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Robust and efficient finite element multigrid and preconditioned minimum residual solvers for the distributed elliptic optimal control problems

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Abstract

In this study, the optimal control problem is considered. This is an important class of partial differential equations constrained optimization problems. The constraint here is an elliptic partial differential equation with Neumann boundary conditions. The discretization of the optimality system produces a block coupled algebraic system of equations of saddle point form. The solution of such systems is a major computational task since they require specialized methods. Constructing robust, fast and efficient solvers for their numerical solution has preoccupied the computational science community for decades and various approaches have been developed. The approaches involve solving simultaneously for all the unknowns using a coupled block system, the segregated approach where a reduced system is solved and the approach of reducing to a fixed point form. Here the minimum residual solver with ideal preconditioning is applied to the unreduced 3 by 3 and reduced 2 by 2 coupled systems and compared to the multigrid method applied to the compact fixed-point form. The two methods are compared numerically in terms of iterative counts and computational times. The numerical results indicate that the two methods produce similar outcomes and the multigrid solver becoming very competitive in terms of the iterative counts though slower than preconditioned minimum residual solver in terms of computational times. For all the approaches, the two methods exhibited mesh and parameter independent convergence. The optimal performance of the two methods is verified computationally and theoretically.

Keywords: Elliptic optimal control problems, partial differential equations (PDEs), saddle point problems, optimality system, finite element method (FEM), multigrid method (MGM), preconditioned minimum residual method (PMINRES).

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1. Introduction

In this paper, we consider a robust and numerical solution of the distributed partial differential equation constrained optimisation problems. The PDE-optimisation problems have come to the forefront of an active field of research in applied mathematics. These are optimisation problems where there is an objective function to minimise with PDEs as constraints. There are many areas in which these problems have been applied in modern science and engineering [1] and have evolved both in theory [21, 40] and in computation [17, 21, 36]. Such interesting applications of these problems are in fluid flow, biological

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and chemical processes, finance and many more. The need for efficient simulations of the optimal control problems stimulates the development and evolvement of the numerical solution in various divisions such as Stokes control [19, 26, 28, 39], Navier-Stokes control [16, 27], wave control [37], parabolic control [19, 20, 38] and elliptic control [3, 4, 13, 14, 35]. The efficient solution of PDE-constrained optimization problems is a highly challenging task computationally. This is especially true for all the problems which typically require the solution of very large matrix systems arising from the discretization of the underlying PDEs. This paper considers optimal control problems constrained by elliptic partial differential equations. The solution process involves formulating a coupled optimality system of partial differential equations comprising of the state, control and adjoint equations. Then the discretization of the system leads to a large, block sparse linear algebraic systems of saddle point form. The properties of the system call for the use of specialised solvers which are robust, fast and efficient.

The formulation of an optimal control problem involves the cost functional to minimize subject to the constraint given by the modeling partial differential equations defined in a bounded domain $\Omega \subset \mathcal{R}^2$,

$$\min_{(u,y)} J(y,u) := \frac{1}{2} \parallel y - y_d \parallel^2_{L^2(\Omega)} + \frac{\delta}{2} \parallel u \parallel^2_{L^2(\Omega)}, \tag{1.1}$$

subject to the constraints

$$-\Delta y + y = f + u, \quad \text{in } \Omega, \qquad \frac{\partial y}{\partial n} = 0, \quad \text{on } \partial\Omega \tag{1.2}$$

with y the state variable, y_d the desired state known over the domain $\bar{\Omega}$ and u the control variable on the right hand side. The parameter δ is called the regularization parameter which measures the cost of the control and is supplied and positive. The main thrust of this work is analyze and apply the numerical methods that produce an appropriate solution to the optimality system that is to find the control and the state variables. The solution of the state variable y must satisfy the PDE over Ω and must be as close as possible to y_d in L^2-norm so that the objective function can be minimised. In this paper we consider a distributed control problem over the domain Ω which will be ill-posed for $\delta = 0$. In general the regularisation parameter needs to be determined, it can be shown that the value $\delta = 1e-2$ is most optimal [22, 32, 33]. The main thrust is to apply and analyse the numerical treatment of the problem for the approximate solution of the problem (1.1)-(1.2), mainly that the numerical solution must be of reasonable accuracy and the computational efficiency of the solver is independent of the parameters, regularization parameter δ and the discretization parameter h.

The appropriate spaces for the model variables are $V = H^1(\Omega)$ and $\mathcal{U} = L^2(\Omega)$. The optimal control problem has a unique solution $(y,u) \in V \times \mathcal{U}$ characterized by the following optimality system called the Karush-Kuhn-Tucker (KKT) system [3, 18, 21]. The first order optimality system of the PDE-optimal control problems consists state equation, adjoint equation and the control equation which is a saddle point problem as given below

1. the adjoint equation

$$-\Delta p + p = y - y_d, \quad \text{in } \Omega, \qquad \frac{\partial p}{\partial n} = 0, \quad \text{on } \partial\Omega, \tag{1.3}$$

   where the variable p is called the Lagrange multiplier which lies in the space $V = H^1(\Omega)$;
2. the control equation

$$\delta u = p; \tag{1.4}$$

3. the state equation

$$-\Delta y + y = f + u, \quad \text{in } \Omega, \qquad \frac{\partial y}{\partial n} = 0, \quad \text{on } \partial\Omega. \tag{1.5}$$

The optimality system is achieved through the Lagrange multiplier method which partitions the model problem into three equations namely in the state y, control u and the adjoint, p as enumerated above. The coupled system is the basis for the construction of the iterative solvers starting with the discretization. The finite element discretization of the optimality system (1.3)-(1.5) leads a to large scale symmetric indefinite linear systems of equations. For many applications these systems cannot be solved using direct numerical linear algebra techniques. Consequently, it is important to have specialised efficient iterative methods for solving these optimality systems. The finite element discretization method described here is based on the references [5–7, 9, 10, 12, 13]. The finite element discretization of the optimal control problem results in large symmetric sparse indefinite system whose condition number grows when the mesh size increases and the regularisation parameter approaches zero. The numerical solution process of such system presents a lot of computational challenges and this has attracted a lot of attention within the scientific computational community to construct fast, efficient solution methods.

Various formulations of the discretized system have been done which are the basis for the construction of robust, fast and efficient solvers to determine the state and the control variables. This paper will pursue the three formulations that are widely considered in literature. The first approach is to apply the multigrid method to a reduced single fixed point from of the equation in the control variable, u [30]. This approach relies on the available exact solvers for the smoothing process. Here we use the workhorse for the direct solver the $PK = LU$ factorisation for the state and adjoint equation with respect to y and p, respectively. The other two approaches are to directly apply the PMINRES to a reduced or not reduced system to solve for the variables. In this paper, the performance of the two methods are compared for the three formulations. So far the comparison of the performance of the multigrid and PMINRES solvers applied to these formulations for this class of optimal control problems has not been done. We have not seen the comparison in literature.

This work on multigrid and preconditioned minimum residual methods has been motivated by the need to effectively and efficiently solve large application problems. The multigrid method has been shown to be very efficient and successful in solving control problems [1–3, 6, 14] and elliptic partial differential equations [3, 8, 25] in an accurate and computationally efficient way. The multigrid method has been applied to problems discretized by the finite difference [3, 4] method and widely by finite element method [4, 9, 10, 30]. The effectiveness of the multigrid method depends on the correct choice of the smoothers [15]. The key features and ingredients of the multigrid method are smoothing and coarse grid correction that involves the inter-grid transfers and a solution correction step. The main results of the work are the convergence of the multigrid method in calculating the optimal control and optimal state variables in an appropriate norm which is based on the smoothing and approximation properties.

In this work we use the $PK = LU$-smoother with special attention to the performance of the multigrid method in number of iterations, computational time and L^2- and H^1-norm errors. The formulations from the finite element discretization is very large, sparse and indefinite such that well established iterative schemes such as Krylov subspace methods become very slow, stagnant or fail to converge if not conveniently preconditioned. Preconditioning enhances the convergence behaviour of the iterative schemes. The candidate method of choice for symmetric indefinite system is the minimum residual introduced [11] which has an advantage that it is parameter free method. The advantage of the PMINRES is robust for all the values of the regularisation parameter while the multigrid method convergence depends on the regularisation parameter and the small values are not satisfactory in practice. In this paper, it will shown numerically that as expected the two solvers show mesh independent convergence.

The rest of the paper is organized as follows. In Section 2 we give the discrete optimality system by finite element method and show the three formulations. In Section 3 the iterative solution techniques, the geometric method, and PMINRES are outlined. The known theoretical convergence analysis results are also outlined. In Section 4 numerical experimental and comparative analysis results on the performance multigrid and PMINRES methods for the three formulations are presented, discussed and the conclusion is given.

2. Discrete optimality system

In this study the finite element method is chosen to discretize the optimality system. The continuous optimality system (1.1)-(1.2) is transformed into the variational formulation. Find the solutions $(y, p, u) \in H^1(\Omega) \times H^1(\Omega) \times L^2(\Omega)$ for all $w \in H_1(\Omega)$ such that

$$a(y, w) = F(w), \quad \forall w \in H^1(\Omega), \tag{2.1}$$

$$a(w, p) = G(w), \quad \forall w \in H^1(\Omega), \tag{2.2}$$

$$\langle u, w\rangle_{L^2(\Omega)} = \frac{-1}{\delta}\langle p, w\rangle_{L^2(\Omega)}, \quad \forall w \in H^1(\Omega), \tag{2.3}$$

where $L^2(\Omega)$ is a space integrable functions defined as:

$$L^2(\Omega) := \{u : \int_\Omega | u |^2 \, dx < \infty\}$$

with norm

$$\| u \|_{L^2(\Omega)} := \{\int_\Omega | u |^2 \, dx\}^{\frac{1}{2}}$$

and $H^1(\Omega)$ is a Sobolev space defined as

$$H^1(\Omega) := \{u \in L^2(\Omega) : \frac{\partial u}{\partial x_j} \in L^2(\Omega), j = 1, 2, 3, \ldots, n\}$$

with norm

$$\| u \|_{H^1(\Omega)} := \{\| u \|^2_{L^2(\Omega)} + \sum_{j=1}^{n} \| \frac{\partial u}{\partial x_j} \|^2_{L^2(\Omega)}\}^{\frac{1}{2}},$$

$$a(y, w) = \int_\Omega \nabla y \cdot \nabla w \mathbf{d}x + \int_\Omega y \cdot w \mathbf{d}x,$$

$$F(w) = \int_\Omega (f + u) \cdot w \mathbf{d}x + \int_\Gamma g \cdot w \mathbf{d}s,$$

$$G(w) = \int_\Omega (y + y_d) \cdot w \mathbf{d}x + \int_\Gamma g \cdot w \mathbf{d}s.$$

The continuous optimality system is transformed to the discrete optimality system using the finite element discretization method. We define the finite dimensional spaces V_h and $\mathcal{U}_h$ subspaces of the appropriate spaces $V = H^1(\Omega)$ and $\mathcal{U} = L^2(\Omega)$ where h is the discretization parameter.

We use shape regular partition of the domain Ω into triangles. Consider a sequence of discretization with different step sizes h and level of refinement l. Fix the coarsest grid mesh size h_0 and define

$$h_l = 2^{-l} h_0, \quad l \in \mathcal{N}_0 = 0, 1, 2, \ldots,$$

where l is the level number which denotes the grid refinement level. The discrete version of the problem (2.1)-(2.3) is: Find discrete solutions $(y_h, p_h, u_h) \in V_h \times V_h \times \mathcal{U}_h$ for $w \in H^1$ such that

$$a(y_h, w_h) = \langle f_h + u_h, w_h\rangle_{L^2(\Omega)}, \quad \forall w_h \in H^1(\Omega),$$

$$a(w_h, p_h) = \langle w_h, y_h - y_{d,h}\rangle_{L^2(\Omega)}, \quad \forall w_h \in H^1(\Omega),$$

$$\langle u_h, w_h\rangle_{L^2(\Omega)} = \frac{-1}{\delta}\langle p_h, w_h\rangle_{L^2(\Omega)}, \quad \forall w_h \in H^1(\Omega).$$

The finite element method entails partitioning of the solution domain Ω into triangles, in our case that is $\Omega = \cup_i \tau_i$, we denote a set of triangular elements by $T_h = \{\tau_1, \tau_2, \tau_3, \ldots\}$ and on each element τ_i and we denote the space $P_k(\tau_i)$ of degree less than or equal to k. We specify

$$\mathbf{V}_h := \{y_h \in V \mid y_h \mid_{\tau_i} \in P_1(\tau_i), \ \ \forall \text{ elements } \tau_i\}.$$

The solution $(y_h, p_h, u_h) \in V_h \times V_h \times \mathcal{U}_h$ is uniquely determined by specifying components of y_h, u_h and p_h on the nodes of the elements. The finite element method results in the coupled linear algebraic system which has to be solved by the appropriate solvers. The resulting discrete KKT system is

$$\begin{pmatrix} M_h & O & L_h^T \\ O & \delta M_h & -M_h^T \\ L_h & -M_h & O \end{pmatrix} \begin{pmatrix} y_h \\ u_h \\ p_h \end{pmatrix} = \begin{pmatrix} -M_h y_{d,h} \\ O \\ M_h f_h \end{pmatrix}, \tag{2.4}$$

where define $L_h = A_h + M_h$ symmetric that is $L_h = L_h^T$ and M_h is symmetric ($M_h = M_h^T$) with

$$A_h = [a_{ij}], \ \ a_{ij} = \int_\Omega (\nabla \phi_i \cdot \nabla \phi_j)_{i,j=1,\ldots,n}, \qquad M_h = [m_{ij}], \ \ m_{ij} = \int_\Omega (\phi_i \cdot \phi_j))_{i,j=1,\ldots,n}.$$

The linear algebraic system can be represented as

$$\mathcal{K}_h \mathbf{x}_h = \mathbf{b}_h, \tag{2.5}$$

where $\mathcal{K}_h := \begin{pmatrix} M_h & O & L_h \\ O & \delta M_h & -M_h \\ L_h & -M_h & O \end{pmatrix}$, $\mathbf{x} := \begin{pmatrix} y_h \\ u_h \\ p_h \end{pmatrix}$ and $\mathbf{b} := \begin{pmatrix} -M_h y_{d,h} \\ O \\ M_h f_h \end{pmatrix}$ is the first formulation of the unreduced system (2.4) is very large and each of the blocks are discretization of the PDE are sparse. The second formulation is the reduced system obtained by substituting u_h in the state equation by means of the control equation $u_h = -\delta^{-1} p_h$, the reduced KKT system can be stated as

$$\tilde{\mathcal{K}}_h = \begin{pmatrix} M_h & L_h \\ L_h & -\delta^{-1} M_h \end{pmatrix} \begin{pmatrix} y_h \\ p_h \end{pmatrix} = \begin{pmatrix} -M_h y_{d,h} \\ M_h f_h \end{pmatrix}. \tag{2.6}$$

The formulations (2.5) and (2.6) are solved using the all at once PMINRES method for the control and the state variables. The third formulation involve further expressing the KKT system in compact fixed point form.

Let the discrete control be v_h already known, the procedure involves calculating the discrete state $y_h(v_h)$ from the discrete control v_h and the discrete adjoint $p_h(v_h)$ from the discrete state $y_h(v_h)$. Then we can define the optimal discrete control u_h for the distributed control problem as

$$u_h = -\delta^{-1} \cdot p_h(v_h).$$

The mapping $u \mapsto y(u) \mapsto p(u) \mapsto -\delta^{-1} p(u)$ is affine and defined on a linear operator $\mathcal{G}$ such that the optimal control in (2.4) can have the representation

$$-\delta^{-1} p(u) = \mathcal{G} u + q.$$

The discrete optimality system (2.4) simplifies to

$$M_h u_h = -\frac{1}{\delta} L_h^{-1} M_h \Big[L_h^{-1} M_h (u_h + f_h) - y_{d,h} \Big]. \tag{2.7}$$

The discrete optimality system (2.7) is equivalent to the formulation in compact form of the fixed point iteration

$$(I_h - \mathcal{G}_h) u_h = q_h \tag{2.8}$$

with

$$\mathcal{G}_h = -\frac{1}{\delta} L_h^{-1} M_h (L_h^{-1} M_h), \qquad q_h = \frac{1}{\delta} L_h^{-1} M_h \Big[L_h^{-1} M_h f_h - y_{d,h} \Big],$$

$\mathcal{G}_h$ is the discrete operator of $\mathcal{G}$. The knowledge of the entries of the matrix $\mathcal{G}_h$ is not necessary except at the coarsest level $(h_0, l = 0)$. The compact fixed point system (2.8) is going to be solved using the multigrid method with PK = LU-decomposition smoother.

3. Iterative solvers

The matrices $\mathcal{K}_h$ (2.5) and $\tilde{\mathcal{K}}_h$ (2.6) are symmetric 3×3 and 2×2 block of saddle point forms respectively. The other formulation is a compact fixed point form (2.8) for the control variable. The three formulations need to be solved efficiently and this section discusses the two methods that are well suited for this purpose.

3.1. Multigrid methods

In this section the multigrid procedure for solving the compact fixed point formulation of state-constrained elliptic optimal control problems is presented. A typical multigrid method uses a sequence of nested discretization grids of increasing fineness $\Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega_l \subset \Omega$. Associated with the sequence of grids is a sequence of finite element spaces $V_1 \subset V_2 \subset \cdots \subset V_l \subset V = H^1(\Omega)$ of subspaces of the finite dimensional subspaces V_h for the control. The subspace V_l is defined on the sequence of grids $l \in \{0, 1, 2, 3, \ldots, l_{max}\}$ with mesh sizes $h_0, h_1, h_2, \ldots, h_{max}$ with $h_{l+1} := \frac{1}{2}h_l$. The multigrid methods belong to a class of optimal order methods for solving linear systems emanating from the discretization techniques like the finite element method [15, 33, 36, 41]. The multigrid methods combine iterative methods to smooth the error with the correction derived from the coarse grid computation. We use the W-cycle multigrid. The starting point of the multigrid concept is the observation that classical iteration methods have some smoothing properties. Let the smoothing algorithm be represented by $\mathcal{S}_l$ such that we get an update

$$u_l^{k+1} = \mathcal{S}_l(u_l^k, q_l), \tag{3.1}$$

where u_l^k is the initial control/control solution at level l. In this work the operator $\mathcal{S}_l$ represents $PK = LU$-decomposition using sequence of the optimality system to smooth the error of the control. The main goal is to find the pair (y_l, u_l) of the discrete control and the discrete state variables at the finest level l. To calculate this, an multigrid algorithm is developed over the discrete compact fixed point equation that characterizes the discrete optimal control.

3.2. Multigrid algorithm

In this section the multigrid algorithm is presented to solve the fixed point formulation of the discrete optimality system. The multigrid method studied here we refer to [14, 29, 36]. The main ingredients of the multigrid iteration are the smoothing and coarse grid correction steps. The coarse grid correction process is carried out by a restriction, coarse grid solve, interpolation. This means that at each grid level l, the discrete problem is represented by (2.8) is solved. Let $l \in \mathcal{N}_0$ be the refinement levels. The summary below gives the synopsis of the steps in the multigrid solver.

- At the coarsest level, $l = 0$, the equation $u_l = \mathcal{G}_l u_l + q_l$ where u_l is the desired control, is solved exactly by $PK = LU$-decomposition of $I_0 - \mathcal{G}_0$. At this level the entries matrix $\mathcal{G}_0$ are known by evaluation of $\mathcal{G}_0 v_0 + q_0$ for $q_0 = 0$ and all unit vectors v_0.
- **Smoothing:** If the level is not the coarsest one, $l > 0$, with the initial control u_l^k at the level l (finest level) we apply (3.1) to get a smoother variable

$$u_l^{k+\frac{1}{2}} = \mathcal{G}_l u_l^k + q_l,$$

where q_l and $\mathcal{G}_l$ are defined in (2.8). The equation involving the expression $\mathcal{G}_l u_l$ separately. The explicit representation of the smoothing operator $\mathcal{S}$ (3.1) is illustrated as follows

1. solve the equation $u_l^{k_1} = \mathcal{G}_l u_l^k$ using $PK = LU$ decomposition of the sequence of linear algebraic systems below
   - choose the initial control value $u_l^{k_0}$ and solve for the state variable y_l using the equation $L_l y_l = M_l u_l^{k_0}$ where L_l and M_l are the stiffness and mass matrices respectively at level l;

- with y_l solve for the adjoint variables using the equation $L_l p_l = M_l y_l$ with same matrices defined above;
- with p_l find the new control using the relation $u_l^{k_1} = -\delta^{-1} p_l$;

2. finally, we get a smoother variable $u_l^{k+\frac{1}{2}} = u_l^{k_1} + q_l$;
3. repeat the smoothing process (1) with the initial $u_l^{k+\frac{1}{2}}$ to the equation, to get $u_l^{k_2} = \mathcal{G}_l u_l^{k+\frac{1}{2}} + q_l$.

- **Calculating the defect:** After the smoothing process we calculate the corresponding defect

$$d_l = (I_l - \mathcal{G}_l)u_l^{k+\frac{1}{2}} - q_l = u_l^{k+\frac{1}{2}} - \mathcal{G}_l u_l^{k+\frac{1}{2}} - q_l = \mathcal{G}_l\left[u_l^k - u_l^{k+\frac{1}{2}}\right] = u_l^{k+\frac{1}{2}} - u_l^{k_2}.$$

- **Restrict the defect:** This is an inter-grid transfer process. The process transfers the defect from the finer grid to a coarser grid. By a suitable restriction $r_{l,l-1} : V_0^l \to V_0^{l-1}$ to a coarser grid with $V_0 = L^2(\Omega)$, we obtain the result

$$d_{l-1} = r_{l,l-1} d_l \in V_0^{l-1}.$$

- Approximate on the coarser grid that is on the level $l-1$ by

$$w_{l-1} = (I_{l-1} - \mathcal{G}_{l-1})^{-1} d_{l-1}$$

two iterations of the multigrid method on the level $l-1$.

- **Prolongate the approximate:** The prolongation/interpolation is an inter-grid transfer process. The process transfers the smooth error from the coarser grid to a finer grid. It is a linear mapping. By a suitable prolongation $p_{l-1,l} : V_0^{l-1} \to V_0^l$ to a finer grid and coarse grid correction, we obtain the result

$$u_l^{k+1} = u_l^{k+\frac{1}{2}} - p_{l-1,l} w_{l-1}.$$

The above description gives the two grid algorithm. Applying the two grid recursively results in multigrid algorithm. Defining the multigrid method (MGM) recursively. We define the algorithm MGM_l at level $l > 0$ by means of the algorithm MGM_{l-1} corresponding to the coarser grid. Now we define the multigrid algorithm.

We define the multigrid algorithm at level l as $MGM_l(u_l^{new}, u_l^{old}, q_l)$, where

- u_l^{new} is the output of one step of the multigrid algorithm at level l;
- u_l^{old} is the input at level l;
- q_l is defined implicitly by $f_l, g_l, y_{d,l}$ at level l;
- $u_l^k := u_l^{old} \mapsto u_l^{k+1} =: u_l^{new}$.

Algorithm 3.1. $MGM_l(u_l^{new}, u_l^{old}, q_l)$.
if $l = 0$ (coarsest grid)

$$u_0 = (I_0 - \mathcal{G}_0)^{-1} q_0$$

else $l > 0$ define $MGM_l(u_l^{new}, u_l^{old}, q_l)$.

1. **Smoothing**

$$\widetilde{u}_l = \mathcal{G}_l u_l^{old} + q_l.$$

- **Defect computation**

$$d_l = \widetilde{u}_l - \mathcal{G}_l \widetilde{u}_l - q_l.$$

2. **Restrict the defect**

$$d_{l-1} = r_{l,l-1} d_l.$$

3. **Approximate solution**
$$v_{l-1} = \mathcal{G}_{l-1} v_{l-1} + d_{l-1}.$$
4. Applying two iterations of MGM_{l-1} at the recursive call:
   - Set $v_{l-1}^{(0)} = 0$.
   - Compute
   $$v_{l-1}^{(1)} = MGM_{l-1}(v_{l-1}^{(1)}, v_{l-1}^{(0)}, d_{l-1}).$$
   - Compute
   $$v_{l-1}^{(2)} = MGM_{l-1}(v_{l-1}^{(2)}, v_{l-1}^{(1)}, d_{l-1}).$$
5. **Correction step**.
   Define the new iterate by
$$u_l^{new} := \widetilde{u}_l - p_{l-1,l} v_{l-1}^{(2)}.$$

In this work we use a multigrid W-cycle which starts at a finest level l. The finest level solution is then transferred to next coarser level (restriction). After some relaxation (smoothing) cycles on the coarse level, the solution is then restricted to next coarser level until the coarsest level is reached. The solution obtained at the coarsest level is than interpolated back to the finer level (prolongation). The solution from this finer level is interpolated to next finer level after some relaxation iterations. The solution is prolonged till the finest level is reached. The whole process is repeated until satisfactory convergence is reached.

3.3. Analysis of the multigrid algorithm

The convergence analysis of the multigrid method presented here follows the format in [13, 36]. An iteration of single multigrid step consists of a combination of smoothing step and a coarse grid correction step. The following relations result from the application of each step of the multigrid method.

- The exact solution is given by the relation
$$u_l = \mathcal{G}_l u_l + q_l.$$
- **Smoothing:** is done by the application of the relation
$$u_l^k \mapsto \widetilde{u}_l := \mathcal{G}_l u_l^k + q_l, \tag{3.2}$$
apply $\mathcal{G}$ m-times $m > 1$ we have the error
$$\widetilde{u}_l - u_l = \mathcal{G}^m (u_l^k - u_l) = \mathcal{G}^m \triangle u_l^k. \tag{3.3}$$
- **Calculating the defect**.
$$d_l = (I_l - \mathcal{G}_l)\widetilde{u}_l - q_l. \tag{3.4}$$
- From the exact solution and the defect relations we get
$$u_l = \widetilde{u_l} - (I_l - \mathcal{G}_l)^{-1} d_l. \tag{3.5}$$
From this relation we have
$$d_l = (I_l - \mathcal{G}_l)(\widetilde{u_l} - u_l). \tag{3.6}$$
- **Coarse grid correction**: Produces the relation for the new iterate
$$\begin{aligned} u_l^{k+1} &= \widetilde{u_l} - p_{l-1,l}(I_{l-1} - \mathcal{G}_{l-1})^{-1} r_{l,l-1} d_l \\ &= \widetilde{u_l} - p_{l-1,l}(I_{l-1} - \mathcal{G}_{l-1})^{-1} r_{l,l-1}(I_l - \mathcal{G}_l)(\widetilde{u_l} - u_l). \end{aligned} \tag{3.7}$$

- The error of the new iterate, subtract the exact solution from both sides

$$\begin{aligned} u_l^{k+1} - u_l &= \widetilde{u_l} - p_{l-1,l}(I_{l-1} - \mathcal{G}_{l-1})^{-1} r_{l,l-1}(I_l - \mathcal{G}_l)(\widetilde{u_l} - u_l) - u_l, \\ \triangle u_l^{k+1} &= \widetilde{u_l} - u_l - p_{l-1,l}(I_{l-1} - \mathcal{G}_{l-1})^{-1} r_{l,l-1}(I_l - \mathcal{G}_l)(\widetilde{u_l} - u_l) \\ &= \mathcal{G}^m \triangle u_l^k - p_{l-1,l}(I_{l-1} - \mathcal{G}_{l-1})^{-1} r_{l,l-1}(I_l - \mathcal{G}_l)\mathcal{G}^m \triangle u_l^k \\ &= \left[I_l - p_{l-1,l}(I_{l-1} - \mathcal{G}_{l-1})^{-1} r_{l,l-1}(I_l - \mathcal{G}_l)\right]\mathcal{G}^m \triangle u_l^k. \end{aligned}$$

- The above relation for the error can be expressed as

$$\triangle u_l^{k+1} = \mathcal{M}_l \triangle u_l^k, \tag{3.8}$$

where $\mathcal{M}_l$ is the iteration matrix given by

$$\mathcal{M}_l = \left[I_l - p_{l-1,l}(I_{l-1} - \mathcal{G}_{l-1})^{-1} r_{l,l-1}(I_l - \mathcal{G}_l)\right]\mathcal{G}^m. \tag{3.9}$$

- The idea is to establish the convergence rates for our multigrid algorithm which are defined by the relation

$$\frac{\| \triangle u_l^{k+1} \|_0}{\| \triangle u_l^k \|_0} = \| \mathcal{M}_l \|_{V_0 \to V_0}. \tag{3.10}$$

The convergence property of the iterative process depends on $\| \mathcal{M}_l \|_{V_0 \to V_0}$.

Theorem 3.2. *Let* $l, l-1 \in \mathcal{N}$ *and* $0 < \sigma \leqslant \frac{n_{l-1}}{n_l} < 1$, *then from the conditions* (3.2)-(3.10) *it holds*

$$\| \mathcal{M}_l \|_{V_0 \to V_0} \leqslant C n_l^{-\sigma}.$$

The method converges for sufficiently large n_l *which depends on the mesh size and* C *is independent of* l *and the smoothing steps. The method converges proportional to the mesh size.*

Conclusion 3.3. The rate of convergence of the multigrid method on the level $l \in \mathcal{N}_0$ is proportional to h_l^β for some $\beta > 0$. This means that the estimate

$$\| u_l^{k+1} - u_l \|_{V_0^l} = C h_l^\beta \| u_l^k - u_l \|_{V_0^l},$$

where u_l is the discrete exact solution, holds for two consecutive iterates. Since u_l is unknown we use the continuous control u to get the convergence rates of the multigrid algorithm and conclude that the rate of convergence is proportional to by a factor h_l^β which means $\beta > 0$.

In this work we consider $V_0 = L^2(\Omega)$ and we also check convergence when $V_0 = H^1(\Omega)$.

3.4. Preconditioned minimum residual solver

In this section we turn our attention to the algorithmic structure of the PMINRES for solving the discretized linear algebraic systems, not reduced (2.5) and reduced (2.6) under consideration in this study. The main goal being to find the pair (y_l, u_l) of the discrete state and the discrete control variables at the level of refinement l. The PMINRES method introduced by [29] has the appeal of not requiring any parameters for making the algorithm efficient. The preconditioned MINRES to solve the saddle point problems was recommended in [11]. The PMINRES was also used in [24] to solve the Stokes equations and we follow the similar format in this paper in the solution of the optimal control problems. The PMINRES method is based on the following residual minimization problem:

Given the initial guess $\mathbf{x}_0$, determine $\mathbf{x}_k \in \mathbf{x}_0 + \mathcal{K}^k(\mathcal{M}; \mathbf{r}_0)$ such that

$$\| \mathcal{M}\mathbf{x}_k - \mathbf{b} \| = \min(\| \mathcal{M}\mathbf{x}_k - \mathbf{b} \| \, | \, \mathbf{x}_0 + \mathcal{K}^k(\mathcal{M}; \mathbf{r}_0)),$$

where $\mathbf{r}_0 = \mathbf{b} - \mathcal{M}\mathbf{x}_0$ and $\mathcal{K}^k(\mathcal{M}, \mathbf{r}_0) = span\left\{\mathbf{r}_0, \mathcal{M}\mathbf{r}_0, \mathcal{M}^2\mathbf{r}_0, \ldots, \mathcal{M}^{k-1}\mathbf{r}_0\right\}$ is the Krylov subspace. Below is the MINRES algorithm for computing the iterate x^k as given in [11].

In this study a symmetric positive definite block preconditioner $\widehat{\mathcal{M}}$ of $\mathcal{M}$ as scrutinized in [24, 31] and $\bar{\mathcal{M}}$ of $\bar{\mathcal{M}}$, respectively is considered. Such block preconditioners are given as

$$\widehat{\mathcal{M}} := \begin{bmatrix} \widehat{A} & O \\ O & \widehat{S} \end{bmatrix}$$

for the reduced formulation where the Schur complement $S = LM^{-1}L^T$. The preconditioners of such form are well studied when coefficient matrix comes form the finite element of the PDEs [11, 29, 31] and were also used in [32, 33] and

$$\bar{\mathcal{M}} = \begin{bmatrix} \widehat{M} & O & O \\ O & \delta\widehat{M} & O \\ O & O & \widehat{S} \end{bmatrix},$$

where the Schur complement $S = LM^{-1}L^T + \frac{1}{\delta}M$. In both cases the ideal preconditioning in used which entails the exact application of the preconditioners $\hat{A}$ and the Schur compliment $\hat{S}$ such that the left preconditioned system becomes

$$\widehat{\mathcal{M}}^{-1}\mathcal{M}x = \widehat{\mathcal{M}}^{-1}b,$$

which is

$$\widetilde{\mathcal{M}}x = \widetilde{b}. \tag{3.11}$$

The residual minimization criteria ia applied to the preconditioned system (3.11):

Given the initial guess $x_0 \in \mathcal{R}^{n+m}$ and initial residual $\widetilde{r}_0 = \widetilde{b} - \widetilde{\mathcal{M}}x_0$, determine $x_k \in x_0 + \mathcal{K}^k(\widetilde{\mathcal{M}}; \widetilde{r}_0)$ such that

$$\| \widetilde{\mathcal{M}}x_k - \widetilde{b} \| = \min(\| \widehat{\mathcal{M}}x_k - \widetilde{b} \| \mid x_0 + \mathcal{K}^k(\widetilde{\mathcal{M}}; \widetilde{r}_0)),$$

where $\widetilde{r}_0 = \widetilde{b} - \widetilde{\mathcal{M}}x_0$ and $\mathcal{K}^k(\widetilde{\mathcal{M}}, \widetilde{r}_0) = \text{span}\left\{\widetilde{r}_0, \widetilde{\mathcal{M}}\widetilde{r}_0, \widetilde{\mathcal{M}}^2\widetilde{r}_0, \ldots, \widetilde{\mathcal{M}}^{k-1}\widetilde{r}_0\right\}$ is the Krylov subspace and $\langle \cdot, \cdot \rangle_{\widehat{M}} = \langle \widehat{M}\cdot, \cdot \rangle$.

Then the preconditioned residual $\widehat{\mathcal{M}}^{-1}(b - \mathcal{M}x)$ is minimized in $\| \cdot \|_{\widehat{M}}$ over transformed Krylov subspace. The implementation of the PMINRES requires per iteration one evaluation of $\widehat{\mathcal{M}}^{-1}z$ for a given z and one multiplication by $\hat{A}$. The evaluation of the preconditioner is achieved by solving a linear system $z = \widehat{\mathcal{M}}y$. The preconditioned minimum residual method is outlined in the algorithm below.

Algorithm 3.4 (The PMINRES algorithm)**.**

$\mathbf{v}_0 = \mathbf{0}, \quad \mathbf{w}_0 = \mathbf{0}, \quad \mathbf{w}_1 = \mathbf{0}$

Choose $\mathbf{x}_0$, compute $\mathbf{v}_1 = \mathbf{b} - \mathcal{M}\mathbf{x}_0$, set $\gamma_1 = \| \mathbf{v}_1 \|$

Solve $\widehat{M}\mathbf{z}_1 = \mathbf{v}_1$ set $\gamma_1 = \sqrt{\langle \mathbf{z}_1, \mathbf{v}_1 \rangle}$

set $\eta = \gamma_1$, $s_0 = s_1 = 0$, $c_0 = c_1 = 1$

for i = 1,2... until convergence do

$\mathbf{z}_i = \frac{\mathbf{z}_i}{\gamma_i}$

$\delta_i = \langle \mathcal{M}\mathbf{z}_i, \mathbf{z}_i \rangle$

$\mathbf{v}_{i+1} = \mathcal{M}\mathbf{v}_i - (\frac{\delta_i}{\gamma_i})\mathbf{v}_i - (\frac{\gamma_i}{\gamma_{i-1}})\mathbf{v}_{i-1}$

Solve $\widehat{M}\mathbf{z}_{i+1} = \mathbf{v}_{i+1}$

$\gamma_1 = \sqrt{\langle \mathbf{z}_{i+1}, \mathbf{v}_{i+1} \rangle}$

$\alpha_0 = c_i\delta_i - c_{i-1}s_i\gamma_i$

$\alpha_1 = \sqrt{\alpha_0^2 + \gamma_{i+1}^2}$

$\alpha_2 = s_i\delta_i + c_{i-1}c_i\gamma_i$
$\alpha_3 = s_{i-1}\gamma_i$
$c_{i+1} = \frac{\alpha_0}{\alpha_1}, \quad s_{i+1} = \frac{\gamma_{i+1}}{\alpha_1}$
$\mathbf{w}_{i+1} = \frac{(\mathbf{z}_i - \alpha_3\mathbf{w}_{i+1} - \alpha_2\mathbf{w}_j)}{\alpha_1}$
$\mathbf{x}_i = \mathbf{x}_{i-1} + c_{i+1}\eta\mathbf{w}_{i+1}$
$\eta = -s_{i+1}\eta$
Test for convergence
end for

The main convergence results for PMINRES method are due to [30, 31, 35]. The convergence analysis is based on the eigenvalue analysis of the preconditioned matrix system which plays a crucial role. The spectral analysis of the preconditioner influences the convergence properties of the iterative scheme connected with the approximations of $\widehat{A}$ and $\widehat{S}$. We seek that the eigenvalues of the preconditioned system are well clustered and distributed provided that the eigenvalues of $\widehat{A}^{-1}A$ and $\widehat{S}^{-1}S$ are so. The convergence of the iterative scheme is driven mostly by the ratio between the largest and smallest eigenvalue of the preconditioned system. The theorem below gives convergent analysis of the PMINRES method.

Theorem 3.5. *Let $\mathcal{M} \in \mathcal{R}^{(n+m)\times(n+m)}$ be symmetric and $\widehat{\mathcal{M}} \in \mathcal{R}^{(n+m)x(n+m)}$ be symmetric and positive definite. For x_k, $k > 0$ computed in the preconditioned MINRES algorithm we define $\widehat{r}_k = \widehat{\mathcal{M}}^{-1}(b - \mathcal{M}x_k)$. Then the following holds*

$$\| \widehat{r}_k \|_{\widehat{\mathcal{M}}} = \min_{p_k \in \mathcal{P}_k; p_0 = 1} \| p_k(\widehat{\mathcal{M}}^{-1}\mathcal{M})\widehat{r}_0 \|_{\widehat{\mathcal{M}}} \leqslant \min_{p_k \in \mathcal{P}_k; p_0 = 1} \max_{\lambda \in \sigma(\widehat{\mathcal{M}}^{-1}\mathcal{M})} |p_k(\lambda)| \, \| \widehat{r}_0 \|_{\widehat{\mathcal{M}}} .$$

The maximum is over the eigenvalues λ of $\widehat{\mathcal{M}}^{-1}\mathcal{M}$ defined in Theorem 3.5. For the proof we refer to [35]. Theorem 3.5 gives that the rate of convergence of the preconditioned MINRES method depends on $\sigma(\widehat{\mathcal{M}}^{-1}\mathcal{M})$.

4. Numerical results

In this section we present numerical results which have been obtained from using the multigrid method to solve the compact fixed point formulations and the PMINRES to solve the reduced and not reduced formulations. The PMINRES is used with the block preconditioner which are approximated by the exact approximations. For the multigrid results of the experiments, use $m_1 = m_2 = 1$ pre- and post-smoothing steps. This means that one multigrid cycle uses $m_1 + m_2$ =2 iterations of the smoothing algorithm on the finest level. Simulations and implementations were performed on a Windows 10 platform with 2.6 GHz speed intel(R) processor by using Matlab 7 programming language.

The Table 1 below shows an example of the refinement levels and the number degrees of freedom on each level which corresponds to the size of each entry of the block matrix.

Table 1: Refinement levels and number of nodes.

Refinement Level (l)	1	2	3	4	5	6	7
Mesh size (h_l)	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$
Nodes (number of grid points)	9	25	289	1089	4225	16641	66049

The accuracy of the approximation is obtained by applying piecewise linear functions is $O(h^2)$ in L^2-norm and $O(h)$ in H^1-norm. We introduce the discrete L^2- and H^1-norms of the error with respect to the optimal control variable u_l. We approximate the element with both the numerical solution u_l and the exact solution u at the centroid $(x(s), y(s))$ of each of the triangles $T_i \in T_h$. Then the L^2-norm of the error is

$$\| u - u_l \|_{L^2} = \sqrt{\sum \text{area}(T_i) \cdot (u(x(s), y(s)) - u_l^n(x(s), y(s)))^2}$$

and n is the n^{th} is the iteration index at level l. Similarly the H^1-norm discretization error at the n^{th}

iteration and l is

$$\| u - u_l \|_{H^1} = \sqrt{\sum \text{area}(T_i) \cdot (u_x(x(s), y(s)) - u^n_{x,l}(x(s), y(s)))^2 - (u_y(x(s), y(s)) - u^n_{y,l}(x(s), y(s)))^2},$$

where u_x, u_y are the partial derivatives of the exact solution and $u_{x,l}, u_{y,l}$ are the gradients of the finite element solution. We have also established that the convergence rate of the multigrid algorithm is proportional to h_l^β, $\beta > 0$. Theoretically $\beta = 2$ for L^2-norm error and $\beta = 1$ for H^1-norm error.

The exact solutions for the distributed optimality system and $\overline{\Omega} = (0,1)^2 \subset \mathcal{R}^2$ are

$$y = \cos(\pi x_1)\cos(\pi x_2), \qquad u = (2\pi^2 + 1)\cos(\pi x_1)\cos(\pi x_2), \qquad p = -\delta(2\pi^2 + 1)\cos(\pi x_1)\cos(\pi x_2).$$

We get the corresponding desired state as

$$y_d = (\delta(2\pi^2 + 1)^2 + 1)\cos(\pi x_1)\cos(\pi x_2).$$

We will consider the initial control $u_0 = 0$. The goal is to find the optimal control so that the optimal state is as close to target state as possible and this is achieved by solving the compact fixed point problem (2.8) and the coupled systems, not reduced (2.5) and reduced (2.6).

For this work the value $\delta = 1e-2$ produces optimal results [22, 32, 33]. It can be shown that for large values of weighting factor the discretization error grows and that at any grid level the methods diverge for the values $\delta \leqslant 1e-3$ and less.

Table 2: The results of MGM, $\delta = 1e-2$, tolerance $= 1e-8$.

level	iter/time	Control	State	L^2− control error	H^1-control error	L^2−error state	Obj. func
l	k(sec)	$\| u_l \|_{L^2}$	$\| y_l \|_{L^2}$	$\| u - u_l \|_{L^2}$	$\| u - u_l \|_{H^1}$	$\| y_d - y_l \|_{L^2}$	$J(y_l, u_l)$
1	2 (0.906)	2.506e-1	1.355e-2	3.0959e-1	7.6929e-2	7.141e-3	4.823e-3
2	6(0.984)	1.120e-1	5.245e-3	7.9596e-1	3.8782e-2	2.123e-3	1.173e-3
3	5 (1.047)	3.468e-2	1.657e-3	2.0051e-2	1.9438e-2	1.142e-4	2.305e-4
4	4 (2.093)	9.441e-3	4.545e-4	5.0226e-3	8.7728e-3	8.333e-6	5.138e-5
5	3(2.047)	2.449e-3	1.180e-4	1.2560e-3	4.8653e-3	5.572e-7	1.252e-5
6	3 (3.172)	6.228e-4	3.002e-5	3.1443e-4	2.4326e-3	3.592e-8	3.132e-6

We present in Table 2 the results on obtained from the form the multigrid methods applied to a compact fixed point formulation.

Table 3: The results of PMINRES for unreduced formulation, $\delta = 1e-2$, tolerance $= 1e-8$.

level	iter/time	Control	State	L^2− control error	H^1-control error	L^2−error state	Obj. func
l	k/sec	$\| u_l \|_{L^2}$	$\| y_l \|_{L^2}$	$\| u - u_l \|_{L^2}$	$\| u - u_l \|_{H^1}$	$\| y_d - y_l \|_{L^2}$	$J(y_l; u_l)$
1	13(0.016)	2.506e-1	1.355e-2	3.0959e-1	7.6919e-2	7.141e-3	4.823e-3
2	13(0.063)	1.12e-1	5.244e-3	7.9596e-2	3.8782e-2	1.123e-3	1.173e-3
3	15(0.125)	3.468e-2	1.65e-3	2.0051e-2	1.9438e-2	1.142e-4	2.305e-4
4	11(0.321)	9.441e-3	4.542e-4	5.0226e-3	9.7728e-3	8.333e-6	5.138e-5
5	9(1.180)	2.449e-3	1.180e-4	1.2560e-3	4.8653e-3	5.572e-7	1.252e-5
6	9(6.390)	6.228e-4	3.002e-5	3.1443e-4	2.4326e-3	3.952e-8	3.132e-6

We present in Table 3 the results on obtained from the form the PMINRES method applied to an unreduced formulation. We present in Table 4 the results on obtained from the form the PMINRES method applied to a reduced formulation.

The results in Tables 2, 3, and Table 4 clearly show that the PMINRES solver applied to the reduced and unreduced coupled systems and the MGM applied to the compact fixed point form produce similar outcomes for the control, state in L^2-norm, similar results also in the L^2- and H^1- norm errors and the minimum value of the objective function. Now what is critical is to compare the methods in terms of iteration counts, computing times and check the characteristics of the convergence errors in L^2- and H^1-norms.

Table 4: The results of PMINRES for reduced formulation, $\delta = 1e-2$, tolerance $= 1e-8$.

level	iter/time	Control	State	L^2- control error	H^1-control error	L^2-error state	Obj. func
l	k/sec	$\| u_l \|_{L^2}$	$\| y_l \|_{L^2}$	$\| u-u_l \|_{L^2}$	$\| u-u_l \|_{H^1}$	$\| y_d-y_l \|_{L^2}$	$J(y_l;u_l)$
1	13(0.031)	2.50e-1	1.355e-2	3.0959e-1	7.6919e-2	7.141e-3	4.823e-3
2	13(0.047)	1.120e-1	5.244e-3	7.9596e-2	3.8782e-2	1.123e-3	1.173e-3
3	15(0.078)	3.468e-2	1.65e-3	2.0051e-2	1.9438e-2	1.142e-4	2.305e-4
4	11(0.23)	9.441e-3	4.542e-4	5.0226e-3	9.7728e-3	8.333e-6	5.138e-5
5	9(0.790)	2.449e-3	1.180e-4	1.2560e-3	4.8653e-3	5.572e-7	1.252e-5
6	7(3.510)	6.228e-4	3.002e-5	3.1443e-3	2.4326e-3	3.952e-8	3.132e-6

Table 5: Number of iterations and CPU time for MGM and PMINRES for comparison, tolerance = 10^{-8}.

Levels	Multigrid		PMINRES reduced		PMINRES not reduced	
	iter	cpu(sec)	iter	cpu(sec)	iter	cpu(sec)
$\frac{1}{8}$	2	0.906	13	0.031	13	0.016
$\frac{1}{16}$	6	0.984	13	0.047	13	0.063
$\frac{1}{32}$	5	1.047	13	0.078	13	0.125
$\frac{1}{64}$	4	2.093	11	0.230	11	0.321
$\frac{1}{128}$	3	2.047	9	0.790	9	1.180
$\frac{1}{256}$	3	3.172	7	3.500	9	6.390

The Table 5 shows the iteration counts and computational times of the two methods for the three formulations. The results in Table 5 shows that both methods are fast, robust and efficient with the changes in the discretization parameter. As the number problem size increases both methods have shown parameter independent convergence and the number of iterations decreases with a small increase in the computational times in seconds. The PMINRES converges in few seconds as compared to the MGM for both formulations. However the MGM is slightly faster in terms of iterative counts. The two methods are recommended for solving large linear indefinite system of saddle point form because the two methods have shown parameter independent convergence.

Table 6: Convergence results $\delta = 1e-2$, tolerance $= 1e-8$.

level	L^2- error control	Ratio	H^1- error	Ratio
l	$\| u-u_l^k \|_{L^2}$	$\frac{\|u-u_{l+1}\|_{L^2}}{\|u-u_l\|_{L^2}}$	$\| u-u_l \|_{H^1}$	$\frac{\|u-u_{l+1}\|_{H^1}}{\|u-u_l\|_{H^1}}$
1	3.0959e-1		7.6919e-2	
2	7.9596e-2	0.2571	3.8782e-2	0.5042
3	2.0051e-2	0.2519	1.9438e-3	0.5012
4	5.0226e-3	0.2505	9.7728e-3	0.5005
5	1.256e-3	0.2501	4.8653e-3	0.5001
6	3.1434e-4	0.2500	2.4326e-3	0.5000

The results in Table 6 summaries the L^2- and H^1-norm errors of the MGM and PMINRES for the three formulations at each grid level. The discretization parameter h_l is the mesh width at level l. The finite element discretization and the two methods used convergence with the second order L^2-norm error and first order H^1-norm error. Therefore the reduction in L^2-norm error is by a factor of $\frac{1}{4}$ and in H^1-norm is by a factor of $\frac{1}{2}$ as indicated in the ratio column. This agrees and confirms the convergence Theorem 3.2 and theoretical results that L^2-norm error shows quadratic convergence that is $O(h^2)$ and H^1-norm error shows linear convergence that is $O(h)$. This means that further refinement reduces the error and as the step size becomes smaller and the iteration error approaches zero. All these confirms what the theory says. This is also clear on Figure 2 on the L^2 and H^1-norm errors.

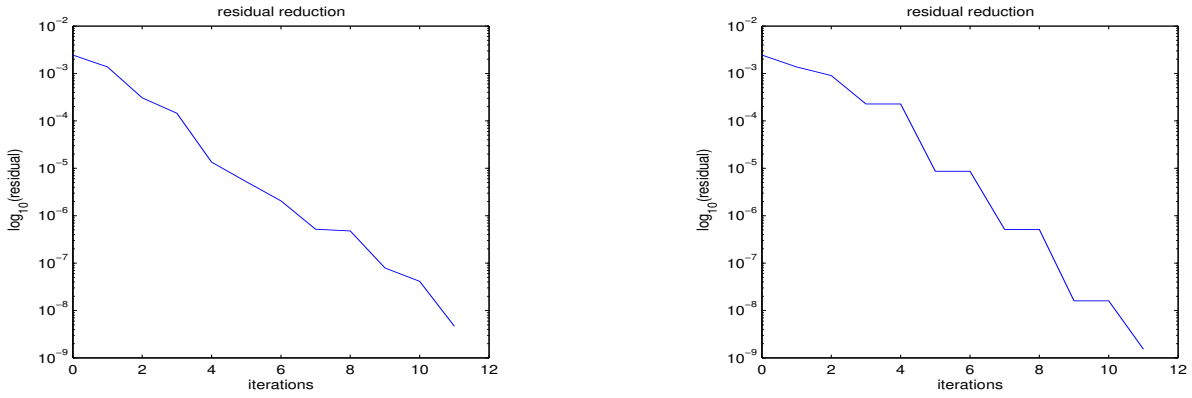

Figure 1: Error reduction from PMINRES reduced (left) and PMINRES unreduced (right) at level 4.

The Figure 1 shows the error reduction from the PMINRES for both reduced and unreduced formulations at level 4. This shows that the error reduction is very fast up to until the maximum number of iterations is reached.

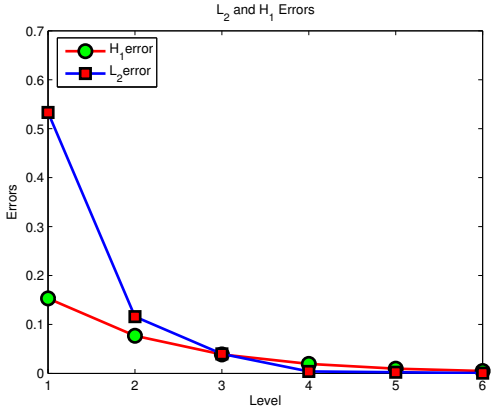

Figure 2: C changes L^2 and H^1-norm errors at level 4.

The Figure 2 shows changes L^2- and H^1-norm errors at levels 4 when the multigrid and the PMINRES schemes are applied.

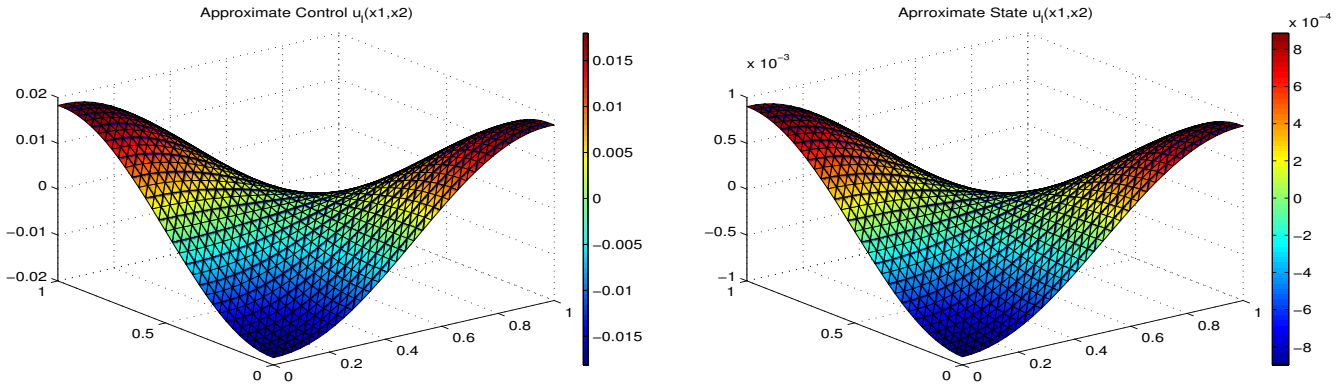

Figure 3: Optimal control (left) and optimal state (right) at level 4.

The Figure 3 shows the snapshot of the optimal control solution and optimal state solution at level 4.

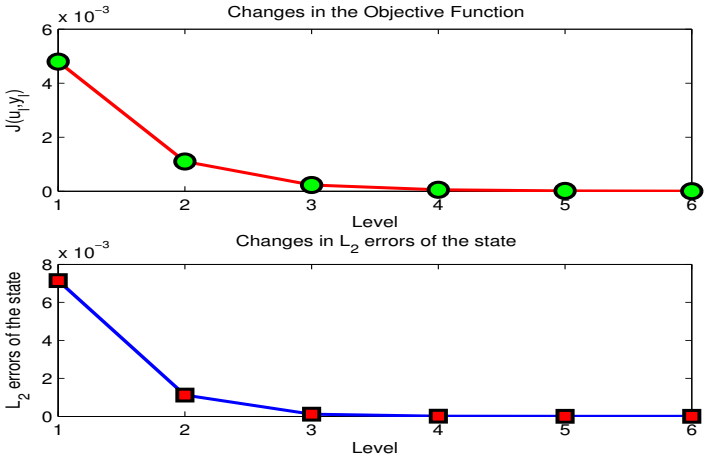

Figure 4: Changes in the objective function (left) and Changes L^2 and H^1-norm errors at level 4.

The Figure 4 shows the changes in the objective function from one level to another. The figure shows that the objective function is minimised. The changes in L^2 error for the state show that the optimal state variable is very close to the deired/target state which is the expected result.

4.1. Conclusion

The main objective of this work was to apply fast and efficient iterative solvers for the distributed elliptic optimal control problem discretized by finite element method. The standard finite element method discretization of the problem produced large, sparse, indefinite linear algebraic system of saddle point form. The system of equations has three formulations which are the basis for numerical analysis of the problem. We presented the optimal MGM with LU-smoother for the compact fixed point form formulation and compare the performance with the PMINRES solver with ideal preconditioning to the reduced and unreduced formulations. We observed that the methods produced similar results of the solution variables but differ in computational times and iterative counts. Both methods are fast and efficient though the MGM converges in less iterative counts and the PMIRES was faster in computational times. We observed that the convergence for the optimal control problem is closely related to those well known for the underlying elliptic partial differential equation which is the constraint. We have also paid particular attention to the L^2- and H^1-norm errors and results show that the two solvers are optimal for the optimal control problems and its convergence in L^2 and H^1 confirms with the theoretical results. The methods have displayed mesh independent convergence.

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