s^{+\infty} g(t-s)\Delta\varphi_1(t)dt\right|^{q'} dsdx\right)^{1/q'}.\end{aligned} \tag{3.6}$$

Using the three estimates (3.4), (3.5), (3.6) with (2.1) we obtain

$$\int_W |v|^p \varphi_1 dxdt + \int_{\mathbb{R}^N} u_1(x)\varphi_1(0,x)\, dx \leqslant A\left(\int_{U_1\cup U_2} |u|^q \varphi_2 dxdt\right)^{1/q}, \tag{3.7}$$

where

$$\begin{aligned}A = \Bigg\{ &\left(\int_{U_1} \varphi_2^{-q'/q} |\varphi_{1tt}|^{q'} dtdx\right)^{1/q'} + \left(\int_{U_2} |M(t)|^{q'} \varphi_2^{-q'/q} |\Delta\varphi_1|^{q'} dxdt\right)^{1/q'} \\ &+ \left(\int_{U_2} \varphi_2^{-q'/q} \left|\int_s^{+\infty} g(t-s)\Delta\varphi_1(t)dt\right|^{q'} dsdx\right)^{1/q'} \Bigg\},\end{aligned}$$

and $W := \operatorname{supp}\varphi_1 \cap \operatorname{supp}\varphi_2$.

Likewise, it is easy to see that

$$\int_W |u|^q \varphi_2 dxdt + \int_{\mathbf{R}^N} v_1(x)\varphi_2(0,x)\, dx \leqslant B\left(\int_{V_1\cup V_2} |v|^p \varphi_1 dxdt\right)^{1/p}, \tag{3.8}$$

where

$$\begin{aligned}B = \Bigg\{ &\left(\int_{V_1} \varphi_1^{-p'/p} |\varphi_{2tt}|^{p'} dtdx\right)^{1/p'} + \left(\int_{V_2} |M(t)|^{p'} \varphi_1^{-p'/p} |\Delta\varphi_2|^{p'} dxdt\right)^{1/p'} \\ &+ \left(\int_{V_2} \varphi_1^{-p'/p} \left|\int_s^{+\infty} h(t-s)\Delta\varphi_2(t)dt\right|^{p'} dsdx\right)^{1/p'} \Bigg\}.\end{aligned}$$

The combinition of (3.7) and (3.8) yields

$$\begin{aligned}\int_{V_1\cup V_2} |v|^p \varphi_1 dxdt \leqslant & -\int_{\mathbb{R}^N} u_1(x)\varphi_1(0,x)\, dx \\ & \times A\left[B\left(\int_{V_1\cup V_2} |v|^p \varphi_1 dxdt\right)^{1/p} - \int_{\mathbb{R}^N} v_1(x)\varphi_2(0,x)\, dx\right]^{1/q},\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}\int_{U_1\cup U_2} |u|^q \varphi_2 dxdt \leqslant & -\int_{\mathbb{R}^N} v_1(x)\varphi_2(0,x)\, dx \\ & \times B\left[A\left(\int_{U_1\cup U_2} |u|^q \varphi_2 dxdt\right)^{1/q} - \int_{\mathbb{R}^N} u_1(x)\varphi_1(0,x)\, dx\right]^{1/p}.\end{aligned} \tag{3.10}$$

From our assumptions on the initial data we deduce that

$$\int_{\mathbb{R}^N} u_1(x)\,\varphi_1(0,x)\,dx \geqslant 0 \quad \text{and} \quad \int_{\mathbb{R}^N} v_1(x)\,\varphi_2(0,x)\,dx \geqslant 0.$$

Hence, relations (3.9) and (3.10) imply that

$$\left(\int_{V_1\cup V_2} |v|^p\,\varphi_1 dxdt\right)^{1-1/pq} \leqslant AB^{1/q} \quad \text{and} \quad \left(\int_{U_1\cup U_2} |u|^q\,\varphi_2 dxdt\right)^{1-1/pq} \leqslant BA^{1/p}.$$

Next, we used the scaled variables $t = R^\theta \tau$ and $x = Ry$ where $\theta = \theta_1 = \theta_2$ to obtain

$$\left(\int_{U_1} \varphi_2^{-q'/q}\,|\varphi_{1tt}|^{q'}\,dtdx\right)^{1/q'} = R^{\frac{N+\theta}{q'}-2\theta}\left(\int_{U_1} \varphi_2^{-q'/q}\,|\varphi_{1\tau\tau}|^{q'}\,d\tau dy\right)^{1/q'}, \tag{3.11}$$

and

$$\left(\int_{U_2} |M(t)|^{q'}\,\varphi_2^{-q'/q}\,|\Delta\varphi_1|^{q'}\,dxdt\right)^{1/q'} \leqslant R^{\frac{N+\theta}{q'}-1}\left(\int_{U_2} \left|\tilde{M}(\tau)\right|^{q'}\,\varphi_2^{-q'/q}\,|\Delta\varphi_1|^{q'}\,d\tau dy\right)^{1/q'}, \tag{3.12}$$

where

$$\tilde{M}(\tau) = \xi_0 + \xi_1\int_{\mathbb{R}^N} |\nabla u|^2\,dy + \xi_2\int_{\mathbb{R}^N} |\nabla v|^2\,dy + \frac{d}{2d\tau}(\xi_3\int_{\mathbb{R}^N} |\nabla u|^2\,dy + \xi_4\int_{\mathbb{R}^N} |\nabla v|^2\,dy).$$

We may rewrite the term containing the memory in the form

$$\int_{U_2} \varphi_2^{-q'/q}\left|\int_t^{+\infty} g(\nu-t)\Delta\varphi_1(\nu)d\nu\right|^{q'}\,dtdx,$$

and use the scaling to get

$$\begin{aligned}\int_{U_2} (\varphi_2)^{-q'/q}\left|\int_t^{+\infty} g(\nu-t)\Delta\varphi(\nu)d\nu\right|^{q'}\,dtdx &= \int_{D_R} |\Delta\phi|^{q'}\,\phi^{-q'/q}\int_0^{2R} (\mu)^{-q'/q}\left|\int_t^{+\infty} g(\nu-t)\mu(\nu)d\nu\right|^{q'}\,dtdx\\ &\leqslant CR^{N+\theta-2q'}\int_\Omega |\Delta\phi|^{q'}\,\varphi^{-\frac{q'}{q}}\left|\int_{R^\theta\tau}^{+\infty} g(\nu-R^\theta\tau)\mu(\nu)d\nu\right|^{q'}\,d\tau dy,\end{aligned}$$

where $\Omega := \{(\tau,y) : 1 \leqslant \tau, |y| \leqslant 2\}$ and $D_R := \{x \in \mathbb{R}^N : R < |x| < 2R\}$.

In light of Hypothesis 2.2 and by using the change of variable $1 + \nu - R^\theta\tau = \eta$ and the fact that μ is non increasing we see that

$$\int_{R^\theta\tau}^{+\infty} g(\nu - R^\theta\tau)\mu(\nu)d\nu \leqslant K\int_1^{+\infty} \frac{\mu(\eta + R^\theta\tau - 1)}{\eta^\rho}d\eta.$$

As $R^\theta\tau \geqslant 1$, $\mu(\eta) = 0$ for $\eta \geqslant 2$, and $\mu(\eta) \leqslant 1$, we have

$$\int_{R^\theta\tau}^{+\infty} g(\nu - R^\theta\tau)\mu(\nu)d\nu \leqslant K\int_1^2 \frac{1}{\eta^\rho}d\eta \leqslant C,$$

and, therefore,

$$\left(\int_{U_2} \varphi_2^{-q'/q}\left|\int_s^{+\infty} g(t-s)\Delta\varphi_1(t)dt\right|^{q'}\,dsdx\right)^{1/q'} \leqslant CR^{\frac{N+\theta}{q'}-2}\left(\int_{U_2} \varphi_2^{-q'/q}\,|\Delta\varphi_1|^{q'}\,d\tau dy\right)^{1/q'}. \tag{3.13}$$

By virtue of (3.12), (3.13), (3.2), and (3.3), we find that

$$A \leqslant C\left(R^{\frac{N+\theta}{q'}-2\theta}+R^{\frac{N+\theta}{q'}-1}+R^{\frac{N+\theta}{q'}-2}\right) \quad \text{and} \quad B \leqslant C\left(R^{\frac{N+\theta}{p'}-2\theta}+R^{\frac{N+\theta}{p'}-1}+R^{\frac{N+\theta}{p'}-2}\right).$$

Relations (3.8) and (3.11) imply that for a sufficiently large R, we have

$$\int_W |u|^q \varphi_2 dxdt \leqslant B\left(\int_{V_1\cup V_2} |v|^p \varphi_1 dxdt\right)^{1/p} \leqslant C\left(AB^p\right)^{\frac{q}{pq-1}}. \tag{3.14}$$

By imposing condition (3.1) and passing to the limit as $R \to \infty$ in (3.14), we obtain

$$\lim \int_W |u|^q \varphi_2 dxdt \leqslant 0.$$

This contradicts our assumption that u is a nontrivial solution. Likewise, using the other estimations, we will reach $v = 0$, which is again a contradiction. This completes the proof. □

4. Necessary conditions for local and global solutions

Now that we have established a range of p and q values that guarantee the nonexistence of solutions to the proposed model (1.1), we move to examine the conditions of the existence of solutions both locally and globally. The following theorem and corrollary convey our results.

Theorem 4.1. *Let the pair (u, v) be a local solution to (1.1) where $T < +\infty$ and $p, q > 1$. Then, there exist constants α and β such that*

$$\lim_{|x|\to\infty} \inf\left(u_1(x)+v_1(x)\right) \leqslant C_{1/4}T^{1-2p'}\left(\alpha T^{2(p'-q')}+\beta\right).$$

Proof. By the definition of a weak solution, for any $\varphi \in C_0^\infty(Q_T) \geqslant 0$, we have

$$\begin{aligned}&\int_{Q_T} |v|^p \varphi dxdt + \int_{\mathbf{R}^N} u_1(x)\varphi(0,x)\,dx \\ &\quad \leqslant \int_{Q_T} |u|\,|\varphi_{tt}|\,dxdt \int_{Q_T} |M(t)|\,|u|\,|\Delta\varphi|\,dxdt + \int_{Q_T} |u(s,x)| \left|\int_s^T g(t-s)\Delta\varphi(t)dt\right| dsdx,\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}&\int_{Q_T} |u|^q \varphi dxdt + \int_{\mathbf{R}^N} v_1(x)\varphi(0,x)\,dx \\ &\quad \leqslant \int_{Q_T} |u|\,|\varphi_{tt}|\,dxdt + \int_{Q_T} |M(t)|\,|u|\,|\Delta\varphi|\,dxdt + \int_{Q_T} |u(s,x)| \left|\int_s^T h(t-s)\Delta\varphi(t)dt\right| dsdx.\end{aligned}$$

Using the ε-Young inequality we can estimate all the terms on the right hand side of (4.1). In fact by writing $|u|\,|\varphi_{tt}| = |u|\,\varphi^{1/q}\varphi^{-1/q}\,|\varphi_{tt}|$, we find that for $\varepsilon > 0$,

$$\int_{Q_T} |u|\,|\varphi_{tt}|\,dtdx \leqslant \varepsilon \int_{Q_T} |u|^q \varphi dtdx + C_\varepsilon \int_{Q_T} \varphi^{-q'/q}\,|\varphi_{tt}|^{q'}\,dtdx, \tag{4.2}$$

and similarly

$$\int_{Q_T} |M(t)|\,|u|\,|\Delta\varphi|\,dtdx \leqslant \varepsilon \int_{Q_T} |u|^q \varphi dxdt + C_\varepsilon \int_{Q_T} |M(t)|^{q'} \varphi^{-q'/q}\,|\Delta\varphi|^{q'}\,dxdt, \tag{4.3}$$

and

$$\int_{Q_T} |u(s,x)| \left| \int_s^T g(t-s)\Delta\varphi(t)dt \right| dsdx \leqslant \varepsilon \int_{Q_T} |u|^q \varphi dsdx + C_\varepsilon \int_{Q_T} \varphi^{-q'/q} \left| \int_s^T g(t-s)\Delta\varphi(t)dt \right|^{q'} dsdx. \tag{4.4}$$

Using (4.1) along with (4.2), (4.3), and (4.4), we can obtain

$$\begin{aligned} &\int_{Q_T} |v|^p \varphi dxdt + \int_{\mathbb{R}^N} u_1(x)\varphi(0,x)\, dx \\ &\leqslant \varepsilon \int_{Q_T} |u|^q \varphi dtdx + C_\varepsilon \int_{Q_T} \left(|\varphi_{tt}|^{q'} + |M(t)|^{q'} |\Delta\varphi|^{q'} + \left| \int_s^T g(t-s)\Delta\varphi(t)dt \right|^{q'} \right) \varphi^{-q'/q}. \end{aligned} \tag{4.5}$$

Following the same steps, we also have

$$\begin{aligned} &\int_{Q_T} |u|^q \varphi dxdt + \int_{\mathbb{R}^N} v_1(x)\varphi(0,x)\, dx \\ &\leqslant \varepsilon \int_{Q_T} |v|^p \varphi dtdx + C_\varepsilon \int_{Q_T} \left(|\varphi_{tt}|^{p'} + |M(t)|^{p'} |\Delta\varphi|^{p'} + \left| \int_s^T h(t-s)\Delta\varphi(t)dt \right|^{p'} \right) \varphi^{-p'/p}. \end{aligned} \tag{4.6}$$

By selecting $\varepsilon \leqslant 1/4$ and in light of (4.5) and (4.6), we deduce that

$$\begin{aligned} &\int_{\mathbb{R}^N} (u_1(x) + v_1(x))\, \varphi(0,x)\, dx \\ &\leqslant C_\varepsilon \int_{Q_T} \left(|\varphi_{tt}|^{q'} + |M(t)|^{q'} |\Delta\varphi|^{q'} + \left| \int_s^T g(t-s)\Delta\varphi(t)dt \right|^{q'} \right) \varphi^{-q'/q} \\ &\quad + C_\varepsilon \int_{Q_T} \left(|\varphi_{tt}|^{p'} + |M(t)|^{p'} |\Delta\varphi|^{p'} + \left| \int_s^T h(t-s)\Delta\varphi(t)dt \right|^{p'} \right) \varphi^{-p'/p}. \end{aligned} \tag{4.7}$$

We choose the test function

$$\varphi(t,x) := \phi\left(\frac{|x|}{R}\right) \mu\left(\frac{t}{T}\right),$$

where $\phi \in C_0^\infty(\mathbb{R}^N)$, $\phi \geqslant 0$, $\operatorname{supp}\phi \subset \left\{x \in \mathbb{R}^N : 1 < |x| < 2\right\}$, $|\Delta\phi| \leqslant k\phi$, and

$$\mu\left(\frac{t}{T}\right) := \begin{cases} 1, & 0 \leqslant t \leqslant T/2, \\ 1 - \frac{(t-T/2)^3}{(T/2)^3}, & T/2 \leqslant t \leqslant T, \\ 0, & t \geqslant T. \end{cases}$$

Next, we estimate the six terms on the right hand side of (4.7). By making the change of variable $t = \tau T$ and making use of the assumptions on φ, we obtain the following inequalities

$$\int_{Q_T} \varphi^{-q'/q} |\varphi_{tt}|^{q'} \leqslant \alpha T^{1-2q'} \int_{\mathbb{R}^N} \phi,$$

$$\int_{Q_T} |M(t)|^{q'} \varphi^{-q'/q} |\Delta\varphi|^{q'} \leqslant T\tilde{M}(T)^{q'} k^{q'} R^{-2q'} \int_{\mathbb{R}^N} \phi,$$

$$\int_{Q_T} \varphi^{-q'/q} \left| \int_s^T g(t-s)\Delta\varphi(t)dt \right|^{q'} \leqslant C k^{q'} R^{-2q'} T^2 \left(\int_0^\infty g(t)dt \right)^{q'} \int_{\mathbb{R}^N} \phi,$$

$$\int_{Q_T} \varphi^{-p'/p} |\varphi_{tt}|^{p'} \leqslant \beta T^{1-2p'} \int_{\mathbb{R}^N} \phi,$$

$$\int_{Q_T} |M(t)|^{p'} \varphi^{-p'/p} |\Delta\varphi|^{p'} \leqslant T\tilde{M}(T)^{p'} k^{p'} R^{-2p'} \int_{\mathbb{R}^N} \phi,$$

$$\int_{Q_T} \varphi^{-p'/p} \left| \int_s^T h(t-s)\Delta\varphi(t)dt \right|^{p'} \leqslant Ck^{p'} R^{-2p'} T^2 \left(\int_0^\infty h(t)dt \right)^{p'} \int_{\mathbb{R}^N} \phi.$$

Substituting these estimates leads to

$$\inf_{|x|>R} (u_1(x)+v_1(x)) \int_{\mathbb{R}^N} \phi \leqslant C_{1/4} \left[\alpha T^{1-2q'} + T\tilde{M}(T)^{q'} k^{q'} R^{-2q'} + Ck^{q'} R^{-2q'} T^2 \right] \int_{\mathbb{R}^N} \phi$$
$$+ C_{1/4} \left[\beta T^{1-2p'} + T\tilde{M}(T)^{p'} k^{p'} R^{-2p'} + Ck^{p'} R^{-2p'} T^2 \right] \int_{\mathbb{R}^N} \phi.$$

By letting $R \to +\infty$, we obtain

$$\liminf_{|x|\to\infty} (u_1(x)+v_1(x)) \leqslant C_{1/4} \left[\alpha T^{1-2q'} + \beta T^{1-2p'} \right]. \tag{4.8}$$

Hence the theorem is proved. □

We can immediately deduce the following result.

Corollary 4.2. *Suppose that* $p, q > 1$ *and* $u_1(x)+v_1(x) \geqslant 0$. *If* (1.1) *admits a global weak solution, then*

$$\liminf_{|x|\to\infty} (u_1(x)+v_1(x)) = 0.$$

Proof. Suppose that (1.1) has a global weak solution and

$$S := \liminf_{|x|\to\infty} (u_1(x)+v_1(x)) > 0.$$

Then, from (4.8), it appears that

$$T \leqslant \max\left\{ \left(\frac{\alpha+\beta}{S} C_{1/4} \right)^{1/(p'-1)}, \left(\frac{\alpha+\beta}{S} C_{1/4} \right)^{1/(q'-1)} \right\},$$

which is a contradiction. □

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