J. Math. Computer Sci., 22 (2021), 128–130
Online: ISSN 2008-949X
Journal of Mathematics and Computer Science
Journal Homepage: www.isr-publications.com/jmcs

Some remarks concerning D^*-metric spaces

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Abstract

In [S. Sedghi, N. Shobe, H. Y. Zhou, Fixed Point Theory Appl., **2007** (2007), 13 pages], Sedghi et al. introduced the notion of D^*-metric space and in [S. Sedghi, N. Shobe, A. Aliouche, Mat. Vesnik, **64** (2012), 258–266] the authors claimed that every G-metric space is D^*-metric. In this short paper we present examples to show that D^*-metric need not be G-metric as well as the G-metric need not be D^*-metric.

Keywords: Metric space, D^*-metric space, G-metric space.

2020 MSC: 47H10, 46B20.

1. Introduction and preliminaries

In 2005, Zead and Sims [5] introduced the notion of G-metric spaces as a generalization of the concept of ordinary metric spaces as follows.

Definition 1.1 ([5])**.** A G-metric space is a pair (A, G), where A is a nonempty set, and $G : A \times A \times A \to [0, \infty)$ such that for all $\kappa, \lambda, \varpi, \zeta \in A$ we have

(G1) $G(\kappa, \lambda, \varpi) = 0$, if $\kappa = \lambda = \varpi$;
(G2) $0 < G(\kappa, \kappa, \lambda)$, for all $\kappa, \lambda \in A$, with $\kappa \neq \lambda$;
(G3) $G(\kappa, \kappa, \lambda) \leqslant G(\kappa, \lambda, \varpi)$, for all $\kappa, \lambda, \varpi \in A$, with $\varpi \neq \lambda$;
(G4) $G(\kappa, \lambda, \varpi) = G(\kappa, \varpi, \lambda) = G(\lambda, \varpi, \kappa) = \cdots$, (symmetry in all three variables); and
(G5) $G(\kappa, \lambda, \varpi) \leqslant G(\kappa, \zeta, \zeta) + G(\zeta, \lambda, \varpi)$, for all $\kappa, \lambda, \varpi, \zeta \in A$, (rectangle inequality).

The function G is called a G-metric on A.

Many authors obtained fixed point results for different contractive mappings in the frame work of G-metric space, for more details we refer the reader to [1–5].

In 2007, Sedghi et al. introduced the concept of D^*-metric space as follows.

Definition 1.2 ([8])**.** A D^*-metric space is a pair (A, D^*) where A is a nonempty set, and $D^* : A \times A \times A \to [0, \infty)$ such that for all $\kappa, \lambda, \varpi, \zeta \in A$ we have

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doi: 10.22436/jmcs.022.02.04

Received: 2020-05-18 *Revised:* 2020-06-03 *Accepted:* 2020-06-15

(D^*1) $D^*(\kappa,\lambda,\varpi) \geqslant 0$;

(D^*2) $D^*(\kappa,\lambda,\varpi) = 0$, iff $\kappa = \lambda = \varpi$;

(D^*3) $D^*(\kappa,\lambda,\varpi) = D^*(\kappa,\varpi,\lambda) = D^*(\lambda,\varpi,\kappa) = \cdots$, (symmetry in all three variables); and

(D^*4) $D^*(\kappa,\lambda,\varpi) \leqslant D^*(\kappa,\lambda,\zeta) + D^*(\zeta,\varpi,\varpi)$, for all $\kappa,\lambda,\varpi,\zeta \in A$.

The function D^* is called a D^*-metric on A.

Many authors obtained fixed point results under some contractive conditions, see [6, 8]. Note that every D^*-metric on A defines a metric d_{D^*} on A by

$$d_{D^*}(\kappa,\lambda) = D^*(\kappa,\lambda,\lambda), \quad \forall \kappa,\lambda \in A.$$

Lemma 1.3 ([8]). *Let (A, D^*) be a D^*-metric space. Then D^* is symmetric, i.e., $D^*(\kappa,\lambda,\lambda) = D^*(\kappa,\kappa,\lambda)$.*

Lemma 1.4 ([8]). *Let A be a D^*-metric space, then the function $D^*(\kappa,\lambda,\varpi)$ is jointly continuous on $A \times A \times A$.*

2. Main results

In [7, Remark 1.3], Sedghi et al. claimed that "every G-metric space is D^*-metric". The following example shows that this claim need not be true in general.

Example 2.1 ([3]). Let $A = \mathbf{N}$, be the set of all natural numbers, and define $G : A \times A \times A \to \mathbf{R}$ such that for all $\kappa,\lambda,\varpi \in A$:

- $G(\kappa,\lambda,\varpi) = 0$, if $\kappa = \lambda = \varpi$;
- $G(\kappa,\lambda,\lambda) = \kappa + \lambda$, if $\kappa < \lambda$;
- $G(\kappa,\lambda,\lambda) = \kappa + \lambda + \frac{1}{2}$, if $\kappa > \lambda$;
- $G(\kappa,\lambda,\varpi) = \kappa + \lambda + \varpi$, if $\kappa \neq \lambda \neq \varpi$ and symmetry in all three variables.

Then, (A, G) is a G-metric space. But if $\kappa < \lambda$, we have $G(\kappa,\lambda,\lambda) = \kappa + \lambda \neq \kappa + \lambda + \frac{1}{2} = G(\lambda,\kappa,\kappa)$ that is G is not symmetric, also triangle inequality of D^*-metric does not satisfy, in fact

$$G(2,2,3) = \frac{11}{2} \nleq 5 = G(2,2,2) + G(2,3,3).$$

So, it is not D^*-metric.

In fact, every non-symmetric G-metric space is not D^*-metric. Now we present an example shows that D^*-metric need not to be G-metric.

Example 2.2. Let $A = \mathbf{R}$ be the set of all real numbers and define $D^* : A \times A \times A \to \mathbf{R}$ such that for all $\kappa,\lambda,\varpi \in A$, $D^*(\kappa,\lambda,\varpi) = |\kappa + \lambda - 2\varpi| + |\lambda + \varpi - 2\kappa| + |\varpi + \kappa - 2\lambda|$. Then (A, D^*) is a D^*-metric space [8]. But D^* is not G-metric since (G3) is not satisfied. In fact, $D^*(5,5,10) = 20 \nleq 18 = D^*(5,10,9)$.

However, the following is an example of both G-metric and D^*-metric.

Example 2.3. Let $A = \mathbf{R}$, be the set of all real numbers, and define $G : A \times A \times A \to [0,\infty)$ such that for all $\kappa,\lambda,\varpi \in A$:

- $G(\kappa,\lambda,\varpi) = 0$, if $\kappa = \lambda = \varpi$;
- $G(\kappa,\lambda,\lambda) = G(\kappa,\kappa,\lambda) = |\kappa| + |\lambda|$, if $\kappa \neq \lambda$;

- $G(\kappa,\lambda,\varpi) = |\kappa| + |\lambda| + |\varpi|$, if $\kappa \neq \lambda \neq \varpi$.

Then, (A, G) is a G-metric space and D^*-metric space.

Proposition 2.4. *Every G-metric space define a D^*-metric space.*

Proof. Let (A, G) be a G-metric space, then G-metric space defines a metric space (A, d_G) by

$$d_G(\kappa,\lambda) = G(\kappa,\lambda,\lambda) + G(\lambda,\kappa,\kappa),$$

hence $D^* : A \times A \times A \to [0,\infty)$ by

$$D^*(\kappa,\lambda,\varpi) = \max\{d_G(\kappa,\lambda), d_G(\lambda,\varpi), d_G(\varpi,\kappa)\},$$

or

$$D^*(\kappa,\lambda,\varpi) = d_G(\kappa,\lambda) + d_G(\lambda,\varpi) + d_G(\varpi,\kappa),$$

for all $\kappa,\lambda,\varpi \in A$ is D^*-metric on A. □

Proposition 2.5. *Every D^*-metric space define a G-metric space.*

Proof. Let (A, D^*) be a D^*-metric space, then D^*-metric space defines a metric space (A, d_{D^*}) by

$$d_{D^*}(\kappa,\lambda) = D^*(\kappa,\lambda,\lambda),$$

hence $G : A \times A \times A \to [0,\infty)$ by

$$G(\kappa,\lambda,\varpi) = \max\{d_{D^*}(\kappa,\lambda), d_{D^*}(\lambda,\varpi), d_{D^*}(\varpi,\kappa)\},$$

or

$$G(\kappa,\lambda,\varpi) = d_{D^*}(\kappa,\lambda) + d_{D^*}(\lambda,\varpi) + d_{D^*}(\varpi,\kappa),$$

for all $\kappa,\lambda,\varpi \in A$ is G-metric on A. □

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