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# The Jensen's inequality and functional form of Jensen's inequality for 3-convex functions at a point

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## Abstract

In this paper, we give the refinement of extension of Jensen's inequality to affine combinations. Furthermore, we present a functional form of Jensen's inequality for continuous 3-convex functions at a point of one variable.

**Keywords:** Affine combination, positive linear functional, convex function, 3-convex functions at a point, Jensen's inequality.

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## 1. Introduction

A subset $A$ of a linear space $X$ is called affine if it contains all binomial affine combination $pa_1 + qa_2$, where $a_1, a_2 \in A$ and $p, q \in \mathbb{R}$ such that $p + q = 1$. The affine hull of a set $A \subseteq X$, which is denoted by $\text{aff}A$ is the smallest affine set that contains $A$. A function $f : A \to \mathbb{R}$ is called affine function, if the following equation holds for all binomial affine combinations of $A$

$$f(pa_1 + qa_2) = pf(a_1) + qf(a_2).$$

A subset $C$ of linear space $X$ is called convex if it contains all binomial convex combinations $pc_1 + qc_2$ of points $c_1, c_2 \in C$, where $p, q \in \mathbb{R}_+$ and $p + q = 1$. The convex hull of $C$ is the smallest convex set that contains $C$ and denoted as $\text{conv}C$. A function $f : C \to \mathbb{R}$ is convex if the inequality

$$f(pc_1 + qc_2) \leqslant pf(c_1) + qf(c_2)$$

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holds for all binomial convex combinations $pc_1 + qc_2$ of points $c_1, c_2 \in C$.

Let $\mathbb{X}$ be a subspace of linear space of all real functions on a nonempty set $\Omega$ and contains unit function $I(x) = 1$ for every $x \in \Omega$. For an interval $\mathcal{I} \subseteq \mathbb{R}$, the subset $\mathbb{X}_{\mathcal{I}} \subseteq \mathbb{X}$ contains all functions with image in $\mathcal{I}$. If $pf_1 + qf_2$ is a convex combination of functions $f_1, f_2 \in \mathbb{X}_{\mathcal{I}}$, then the number (convex combination) $pf_1(x) + qf_2(x)$ is in $\mathcal{I}$ for every $x \in \Omega$, which indicates that functions set $\mathbb{X}_{\mathcal{I}}$ is convex. A linear functional $L : \mathbb{X} \to \mathbb{R}$ is positive (non-negative) if $L(f) > 0$ ($L(f) \geqslant 0$) for every non-negative function $f \in \mathbb{X}$, and unital (normalized) if $L(1) = 1$. In 2015, Pavić [4] gave the extension of Jensen's inequality to affine combinations in the following form.

**Theorem 1.1.** *Let $p_\zeta, q_\eta, r_\xi \geqslant 0$ be coefficients such that their sum $p = \sum_{\zeta=1}^{n} p_\zeta$, $q = \sum_{\eta=1}^{m} q_\eta$, $r = \sum_{\xi=1}^{l} r_\xi$ satisfy $p + q - r = 1$ for $p, q \in (0, 1]$ and $a_\zeta, b_\eta, c_\xi \in \mathbb{R}$ be points such that $c_\xi \in \operatorname{conv}\{a, b\}$, where*

$$a = \frac{1}{p} \sum_{\zeta=1}^{n} p_\zeta a_\zeta \ , \ b = \frac{1}{q} \sum_{\eta=1}^{m} q_\eta b_\eta.$$

*Then the affine combination*

$$\sum_{\zeta=1}^{n} p_\zeta a_\zeta + \sum_{\eta=1}^{m} q_\eta b_\eta - \sum_{\xi=1}^{l} r_\xi c_\xi$$

*belongs to $\operatorname{conv}\{a, b\}$, and for every convex functions $f : \operatorname{conv}\{a_\zeta, b_\eta\} \to \mathbb{R}$ satisfies the inequality*

$$f\left(\sum_{\zeta=1}^{n} p_\zeta a_\zeta + \sum_{\eta=1}^{m} q_\eta b_\eta - \sum_{\xi=1}^{l} r_\xi c_\xi\right) \leqslant \sum_{\zeta=1}^{n} p_\zeta f(a_\zeta) + \sum_{\eta=1}^{m} q_\eta f(b_\eta) - \sum_{\xi=1}^{l} r_\xi f(c_\xi). \tag{1.1}$$

In 2014, Pavić [3] gave the functional form of jensen's inequality for the continuous convex functions of one variable in the following form.

**Theorem 1.2.** *Let $[a, b] \subseteq \mathcal{I}$, for closed interval $\mathcal{I}$ and functions $g \in \mathbb{X}_{[a,b]}$, $h \in \mathbb{X}_{\mathcal{I} \setminus (a,b)}$ are defined, respectively. Furthermore, let $f : \mathcal{I} \to \mathbb{R}$ be a continuous convex function such that $f(g), f(h) \in \mathbb{X}$. If a pair of unital positive linear functionals $L, H : \mathbb{X} \to \mathbb{R}$ satisfies*

$$L(g) = H(h), \tag{1.2}$$

*then*

$$L(f(g)) \leqslant H(f(h)). \tag{1.3}$$

Furthermore, Pavić [3] also gave some consequent results based on the following corollaries:

**Corollary 1.3.** *Let $f : \mathcal{I} \to \mathbb{R}$ be a continuous convex function such that $f(g) \in \mathbb{X}$. If a unital positive linear functional $L : \mathbb{X} \to \mathbb{R}$ satisfies (1.2) and (1.3) for $L = H$, then*

$$f(L(g)) \leqslant L(f(g)).$$

**Corollary 1.4.** *Consider a sequence of closed intervals $[a_1, b_1] \subseteq \cdots \subseteq [a_{n-1}, b_{n-1}] \subseteq \mathcal{I}$, function $g_1 \in \mathbb{X}_{[a_1,b_1]}$, functions $g_\xi \in \mathbb{X}_{[a_\xi,b_\xi] \setminus (a_{\xi-1},b_{\xi-1})}$ for $\xi = 2, \ldots, n-1$, and functions $g_n \in \mathbb{X}_{\mathcal{I} \setminus (a_{n-1},b_{n-1})}$. Let $f : \mathcal{I} \to \mathbb{R}$ be a continuous convex function such that $f(g_\zeta) \in \mathbb{X}$. If an $n$-tuple of unital positive linear functionals $L_\zeta : \mathbb{X} \to \mathbb{R}$ satisfies*

$$L_\zeta(g_\zeta) = L_{\zeta+1}(g_{\zeta+1}) \ \text{ for } \zeta = 1, \ldots, n-1,$$

*then*

$$L_\zeta(f(g_\zeta)) \leqslant L_{\zeta+1}(f(g_{\zeta+1})) \ \text{ for } \zeta = 1, \ldots, n-1.$$

**Corollary 1.5.** *Consider a closed interval $\mathcal{I} \subseteq \mathbb{R}$ and $g_1, \ldots, g_n \in \mathbb{X}_{\mathcal{I}}$. If $f : \mathcal{I} \to \mathbb{R}$ be a continuous convex function such that $f(g_\zeta) \in \mathbb{X}$, then every $n$-tuple of positive linear functionals $L_\zeta : \mathbb{X} \to \mathbb{R}$ with $\sum_{\zeta=1}^{n} L_\zeta(1) = 1$ satisfies the inclusion*

$$\sum_{\zeta=1}^{n} L_\zeta(g_\zeta) \in \mathcal{I}$$

*and the inequality*

$$f\left(\sum_{\zeta=1}^{n} L_\zeta(g_\zeta)\right) \leqslant \sum_{\zeta=1}^{n} L_\zeta(f(g_\zeta)).$$

**Theorem 1.6.** *Consider a closed interval $\mathcal{I} \subseteq \mathbb{R}$, $[a, b] \subseteq \mathcal{I}$, $g_1, \ldots, g_n \in \mathbb{X}_{[a,b]}$, $h_1, \ldots, h_m \in \mathbb{X}_{\mathcal{I}\setminus(a,b)}$. Let $f : \mathcal{I} \to \mathbb{R}$ be a continuous convex function such that $f(g_\zeta), f(h_\eta) \in \mathbb{X}$. If a pair of n-tuple of positive linear functionals $L_\zeta, H_\eta : \mathbb{X} \to \mathbb{R}$ with $\sum_{\zeta=1}^{n} L_\zeta(1) = \sum_{\eta=1}^{m} H_\eta(1) = 1$ satisfies*

$$\sum_{\zeta=1}^{n} L_\zeta(g_\zeta) = \sum_{\eta=1}^{m} H_\eta(h_\eta),$$

*then*

$$\sum_{\zeta=1}^{n} L_\zeta(f(g_\zeta)) \leqslant \sum_{\eta=1}^{m} H_\eta(f(h_\eta)).$$

In [1], Baloch et al defined a new class of functions which is defined as follow.

**Definition 1.7.** Let $c \in \mathcal{I}^\circ$, where $\mathcal{I}$ is an arbitrary interval (open, closed or semi-open in either direction) in $\mathbb{R}$ and $\mathcal{I}^\circ$ is its interior. We say that $f : \mathcal{I} \to \mathbb{R}$ is 3-convex function at point c (respectively 3-concave function at point c) if there exists a constant $A$ such that the function $F(x) = f(x) - \frac{A}{2}x^2$ is concave (resp. convex) on $\mathcal{I} \cap (-\infty, c]$ and convex (resp. concave) on $\mathcal{I} \cap [c, \infty)$.

The function is 3-convex on an interval if and only if it is 3-convex at every point of the interval (see [1]) and the class of 3-convex functions at point c is denoted as $K_c^1(\mathcal{I})$ ($K_c^2(\mathcal{I})$) (respectively 3-concave functions at point c).

## 2. Main results

**Theorem 2.1.** *Let $p_\zeta, q_\eta, r_\xi \geqslant 0$ and $\lambda_\zeta, \mu_\eta, \nu_\xi \geqslant 0$ be coefficients such that their sum $p = \sum_{\zeta=1}^{n} p_\zeta$, $q = \sum_{\eta=1}^{m} q_\eta$, $r = \sum_{\xi=1}^{l} r_\xi$ satisfy $p + q - r = 1$ and $p, q \in (0, 1]$; $\lambda = \sum_{\zeta=1}^{n} \lambda_\zeta$, $\mu = \sum_{\eta=1}^{m} \mu_\eta$, $\nu = \sum_{\xi=1}^{l} \nu_\xi$ satisfy $\lambda + \mu - \nu = 1$ and $\lambda, \mu \in (0, 1]$. Let $a_\zeta, b_\eta, c_\xi \in I \subseteq \mathbb{R}$ be points such that $c_\xi \in \operatorname{conv}\{a, b\}$ and also $r_\zeta, s_\eta, t_\xi \in I \subseteq \mathbb{R}$ be points such that $t_\xi \in \operatorname{conv}\{r, s\}$, where*

$$a = \frac{1}{p}\sum_{\zeta=1}^{n} p_\zeta a_\zeta \ , \ b = \frac{1}{q}\sum_{\eta=1}^{m} q_\eta b_\eta \ , \ r = \frac{1}{p}\sum_{\zeta=1}^{n} \lambda_\zeta r_\zeta \ , \ s = \frac{1}{\mu}\sum_{\eta=1}^{m} \mu_\eta s_\eta,$$

*if*

$$\begin{aligned}\sum_{\zeta=1}^{n} p_\zeta(a_\zeta)^2 + \sum_{\eta=1}^{m} q_\eta(b_\eta)^2 - \sum_{\xi=1}^{l} r_\xi(c_\xi)^2 \left(\sum_{\zeta=1}^{n} p_\zeta a_\zeta + \sum_{\eta=1}^{m} q_\eta b_\eta - \sum_{\xi=1}^{l} r_\xi c_\xi\right)^2 \\ = \sum_{\zeta=1}^{n} \lambda_\zeta(r_\zeta)^2 + \sum_{\eta=1}^{m} \mu_\eta(s_\eta)^2 - \sum_{\xi=1}^{l} \nu_\xi(t_\xi)^2 - \left(\sum_{\zeta=1}^{n} \lambda_\zeta r_\zeta + \sum_{\eta=1}^{m} \mu_\eta s_\eta - \sum_{\xi=1}^{l} \nu_\xi t_\xi\right)^2\end{aligned} \tag{2.1}$$

*and also there exists* $c \in \mathcal{I}^\circ$ *such that*

$$\max\{\max_\zeta\{a_\zeta\}, \max_\eta\{b_\eta\}, \max_\xi\{c_\xi\}\} \leqslant c \leqslant \min\{\min_\zeta\{r_\zeta\}, \min_\eta\{s_\eta\}, \min_\xi\{t_\xi\}\},$$

*then for every* $f \in K_c^1(\mathcal{I})$, *the following inequality holds*

$$\begin{aligned}\sum_{\zeta=1}^{n} p_\zeta f(a_\zeta) + \sum_{\eta=1}^{m} q_\eta f(b_\eta) - \sum_{\xi=1}^{l} r_\xi f(c_\xi) - f\left(\sum_{\zeta=1}^{n} p_\zeta a_\zeta + \sum_{\eta=1}^{m} q_\eta b_\eta - \sum_{\xi=1}^{l} r_\xi c_\xi\right)& \\ \leqslant \sum_{\zeta=1}^{n} p_\zeta f(r_\zeta) + \sum_{\eta=1}^{m} q_\eta f(s_\eta) - \sum_{\xi=1}^{l} r_\xi f(t_\xi) - f\left(\sum_{\zeta=1}^{n} p_\zeta r_\zeta + \sum_{\eta=1}^{m} q_\eta s_\eta - \sum_{\xi=1}^{l} r_\xi t_\xi\right)&.\end{aligned} \tag{2.2}$$

*Proof.* Since $f \in K_c^1(\mathcal{I})$, then there exists a constant $A$ such that $F(x) = f(x) - \frac{A}{2}x^2$ is concave on $I \cap (-\infty, c]$ and for $a_\zeta, b_\eta, c_\xi \in I \cap (-\infty, c]$ be points such that $c_\xi \in \mathrm{conv}\{a, b\}$, so by using inequality (1.1) we have

$$\begin{aligned}0 &\geqslant \sum_{\zeta=1}^{n} p_\zeta F(a_\zeta) + \sum_{\eta=1}^{m} q_\eta F(b_\eta) - \sum_{\xi=1}^{l} r_\xi F(c_\xi) - F\left(\sum_{\zeta=1}^{n} p_\zeta a_\zeta + \sum_{\eta=1}^{m} q_\eta b_\eta - \sum_{\xi=1}^{l} r_\xi c_\xi\right) \\ &= \sum_{\zeta=1}^{n} p_\zeta f(a_\zeta) + \sum_{\eta=1}^{m} q_\eta f(b_\eta) - \sum_{\xi=1}^{l} r_\xi f(c_\xi) d - f\left(\sum_{\zeta=1}^{n} p_\zeta a_\zeta + \sum_{\eta=1}^{m} q_\eta b_\eta - \sum_{\xi=1}^{l} r_\xi c_\xi\right) \\ &\quad - \frac{A}{2}\left\{\sum_{\zeta=1}^{n} p_\zeta (a_\zeta)^2 + \sum_{\eta=1}^{m} q_\eta (b_\eta)^2 - \sum_{\xi=1}^{l} r_\xi (c_\xi)^2\right\} + \frac{A}{2}\left(\sum_{\zeta=1}^{n} p_\zeta a_\zeta + \sum_{\eta=1}^{m} q_\eta b_\eta - \sum_{\xi=1}^{l} r_\xi c_\xi\right)^2.\end{aligned}$$

Also, since $f \in K_c^1(\mathcal{I})$ then there exists a constant $A$ such that $F(x) = f(x) - \frac{A}{2}x^2$ is convex on $I \cap [c, \infty)$, hence for $r_\zeta, s_\eta, t_\xi \in I \cap [c, \infty)$ be points such that $t_\xi \in \mathrm{conv}\{r, s\}$, so by using inequality (1.1), we have

$$\begin{aligned}0 &\leqslant \sum_{\zeta=1}^{n} p_\zeta F(r_\zeta) + \sum_{\eta=1}^{m} q_\eta F(s_\eta) - \sum_{\xi=1}^{l} r_\xi F(t_\xi) - F\left(\sum_{\zeta=1}^{n} p_\zeta r_\zeta + \sum_{\eta=1}^{m} q_\eta s_\eta - \sum_{\xi=1}^{l} r_\xi t_\xi\right) \\ &= \sum_{\zeta=1}^{n} p_\zeta f(r_\zeta) + \sum_{\eta=1}^{m} q_\eta f(s_\eta) - \sum_{\xi=1}^{l} r_\xi f(t_\xi) - f\left(\sum_{\zeta=1}^{n} p_\zeta r_\zeta + \sum_{\eta=1}^{m} q_\eta s_\eta - \sum_{\xi=1}^{l} r_\xi t_\xi\right) \\ &\quad - \frac{A}{2}\left\{\sum_{\zeta=1}^{n} p_\zeta (r_\zeta)^2 + \sum_{\eta=1}^{m} q_\eta (s_\eta)^2 - \sum_{\xi=1}^{l} r_\xi (t_\xi)^2\right\} + \frac{A}{2}\left(\sum_{\zeta=1}^{n} p_\zeta r_\zeta + \sum_{\eta=1}^{m} q_\eta s_\eta - \sum_{\xi=1}^{l} r_\xi t_\xi\right)^2.\end{aligned}$$

From above, we have

$$\begin{aligned}&\sum_{\zeta=1}^{n} p_\zeta f(a_\zeta) + \sum_{\eta=1}^{m} q_\eta f(b_\eta) - \sum_{\xi=1}^{l} r_\xi f(c_\xi) - f\left(\sum_{\zeta=1}^{n} p_\zeta a_\zeta + \sum_{\eta=1}^{m} q_\eta b_\eta - \sum_{\xi=1}^{l} r_\xi c_\xi\right) \\ &\qquad - \frac{A}{2}\left\{\sum_{\zeta=1}^{n} p_\zeta (a_\zeta)^2 + \sum_{\eta=1}^{m} q_\eta (b_\eta)^2 - \sum_{\xi=1}^{l} r_\xi (c_\xi)^2\right\} + \frac{A}{2}\left(\sum_{\zeta=1}^{n} p_\zeta a_\zeta + \sum_{\eta=1}^{m} q_\eta b_\eta - \sum_{\xi=1}^{l} r_\xi c_\xi\right)^2 \\ &\quad \leqslant 0 \\ &\quad \leqslant \sum_{\zeta=1}^{n} p_\zeta f(r_\zeta) + \sum_{\eta=1}^{m} q_\eta f(s_\eta) - \sum_{\xi=1}^{l} r_\xi f(t_\xi) - f\left(\sum_{\zeta=1}^{n} p_\zeta r_\zeta + \sum_{\eta=1}^{m} q_\eta s_\eta - \sum_{\xi=1}^{l} r_\xi t_\xi\right)\end{aligned} \tag{2.3}$$

$$-\frac{A}{2}\left\{\sum_{\zeta=1}^{n}p_\zeta(r_\zeta)^2+\sum_{\eta=1}^{m}q_\eta(s_\eta)^2-\sum_{\xi=1}^{l}r_\xi(t_\xi)^2\right\}+\frac{A}{2}\left(\sum_{\zeta=1}^{n}p_\zeta r_\zeta+\sum_{\eta=1}^{m}q_\eta s_\eta-\sum_{\xi=1}^{l}r_\xi t_\xi\right)^2.$$

So

$$\begin{aligned}
&\sum_{\zeta=1}^{n}p_\zeta f(a_\zeta)+\sum_{\eta=1}^{m}q_\eta f(b_\eta)-\sum_{\xi=1}^{l}r_\xi f(c_\xi)-f\left(\sum_{\zeta=1}^{n}p_\zeta a_\zeta+\sum_{\eta=1}^{m}q_\eta b_\eta-\sum_{\xi=1}^{l}r_\xi c_\xi\right)\\
&\quad-\frac{A}{2}\left\{\sum_{\zeta=1}^{n}p_\zeta(a_\zeta)^2+\sum_{\eta=1}^{m}q_\eta(b_\eta)^2-\sum_{\xi=1}^{l}r_\xi(c_\xi)^2\right\}+\frac{A}{2}\left(\sum_{\zeta=1}^{n}p_\zeta a_\zeta+\sum_{\eta=1}^{m}q_\eta b_\eta-\sum_{\xi=1}^{l}r_\xi c_\xi\right)^2\\
&\leqslant\sum_{\zeta=1}^{n}p_\zeta f(r_\zeta)+\sum_{\eta=1}^{m}q_\eta f(s_\eta)-\sum_{\xi=1}^{l}r_\xi f(t_\xi)-f\left(\sum_{\zeta=1}^{n}p_\zeta r_\zeta+\sum_{\eta=1}^{m}q_\eta s_\eta-\sum_{\xi=1}^{l}r_\xi t_\xi\right)\\
&\quad-\frac{A}{2}\left\{\sum_{\zeta=1}^{n}p_\zeta(r_\zeta)^2+\sum_{\eta=1}^{m}q_\eta(s_\eta)^2-\sum_{\xi=1}^{l}r_\xi(t_\xi)^2\right\}+\frac{A}{2}\left(\sum_{\zeta=1}^{n}p_\zeta r_\zeta+\sum_{\eta=1}^{m}q_\eta s_\eta-\sum_{\xi=1}^{l}r_\xi t_\xi\right)^2,
\end{aligned}$$

by using (2.1), we get (2.2). □

*Remark* 2.2. From the proof of Theorem 2.1, we have

$$\begin{aligned}
&\sum_{\zeta=1}^{n}p_\zeta f(a_\zeta)+\sum_{\eta=1}^{m}q_\eta f(b_\eta)-\sum_{\xi=1}^{l}r_\xi f(c_\xi)-f\left(\sum_{\zeta=1}^{n}p_\zeta a_\zeta+\sum_{\eta=1}^{m}q_\eta b_\eta-\sum_{\xi=1}^{l}r_\xi c_\xi\right)\\
&\quad\leqslant\frac{A}{2}\left\{\sum_{\zeta=1}^{n}p_\zeta(a_\zeta)^2+\sum_{\eta=1}^{m}q_\eta(b_\eta)^2-\sum_{\xi=1}^{l}r_\xi(c_\xi)^2\right\}-\frac{A}{2}\left\{\left(\sum_{\zeta=1}^{n}p_\zeta a_\zeta+\sum_{\eta=1}^{m}q_\eta b_\eta-\sum_{\xi=1}^{l}r_\xi c_\xi\right)^2\right\}
\end{aligned}\tag{2.4}$$

and

$$\begin{aligned}
&\sum_{\zeta=1}^{n}p_\zeta f(r_\zeta)+\sum_{\eta=1}^{m}q_\eta f(s_\eta)-\sum_{\xi=1}^{l}r_\xi f(t_\xi)-f\left(\sum_{\zeta=1}^{n}p_\zeta r_\zeta+\sum_{\eta=1}^{m}q_\eta s_\eta-\sum_{\xi=1}^{l}r_\xi t_\xi\right)\\
&\quad\geqslant\frac{A}{2}\left\{\sum_{\zeta=1}^{n}p_\zeta(r_\zeta)^2+\sum_{\eta=1}^{m}q_\eta(s_\eta)^2-\sum_{\xi=1}^{l}r_\xi(t_\xi)^2\right\}-\frac{A}{2}\left\{\left(\sum_{\zeta=1}^{n}p_\zeta r_\zeta+\sum_{\eta=1}^{m}q_\eta s_\eta-\sum_{\xi=1}^{l}r_\xi t_\xi\right)^2\right\}.
\end{aligned}\tag{2.5}$$

So under assumption (2.1), we can get an improvement of (2.2) as follows

$$\begin{aligned}
&\sum_{\zeta=1}^{n}p_\zeta f(a_\zeta)+\sum_{\eta=1}^{m}q_\eta f(b_\eta)-\sum_{\xi=1}^{l}r_\xi f(c_\xi)-f\left(\sum_{\zeta=1}^{n}p_\zeta a_\zeta+\sum_{\eta=1}^{m}q_\eta b_\eta-\sum_{\xi=1}^{l}r_\xi c_\xi\right)\\
&\quad\leqslant\frac{A}{2}\left\{\sum_{\zeta=1}^{n}p_\zeta(a_\zeta)^2+\sum_{\eta=1}^{m}q_\eta(b_\eta)^2-\sum_{\xi=1}^{l}r_\xi(c_\xi)^2\right\}-\frac{A}{2}\left(\sum_{\zeta=1}^{n}p_\zeta a_\zeta+\sum_{\eta=1}^{m}q_\eta b_\eta-\sum_{\xi=1}^{l}r_\xi c_\xi\right)^2\\
&\quad=\frac{A}{2}\left\{\sum_{\zeta=1}^{n}p_\zeta(r_\zeta)^2+\sum_{\eta=1}^{m}q_\eta(s_\eta)^2-\sum_{\xi=1}^{l}r_\xi(t_\xi)^2\right\}-\frac{A}{2}\left(\sum_{\zeta=1}^{n}p_\zeta r_\zeta+\sum_{\eta=1}^{m}q_\eta s_\eta-\sum_{\xi=1}^{l}r_\xi t_\xi\right)^2\\
&\quad\leqslant\sum_{\zeta=1}^{n}p_\zeta f(r_\zeta)+\sum_{\eta=1}^{m}q_\eta f(s_\eta)-\sum_{\xi=1}^{l}r_\xi f(t_\xi)-f\left(\sum_{\zeta=1}^{n}p_\zeta r_\zeta+\sum_{\eta=1}^{m}q_\eta s_\eta-\sum_{\xi=1}^{l}r_\xi t_\xi\right).
\end{aligned}\tag{2.6}$$

Assume that $\tilde{a} = \max_{\zeta}\{a_{\zeta}\}, \tilde{b} = \max_{\eta}\{b_{\eta}\}, \tilde{c} = \max_{\xi}\{c_{\xi}\}$ and $\tilde{r} = \min_{\zeta}\{r_{\zeta}\}, \tilde{s} = \min_{\eta}\{s_{\eta}\}, \tilde{t} = \min_{\xi}\{t_{\xi}\}$. Also, let $\tilde{\tilde{a}} = \max\{\tilde{a}, \tilde{b}, \tilde{c}\}$ and $\tilde{\tilde{r}} = \min\{\tilde{r}, \tilde{s}, \tilde{t}\}$. Now, we give the next result which weakens the assumption (2.1) such that inequality (2.4) holds again.

**Theorem 2.3.** *Let $p_{\zeta}, q_{\eta}, r_{\xi} \geqslant 0$ and $\lambda_{\zeta}, \mu_{\eta}, \nu_{\xi} \geqslant 0$ be coefficients such that their sum $p = \sum_{\zeta=1}^{n} p_{\zeta}$, $q = \sum_{\eta=1}^{m} q_{\eta}$, $r = \sum_{\xi=1}^{l} r_{\xi}$ satisfy $p + q - r = 1$ and $p, q \in (0, 1]$; $\lambda = \sum_{\zeta=1}^{n} \lambda_{\zeta}$, $\mu = \sum_{\eta=1}^{m} \mu_{\eta}$, $\nu = \sum_{\xi=1}^{l} \nu_{\xi}$ satisfy $\lambda + \mu - \nu = 1$ and $\lambda, \mu \in (0, 1]$. Let $a_{\zeta}, b_{\eta}, c_{\xi} \in I \subseteq \mathbb{R}$ be points such that $c_{\xi} \in \operatorname{conv}\{a, b\}$ and $r_{\zeta}, s_{\eta}, t_{\xi} \in I \subseteq \mathbb{R}$ be points such that $t_{\xi} \in \operatorname{conv}\{r, s\}$, where*

$$a = \frac{1}{p}\sum_{\zeta=1}^{n} p_{\zeta}a_{\zeta}\ ,\ b = \frac{1}{q}\sum_{\eta=1}^{m} q_{\eta}b_{\eta}\ ,\ r = \frac{1}{p}\sum_{\zeta=1}^{n} \lambda_{\zeta}r_{\zeta}\ ,\ s = \frac{1}{\mu}\sum_{\eta=1}^{m} \mu_{\eta}s_{\eta}.$$

*such that*

$$\tilde{\tilde{a}} \leqslant \tilde{\tilde{r}}$$

*and $f \in K_1^c(\mathcal{I})$ for some $c \in [\tilde{\tilde{a}}, \tilde{\tilde{r}}]$. Then if*

(a)

$$f''_{-}(\tilde{\tilde{a}}) \geqslant 0$$

*and*

$$\sum_{\zeta=1}^{n} p_{\zeta}(a_{\zeta})^2 + \sum_{\eta=1}^{m} q_{\eta}(b_{\eta})^2 - \sum_{\xi=1}^{l} r_{\xi}(c_{\xi})^2 - \left(\sum_{\zeta=1}^{n} p_{\zeta}a_{\zeta} + \sum_{\eta=1}^{m} q_{\eta}b_{\eta} - \sum_{\xi=1}^{l} r_{\xi}c_{\xi}\right)^2$$
$$\leqslant \sum_{\zeta=1}^{n} \lambda_{\zeta}(r_{\zeta})^2 + \sum_{\eta=1}^{m} \mu_{\eta}(s_{\eta})^2 - \sum_{\xi=1}^{l} \nu_{\xi}(t_{\xi})^2 - \left(\sum_{\zeta=1}^{n} \lambda_{\zeta}r_{\zeta} + \sum_{\eta=1}^{m} \mu_{\eta}s_{\eta} - \sum_{\xi=1}^{l} \nu_{\xi}t_{\xi}\right)^2,$$

*or*

(b)

$$f''_{+}(\tilde{\tilde{r}}) \leqslant 0$$

*and*

$$\sum_{\zeta=1}^{n} p_{\zeta}(a_{\zeta})^2 + \sum_{\eta=1}^{m} q_{\eta}(b_{\eta})^2 - \sum_{\xi=1}^{l} r_{\xi}(c_{\xi})^2 - \left(\sum_{\zeta=1}^{n} p_{\zeta}a_{\zeta} + \sum_{\eta=1}^{m} q_{\eta}b_{\eta} - \sum_{\xi=1}^{l} r_{\xi}c_{\xi}\right)^2$$
$$\geqslant \sum_{\zeta=1}^{n} \lambda_{\zeta}(r_{\zeta})^2 + \sum_{\eta=1}^{m} \mu_{\eta}(s_{\eta})^2 - \sum_{\xi=1}^{l} \nu_{\xi}(t_{\xi})^2 - \left(\sum_{\zeta=1}^{n} \lambda_{\zeta}r_{\zeta} + \sum_{\eta=1}^{m} \mu_{\eta}s_{\eta} - \sum_{\xi=1}^{l} \nu_{\xi}t_{\xi}\right)^2$$

*or*

(c)

$$f''_{-}(\tilde{\tilde{a}}) < 0 < f''_{+}(\tilde{\tilde{r}}) \ \text{ and } \ f \text{ is 3-convex},$$

*then (2.2) holds.*

*Proof.* The idea of proof is similar to proof of Theorem 2.1. Hence, by proceeding as in the proof of Theorem 2.1, from the inequality (2.3), we have

$$\frac{A}{2}\left[\sum_{\zeta=1}^{n} p_{\zeta}(r_{\zeta})^2 + \sum_{\eta=1}^{m} q_{\eta}(s_{\eta})^2 - \sum_{\xi=1}^{l} r_{\xi}(t_{\xi})^2\right] - \frac{A}{2}\left(\sum_{\zeta=1}^{n} p_{\zeta}r_{\zeta} + \sum_{\eta=1}^{m} q_{\eta}s_{\eta} - \sum_{\xi=1}^{l} r_{\xi}t_{\xi}\right)^2$$

$$
\begin{aligned}
&-\frac{A}{2}\left[\sum_{\zeta=1}^{n} p_\zeta (a_\zeta)^2+\sum_{\eta=1}^{m} q_\eta (b_\eta)^2-\sum_{\xi=1}^{l} r_\xi (c_\xi)^2\right]+\frac{A}{2}\left(\sum_{\zeta=1}^{n} p_\zeta a_\zeta+\sum_{\eta=1}^{m} q_\eta b_\eta-\sum_{\xi=1}^{l} r_\xi c_\xi\right)^2\\
&\leqslant \sum_{\zeta=1}^{n} p_\zeta f(r_\zeta)+\sum_{\eta=1}^{m} q_\eta f(s_\eta)-\sum_{\xi=1}^{l} r_\xi f(t_\xi)-f\left(\sum_{\zeta=1}^{n} p_\zeta r_\zeta+\sum_{\eta=1}^{m} q_\eta s_\eta-\sum_{\xi=1}^{l} r_\xi t_\xi\right)\\
&-\left\{\sum_{\zeta=1}^{n} p_\zeta f(a_\zeta)+\sum_{\eta=1}^{m} q_\eta f(b_\eta)-\sum_{\xi=1}^{l} r_\xi f(c_\xi)\right\}+f\left(\sum_{\zeta=1}^{n} p_\zeta a_\zeta+\sum_{\eta=1}^{m} q_\eta b_\eta-\sum_{\xi=1}^{l} r_\xi c_\xi\right).
\end{aligned}
$$

Now, due to the concavity of $F$ on $I\cap(-\infty,c]$ and convexity of $F$ on $I\cap[c,\infty)$, so for every distinct points $x_\eta\in[a,\tilde{a}]$ and $y_\eta\in[\tilde{r},b]$, $j=1,2,3$, we have

$$[x_1,x_2,x_3]f\leqslant A\leqslant[y_1,y_2,y_3]f.$$

Letting $x_\eta\nearrow\tilde{a}$ and $y_\eta\searrow\tilde{r}$, we get (if exists)

$$f''_{-}(\tilde{a})\leqslant A\leqslant f''_{+}(\tilde{r}).$$

Therefore, if assumptions (a) or (b) holds, then

$$
\begin{aligned}
&\frac{A}{2}\left[\sum_{\zeta=1}^{n} p_\zeta (r_\zeta)^2+\sum_{\eta=1}^{m} q_\eta (s_\eta)^2-\sum_{\xi=1}^{l} r_\xi (t_\xi)^2\right]-\frac{A}{2}\left(\sum_{\zeta=1}^{n} p_\zeta r_\zeta+\sum_{\eta=1}^{m} q_\eta s_\eta-\sum_{\xi=1}^{l} r_\xi t_\xi\right)^2\\
&-\frac{A}{2}\left\{\sum_{\zeta=1}^{n} p_\zeta (a_\zeta)^2+\sum_{\eta=1}^{m} q_\eta (b_\eta)^2-\sum_{\xi=1}^{l} r_\xi (c_\xi)^2\right\}+\frac{A}{2}\left(\sum_{\zeta=1}^{n} p_\zeta a_\zeta+\sum_{\eta=1}^{m} q_\eta b_\eta-\sum_{\xi=1}^{l} r_\xi c_\xi\right)^2
\end{aligned}
$$

is positive and we conclude the result. If the assumption (c) holds, the $f''_{-}$ is left continuous, $f''_{+}$ is right continuous, they are both non-decreasing and $f''_{-}\leqslant f''_{+}$. Therefore, there exists $\tilde{c}\in[\tilde{a},\tilde{r}]$ such that $f\in K_1^{\tilde{c}}(I)$ with associated constant $\tilde{A}=0$ and again, we can deduce the result. □

*Remark* 2.4. Again from the proof of Theorem 2.1, we obtain the inequalities (2.4) and (2.5). Now, under assumption (a), (b), or (c) of Theorem 2.3, $A$ is positive or negative or zero, respectively due to argument discussed in the proof. Therefore, we get a better improvement of (2.2) then (2.6) in this case as follows

$$
\begin{aligned}
&\sum_{\zeta=1}^{n} p_\zeta f(a_\zeta)+\sum_{\eta=1}^{m} q_\eta f(b_\eta)-\sum_{\xi=1}^{l} r_\xi f(c_\xi)-f\left(\sum_{\zeta=1}^{n} p_\zeta a_\zeta+\sum_{\eta=1}^{m} q_\eta b_\eta-\sum_{\xi=1}^{l} r_\xi c_\xi\right)\\
&\leqslant \frac{A}{2}\left\{\sum_{\zeta=1}^{n} p_\zeta (a_\zeta)^2+\sum_{\eta=1}^{m} q_\eta (b_\eta)^2-\sum_{\xi=1}^{l} r_\xi (c_\xi)^2\right\}-\frac{A}{2}\left(\sum_{\zeta=1}^{n} p_\zeta a_\zeta+\sum_{\eta=1}^{m} q_\eta b_\eta-\sum_{\xi=1}^{l} r_\xi c_\xi\right)^2\\
&\leqslant \frac{A}{2}\left\{\sum_{\zeta=1}^{n} p_\zeta (r_\zeta)^2+\sum_{\eta=1}^{m} q_\eta (s_\eta)^2-\sum_{\xi=1}^{l} r_\xi (t_\xi)^2\right\}-\frac{A}{2}\left(\sum_{\zeta=1}^{n} p_\zeta r_\zeta+\sum_{\eta=1}^{m} q_\eta s_\eta-\sum_{\xi=1}^{l} r_\xi t_\xi\right)^2\\
&\leqslant \sum_{\zeta=1}^{n} p_\zeta f(r_\zeta)+\sum_{\eta=1}^{m} q_\eta f(s_\eta)-\sum_{\xi=1}^{l} r_\xi f(t_\xi)-f\left(\sum_{\zeta=1}^{n} p_\zeta r_\zeta+\sum_{\eta=1}^{m} q_\eta s_\eta-\sum_{\xi=1}^{l} r_\xi t_\xi\right).
\end{aligned}
$$

Under the assumption of Theorem 2.1 with $f\in K_2^c(\mathcal{I})$, the reverse of inequality (2.2) holds. Now, we give only the statement by weakening assumptions of Theorem 2.1 under which the reverse of inequality (2.2) holds for $f\in K_2^c(I)$.

**Theorem 2.5.** *Let* $p_\zeta, q_\eta, r_\xi \geqslant 0$ *and* $\lambda_\zeta, \mu_\eta, \nu_\xi \geqslant 0$ *be coefficients such that their sum* $p = \sum_{\zeta=1}^{n} p_\zeta$, $q = \sum_{\eta=1}^{m} q_\eta$, $r = \sum_{\xi=1}^{l} r_\xi$ *satisfy* $p + q - r = 1$ *and* $p, q \in (0, 1]$; $\lambda = \sum_{\zeta=1}^{n} \lambda_\zeta$, $\mu = \sum_{\eta=1}^{m} \mu_\eta$, $\nu = \sum_{\xi=1}^{l} \nu_\xi$ *satisfy* $\lambda + \mu - \nu = 1$ *and* $\lambda, \mu \in (0, 1]$. *Let* $a_\zeta, b_\eta, c_\xi \in I \subseteq \mathbb{R}$ *be points such that* $c_\xi \in \operatorname{conv}\{a, b\}$ *and* $r_\zeta, s_\eta, t_\xi \in I \subseteq \mathbb{R}$ *be points such that* $t_\xi \in \operatorname{conv}\{r, s\}$ , *where*

$$a = \frac{1}{p}\sum_{\zeta=1}^{n} p_\zeta a_\zeta \ , \ b = \frac{1}{q}\sum_{\eta=1}^{m} q_\eta b_\eta \ , \ r = \frac{1}{p}\sum_{\zeta=1}^{n} \lambda_\zeta r_\zeta \ , \ s = \frac{1}{\mu}\sum_{\eta=1}^{m} \mu_\eta s_\eta,$$

*such that*

$$\tilde{\tilde{a}} \leqslant \tilde{\tilde{r}}$$

*and* $f \in K_2^c(\mathcal{I})$ *for some* $c \in [\tilde{\tilde{a}}, \tilde{\tilde{r}}]$. *Then if*

(a)

$$f''_-(\tilde{\tilde{a}}) \leqslant 0$$

*and*

$$\sum_{\zeta=1}^{n} p_\zeta (a_\zeta)^2 + \sum_{\eta=1}^{m} q_\eta (b_\eta)^2 - \sum_{\xi=1}^{l} r_\xi (c_\xi)^2 - \left( \sum_{\zeta=1}^{n} p_\zeta a_\zeta + \sum_{\eta=1}^{m} q_\eta b_\eta - \sum_{\xi=1}^{l} r_\xi c_\xi \right)^2$$
$$\geqslant \sum_{\zeta=1}^{n} \lambda_\zeta (r_\zeta)^2 + \sum_{\eta=1}^{m} \mu_\eta (s_\eta)^2 - \sum_{\xi=1}^{l} \nu_\xi (t_\xi)^2 - \left( \sum_{\zeta=1}^{n} \lambda_\zeta r_\zeta + \sum_{\eta=1}^{m} \mu_\eta s_\eta - \sum_{\xi=1}^{l} \nu_\xi t_\xi \right)^2,$$

*or*

(b)

$$f''_+(\tilde{\tilde{r}}) \geqslant 0$$

*and*

$$\sum_{\zeta=1}^{n} p_\zeta (a_\zeta)^2 + \sum_{\eta=1}^{m} q_\eta (b_\eta)^2 - \sum_{\xi=1}^{l} r_\xi (c_\xi)^2 - \left( \sum_{\zeta=1}^{n} p_\zeta a_\zeta + \sum_{\eta=1}^{m} q_\eta b_\eta - \sum_{\xi=1}^{l} r_\xi c_\xi \right)^2$$
$$\leqslant \sum_{\zeta=1}^{n} \lambda_\zeta (r_\zeta)^2 + \sum_{\eta=1}^{m} \mu_\eta (s_\eta)^2 - \sum_{\xi=1}^{l} \nu_\xi (t_\xi)^2 - \left( \sum_{\zeta=1}^{n} \lambda_\zeta r_\zeta + \sum_{\eta=1}^{m} \mu_\eta s_\eta - \sum_{\xi=1}^{l} \nu_\xi t_\xi \right)^2,$$

*or*

(c)

$$f''_-(\tilde{\tilde{a}}) < 0 < f''_+(\tilde{\tilde{r}}) \ \text{ and } \ f \text{ is 3-concave},$$

*then reverse of* (2.2) *holds.*

*Remark* 2.6. From the proof of the Theorem 2.5, we obtain the reverse of inequalities (2.4) and (2.5). Now, due to the convexity of $F$ on $I \cap (-\infty, c]$ and concavity of $F$ on $I \cap [c, \infty)$, for every distinct points $x_\eta \in [a, \tilde{\tilde{a}}]$ and $y_\eta \in [\tilde{\tilde{r}}, b]$, $\eta = 1, 2, 3$, we have

$$[x_1, x_2, x_3]f \geqslant A \geqslant [y_1, y_2, y_3]f.$$

Letting $x_\eta \nearrow \tilde{\tilde{a}}$ and $y_\eta \searrow \tilde{\tilde{r}}$, we get (if exists)

$$f''_-(\tilde{\tilde{a}}) \geqslant A \geqslant f''_+(\tilde{\tilde{r}}).$$

Now, under assumption (a), (b), or (c) of Theorem 2.3, $A$ is negative or positive or zero, respectively due to argument discussed above. Therefore, we get a better improvement in this case as follows

$$\begin{aligned}
&\sum_{\zeta=1}^{n} p_\zeta f(a_\zeta) + \sum_{\eta=1}^{m} q_\eta f(b_\eta) - \sum_{\xi=1}^{l} r_\xi f(c_\xi) - f\left(\sum_{\zeta=1}^{n} p_\zeta a_\zeta + \sum_{\eta=1}^{m} q_\eta b_\eta - \sum_{\xi=1}^{l} r_\xi c_\xi\right) \\
&\quad \geqslant \frac{A}{2}\left\{\sum_{\zeta=1}^{n} p_\zeta (a_\zeta)^2 + \sum_{\eta=1}^{m} q_\eta (b_\eta)^2 - \sum_{\xi=1}^{l} r_\xi (c_\xi)^2\right\} - \frac{A}{2}\left\{\left(\sum_{\zeta=1}^{n} p_\zeta a_\zeta + \sum_{\eta=1}^{m} q_\eta b_\eta - \sum_{\xi=1}^{l} r_\xi c_\xi\right)^2\right\} \\
&\quad \geqslant \frac{A}{2}\left\{\sum_{\zeta=1}^{n} p_\zeta (r_\zeta)^2 + \sum_{\eta=1}^{m} q_\eta (s_\eta)^2 - \sum_{\xi=1}^{l} r_\xi (t_\xi)^2\right\} - \frac{A}{2}\left\{\left(\sum_{\zeta=1}^{n} p_\zeta r_\zeta + \sum_{\eta=1}^{m} q_\eta s_\eta - \sum_{\xi=1}^{l} r_\xi t_\xi\right)^2\right\} \\
&\quad \geqslant \sum_{\zeta=1}^{n} p_\zeta f(r_\zeta) + \sum_{\eta=1}^{m} q_\eta f(s_\eta) - \sum_{\xi=1}^{l} r_\xi f(t_\xi) - f\left(\sum_{\zeta=1}^{n} p_\zeta r_\zeta + \sum_{\eta=1}^{m} q_\eta s_\eta - \sum_{\xi=1}^{l} r_\xi t_\xi\right).
\end{aligned}$$

**Theorem 2.7.** *Let $\mathcal{I} \subseteq \mathbb{R}$ be a closed interval, let $[a, b] \subseteq \mathcal{I}$, let function $g_\zeta \in \mathbb{X}_{[a,b]}$ and function $h_\zeta \in \mathbb{X}_{\mathcal{I}\setminus(a,b)}$ for $\zeta = 1, 2$. Let $f \in K_1^c(\mathcal{I})$ be continuous function such that $f(g_\zeta), f(h_\zeta) \in \mathbb{X}$. If a pair of unital positive linear functionals $L, H : \mathbb{X} \to \mathbb{R}$ satisfies*

$$L(g_\zeta) = H(h_\zeta) \ \text{ and } \ H(h_1^2) - L(g_1^2) = H(h_2^2) - L(g_2^2), \ \ \zeta = 1, 2, \tag{2.7}$$

*then inequality*

$$H(f(h_1)) - L(f(g_1)) \leqslant H(f(h_2)) - L(f(g_2)) \tag{2.8}$$

*holds.*

*Proof.* Since $f \in K_1^c(\mathcal{I})$, there exists a constant $A$ such that $F(x) = f(x) - \frac{A}{2}x^2$ is concave on $\mathcal{I} \cap (-\infty, c]$, therefore by reverse of (1.3) for $F$ on $\mathcal{I} \cap (-\infty, c]$, we get

$$0 \geqslant H(F(h_1)) - L(F(g_1)) = H(f(h_1)) - L(f(g_1)) - \frac{A}{2}(H(h_1^2) - L(g_1^2)).$$

Also, since $F(x) = f(x) - \frac{A}{2}x^2$ is convex on $\mathcal{I} \cap [c, \infty)$, therefore by (1.3) for $F$ on $\mathcal{I} \cap (-\infty, c]$, we get

$$0 \leqslant H(F(h_2)) - L(F(g_2)) = H(f(h_2)) - L(f(g_2)) - \frac{A}{2}(H(h_2^2) - L(g_2^2)).$$

From above, we have

$$H(f(h_1)) - L(f(g_1)) - \frac{A}{2}(H(h_1^2) - L(g_1^2)) \leqslant 0 \leqslant H(f(h_2)) - L(f(g_2)) - \frac{A}{2}(H(h_2^2) - L(g_2^2)).$$

So

$$H(f(h_1)) - L(f(g_1)) - \frac{A}{2}(H(h_1^2) - L(g_1^2)) \leqslant H(f(h_2)) - L(f(g_2)) - \frac{A}{2}(H(h_2^2) - L(g_2^2)),$$

therefore, by the use of (2.7), we get (2.8). □

*Remark* 2.8. From the proof of the Theorem 2.7, we have

$$H(f(h_1)) - L(f(g_1)) \leqslant \frac{A}{2}(H(h_1^2) - L(g_1^2))$$

and

$$H(f(h_2)) - L(f(g_2)) \geqslant \frac{A}{2}(H(h_2^2) - L(g_2^2)).$$

So, under assumption (2.7), we can get a better improvement of (2.8) as follow

$$H(f(h_1)) - L(f(g_1)) \leqslant \frac{A}{2}(H(h_1^2) - L(g_1^2)) \left(= \frac{A}{2}(H(h_2^2) - L(g_2^2))\right) \leqslant H(f(h_2)) - L(f(g_2)).$$

**Corollary 2.9.** *Let $\mathcal{I} \subseteq \mathbb{R}$ be a closed interval, let $[a,b] \subseteq \mathcal{I}$, let function $g_\zeta \in \mathbb{X}_{[a,b]}$ for $\zeta = 1,2$. Let $f \in K_1^c(\mathcal{I})$ be continuous function such that $f(g_\zeta) \in \mathbb{X}$. If a unital positive linear functionals $L : \mathbb{X} \to \mathbb{R}$ satisfies implication (2.7) $\Rightarrow$ (2.8) for $L = H$ such that*

$$L(g_1^2) - (L(g_1))^2 = L(g_2^2) - (L(g_2))^2,$$

*then following inequality holds*

$$L(f(g_1)) - f(L(g_1)) \leqslant L(f(g_2)) - f(L(g_2)).$$

**Corollary 2.10.** *Let $[a_1, b_1] \subseteq \cdots \subseteq [a_{n-1}, b_{n-1}] \subseteq \mathcal{I}$ and let functions $g_1, h_1 \in \mathbb{X}_{[a_1,b_1]}$, let $g_\xi$, $h_\xi \in \mathbb{X}_{[a_\xi,b_\xi]\setminus(a_{\xi-1},b_{\xi-1})}$ for $\xi = 2, \dots, n-1$, and functions $g_n, h_n \in \mathbb{X}_{\mathcal{I}\setminus(a_{n-1},b_{n-1})}$. Let $f \in K_1^c(\mathcal{I})$ be continuous function such that $f(g_\zeta), f(h_\zeta) \in \mathbb{X}$. If an n-tuple of unital positive linear functionals $L_\zeta : \mathbb{X} \to \mathbb{R}$ satisfies*

$$L_\zeta(g_\zeta) = L_{\zeta+1}(g_{\zeta+1}) \ \text{ and } \ L_\zeta(h_\zeta) = L_{\zeta+1}(h_{\zeta+1}) \ \text{ for } \zeta = 1, \dots, n-1,$$

*such that*

$$L_{\zeta+1}(g_{\zeta+1}^2) - L_\zeta(g_\zeta^2) = L_{\zeta+1}(h_{\zeta+1}^2) - L_\zeta(h_\zeta^2),$$

*then*

$$L_{\zeta+1}f(g_{\zeta+1}) - L_\zeta f(g_\zeta) \leqslant L_{\zeta+1}f(h_{\zeta+1}) - L_\zeta f(h_\zeta) \ \text{ for } \zeta = 1, \dots, n-1.$$

**Corollary 2.11.** *Let functions $g_\zeta, h_\zeta \in \mathbb{X}_\mathcal{I}$ for $\zeta = 1, \dots, n$. Let $f \in K_1^c(\mathcal{I})$ be continuous function such that $f(g_\zeta), f(h_\zeta) \in \mathbb{X}$. Then every $n$-tuple of positive linear functionals $L_\zeta : \mathbb{X} \to \mathbb{R}$ with $\sum_{\zeta=1}^n L_\zeta(1) = 1$ such that*

$$\sum_{\zeta=1}^n L_\zeta((g_\zeta)^2) - \Big(\sum_{\zeta=1}^n L_\zeta(g_\zeta)\Big)^2 = \sum_{\zeta=1}^n L_\zeta((h_\zeta)^2) - \Big(\sum_{\zeta=1}^n L_\zeta(h_\zeta)\Big)^2,$$

*satisfies the inclusion*

$$\sum_{\zeta=1}^n L_\zeta(g_\zeta), \sum_{\zeta=1}^n L_\zeta(h_\zeta) \in \mathcal{I}$$

*and the inequality*

$$\sum_{\zeta=1}^n L_\zeta(f(g_\zeta)) - f\Big(\sum_{\zeta=1}^n L_\zeta(g_\zeta)\Big) \leqslant \sum_{\zeta=1}^n L_\zeta(f(h_\zeta)) - f\Big(\sum_{\zeta=1}^n L_\zeta(h_\zeta)\Big).$$

**Theorem 2.12.** *Let $[a,b] \subset \mathcal{I}$, $g_\zeta, g_\zeta^* \in \mathbb{X}_{[a,b]}$ for $\zeta = 1, \dots, n$ and $h_\eta, h_\eta^* \in \mathbb{X}_{\mathcal{I}\setminus(a,b)}$ for $\eta = 1, \dots, m$. Also, let $f \in K_1^c(\mathcal{I})$ be continuous function such that $f(g_\zeta), f(g_\zeta^*), f(h_\eta), f(h_\eta^*) \in \mathbb{X}$. If two pair of $n$-tuple of positive linear functionals $L_\zeta, L_\zeta^*, H_\eta, H_\eta^* : \mathbb{X} \to \mathbb{R}$ with $\sum_{\zeta=1}^n L_\zeta(1) = \sum_{1=1}^n L_\zeta^*(1) = \sum_{\eta=1}^m H_\eta(1) = \sum_{\eta=1}^m H_\eta^*(1) = 1$ satisfy*

$$\sum_{\eta=1}^m H_\eta(h_\eta) = \sum_{\zeta=1}^n L_\zeta(g_\zeta) \ \text{ and } \ \sum_{\eta=1}^m H_\eta^*(h_\eta^*) = \sum_{\zeta=1}^n L_\zeta^*(g_\zeta^*)$$

*and*

$$\sum_{\eta=1}^m H_\eta((h_\eta)^2) - \sum_{\zeta=1}^n L_\zeta((g_\zeta)^2) = \sum_{\eta=1}^m H_\eta^*((h_\eta^*)^2) - \sum_{\zeta=1}^n L_\zeta^*((g_\zeta^*)^2),$$

*then*

$$\sum_{\eta=1}^m H_\eta f(h_\eta) - \sum_{\zeta=1}^n L_\zeta f(g_\zeta) \leqslant \sum_{\eta=1}^m H_\eta^* f(h_\eta^*) - \sum_{\zeta=1}^n L_\zeta^* f(g_\zeta^* s).$$

## 3. Conclusion

In this paper, we presented more generalized extension of Jensen's type inequality to affine combinations and functional form of Jensen's type inequality for non-convex functions. We hope that our results will stimulate other researcher to explore more exciting results in literature.

## Data availability

No data were used to support this study.

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## References

[1] I. A. Baloch, J. Pečarič, M. Praljak, *Generalization of Levinson's inequality*, J. Math. Inequal., **9** (2015), 571–586. 1, 1
[2] C. P. Niculescu, L.-E. Persson, *Convex functions and their applications*, Springer-Verlag, New York, (2006).
[3] Z. Pavić, *Generalization of the Functional form of Jensen's inequality*, J. Adv. Inequal. Appl., **2014** (2014), 12 pages. 1, 1
[4] Z. Pavić, *Certain inequalities for convex functions*, J. Math. Inequal., **9** (2015), 1349–1364. 1
[5] J. E. Pečarič, F. Proschan, Y. L. Tong, *Convex Functions, Partial Orderings and Statistical Applications*, Academic Press, Boston, (1991).