Certain results of 2-variable q-generalized tangent-Apostol-type polynomials

Ghazala Yasmin[a], Abdulghani Muhyi[a,b,*]

[a]*Department of Applied Mathematics, Aligarh Muslim University, Aligarh 202002, India.*
[b]*Department of Mathematics, Hajjah University, Hajjah, Yemen.*

Abstract

The present article aims to introduce and investigate a new class of q-hybrid special polynomials, namely 2-variable q-generalized tangent-Apostol-type polynomials. The generating function, series definition and many other useful relations and identities of this class are established. In addition, certain members of 2-variable q-generalized tangent-Apostol-type family are investigated and some properties of these members are obtained. The graphical representations of these members are shown for several values of indices with the help of Matlab. Further, the distributions of zeros of these members are displayed.

Keywords: q-calculus, q-tangent polynomials and numbers, q-Apostol-type polynomials and numbers, generating functions.

2020 MSC: 05A30, 11B83, 11B68.

1. Introduction

The q-calculus subject leads to a new noteworthy method for computations and classifications of q-special functions. It has gained prominence and numerous popularity during the last three decades or so (see [2, 5, 11, 14–16]). The contemporary interest in the subject is due to the fact that q-series has popped in such diverse areas as quantum groups, statistical mechanics, transcendental number theory, etc. The notations and definitions related to q-calculus recalled here are taken from [3] (see also [6, 10]), which will be used throughout this work.

The q-analogues of a number $\ell \in \mathbb{C}$ and of the factorial function are specified as follows:

$$[\ell]_q = \frac{1-q^\ell}{1-q}, \quad (q \in \mathbb{C}\backslash\{1\})$$

and

$$[\kappa]_q! = \prod_{l=1}^{\kappa}[l]_q = [1]_q[2]_q[3]_q \cdots [\kappa]_q, \ [0]_q! = 1, \ \kappa \in \mathbb{N}, q \in \mathbb{C}\backslash\{0,1\}.$$

The q-binomial coefficient $\binom{\kappa}{l}_q$ is specified as

$$\binom{\kappa}{l}_q = \frac{[\kappa]_q!}{[l]_q! \, [\kappa - l]_q!}, \quad l = 0, 1, 2, \ldots, \kappa; \ \kappa \in \mathbb{N}_0.$$

The q-power basis is specified as

$$(u + v)_q^\kappa = \sum_{l=0}^\kappa \binom{\kappa}{l}_q q^{\frac{l(l-1)}{2}} u^{\kappa-l} v^l. \tag{1.1}$$

The q-derivative of a function f at a point $\tau \in \mathbb{C}\backslash\{0\}$ is given as

$$D_q f(\tau) = \frac{f(\tau) - f(q\tau)}{\tau - q\tau}, \quad 0 < |q| < 1.$$

The functions

$$e_q(\tau) = \sum_{\kappa=0}^\infty \frac{\tau^\kappa}{[\kappa]_q!}, \quad 0 < |q| < 1, |\tau| < |1 - q|^{-1}, \tag{1.2}$$

$$E_q(\tau) = \sum_{\kappa=0}^\infty q^{\frac{1}{2}\kappa(\kappa-1)} \frac{\tau^\kappa}{[\kappa]_q!}, \quad 0 < |q| < 1, \ \tau \in \mathbb{C} \tag{1.3}$$

are called q-exponential functions and satisfy the following relations:

$$D_q e_q(\tau) = e_q(\tau), \ D_q E_q(\tau) = E_q(q\tau), \qquad e_q(\tau) E_q(-\tau) = E_q(\tau) e_q(-\tau) = 1. \tag{1.4}$$

The class of the q-special polynomials such as q-tangent polynomials, q-Bernoulli polynomials, q-Euler polynomials and q-Genocchi polynomials is an expanding field in mathematics [1, 5, 11–13, 15].

Numerous properties of tangent numbers and polynomials and their q-analogue have been studied and investigated by many researchers (see[7, 25, 26]). Further, these numbers and polynomials have enormous applications in analytic number theory, physics and the other related areas.

The tangent polynomials $T_\kappa(x)$ [25] are specified by means of the following generating function

$$\left(\frac{2}{e^{2\tau} + 1}\right) e^{x\tau} = \sum_{\kappa=0}^\infty T_\kappa(x) \frac{\tau^\kappa}{\kappa!}, \quad (|2\tau| < \pi),$$

where

$$\left(\frac{2}{e^{2\tau} + 1}\right) = \sum_{\kappa=0}^\infty T_\kappa \frac{\tau^\kappa}{\kappa!}, \quad (|2\tau| < \pi)$$

and $T_\kappa := T_\kappa(0)$ denotes the tangent numbers.

The q-analogue of the tangent polynomials $T_\kappa(x)$ is denoted by $T_{\kappa,q}(x)$ and specified by means of the generating function (see [27])

$$\left(\frac{2}{e_q(2\tau) + 1}\right) e_q(x\tau) = \sum_{\kappa=0}^\infty T_{\kappa,q}(x) \frac{\tau^\kappa}{[\kappa]_q!}, \quad (|2\tau| < \pi),$$

where $T_{\kappa,q} := T_{\kappa,q}(0)$ denotes the q-tangent numbers.

In [7], Bildirici et al. introduced the generalized tangent polynomials $C_{\kappa,m}(x)$ by means of the following generating function

$$\left(\frac{2}{e^{m\tau} + 1}\right) e^{x\tau} = \sum_{\kappa=0}^\infty C_{\kappa,m}(x) \frac{\tau^\kappa}{\kappa!}, \quad (|m\tau| < \pi, \ m \in \mathbb{R}^+). \tag{1.5}$$

The Apostol-type numbers and polynomials were first studied by Apostol [4] and further investigated by Srivastava [20]. The generalization of Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials were introduced and investigated widely by Luo and Srivastava [20] and Luo [18, 19]. Motivated by these works, many researchers have introduced and investigated the q-analogue of the Apostol-type polynomials (see [8, 17, 21, 23]). Recently, Kurt [17] introduced the q-Apostol-type polynomials $\mathsf{P}_{\kappa,\lambda,q}^{(\alpha)}(x,y;\delta,a,b)$ by the following generating functions

$$\left(\frac{2^{1-\delta}\,\tau^{\delta}}{\lambda^{b}e_{q}(\tau)-a^{b}}\right)^{\alpha}e_{q}(x\tau)E_{q}(y\tau)=\sum_{\kappa=0}^{\infty}\mathsf{P}_{\kappa,\lambda,q}^{(\alpha)}(x,y;\delta,a,b)\frac{\tau^{\kappa}}{[\kappa]_{q}!},$$

$$(|\tau|<2\pi \text{ when } \lambda=a;\ |\tau|<\left|b\log\left(\frac{\lambda}{a}\right)\right| \text{ when } \lambda\neq a;\alpha,\delta\in\mathbb{N}_{0},\ a,b\in\mathbb{R}^{+};\ \lambda\in\mathbb{C})$$

(1.6)

and

$$\mathsf{P}_{\kappa,\lambda,q}^{(\alpha)}(x,y;1,1,1)=\mathfrak{B}_{\kappa,q}^{(\alpha)}(x,y,\lambda), \text{(2-variable q-Apostol-Bernoulli polynomials (see [17, 23])),}$$

$$\mathsf{P}_{\kappa,\lambda,q}^{(\alpha)}(x,y;0,-1,1)=\mathcal{E}_{\kappa,q}^{(\alpha)}(x,y;\lambda), \text{(2-variable q-Apostol-Euler polynomials (see [17, 23])),}$$

$$\mathsf{P}_{\kappa,\frac{\lambda}{2},q}^{(\alpha)}\left(x,y;1,-\frac{1}{2},1\right)=\mathcal{G}_{\kappa,q}^{(\alpha)}(x,y;\lambda), \text{(2-variable q-Apostol-Genocchi polynomials (see [17, 23])),}$$

where

$$\left(\frac{\tau}{\lambda e_{q}(\tau)-1}\right)^{\alpha}e_{q}(x\tau)E_{q}(y\tau)=\sum_{\kappa=0}^{\infty}\mathfrak{B}_{\kappa,q}^{(\alpha)}(x,y;\lambda)\frac{\tau^{\kappa}}{[\kappa]_{q}!},$$

(1.7)

$$\left(\frac{2}{\lambda e_{q}(\tau)+1}\right)^{\alpha}e_{q}(x\tau)E_{q}(y\tau)=\sum_{\kappa=0}^{\infty}\mathcal{E}_{\kappa,q}^{(\alpha)}(x,y;\lambda)\frac{\tau^{\kappa}}{[\kappa]_{q}!},$$

(1.8)

$$\left(\frac{2\tau}{\lambda e_{q}(\tau)+1}\right)^{\alpha}e_{q}(x\tau)E_{q}(y\tau)=\sum_{\kappa=0}^{\infty}\mathcal{G}_{\kappa,q}^{(\alpha)}(x,y;\lambda)\frac{\tau^{\kappa}}{[\kappa]_{q}!}$$

(1.9)

and $\mathfrak{B}_{\kappa,q}^{(1)}(0,0;1):=\mathfrak{B}_{\kappa,q}$, $\mathcal{E}_{\kappa,q}^{(1)}(0,0;1):=\mathcal{E}_{\kappa,q}$ and $\mathcal{G}_{\kappa,q}^{(1)}(0,0;1):=\mathcal{G}_{\kappa,q}$ denote the q-Bernoulli, q-Euler and q-Genocchi numbers, respectively.

Motivated by the above works and using replacement technique (see [28]), in the present article we introduce the class of 2-variable q-generalized tangent-Apostol-type polynomials (2Vq-GTATP) by means of generating functions and series definition. Further, some significant and useful results are obtained for this class and some of its members. In Section 2, we start with introducing the q-generalized tangent polynomials. Thereafter, the 2-variable q-generalized tangent-Apostol-type polynomials are introduced by means of the generating function and series definition. Also, we consider some members of the 2-variable q-generalized tangent-Apostol-type polynomials and obtain generating functions and some other important properties of these members. In Section 3, few values and expressions of q-tangent numbers and polynomials, q-Bernoulli numbers and polynomials, q-Euler numbers and polynomials and q-Genocchi numbers and polynomials are obtained and used to display the graphs of some members belonging to the 2-variable q-generalized tangent-Apostol-type family for different values of indices. Further, the zeros of some members of 2-variable q-generalized tangent-Apostol-type polynomial are obtained and plotted by using Matlab.

2. 2-variable q-generalized tangent-Apostol-type polynomials

In order to introduce the 2-variable q-generalized tangent-Apostol-type polynomials, we first define the q-generalized tangent polynomials (q-GTP) according to generating function (1.5) as:

Definition 2.1. The q-analogue of the generalized tangent polynomials is defined as:

$$\left(\frac{2}{e_q(m\tau)+1}\right)e_q(x\tau) = \sum_{\kappa=0}^{\infty} C_{\kappa,m,q}(x)\frac{\tau^\kappa}{[\kappa]_q!}, \quad (|m\tau| < \pi, \ m \in \mathbb{R}^+). \tag{2.1}$$

We have:

(i) in the case when $m = 1, C_{\kappa,m,q}(x) = \mathcal{E}_{\kappa,q}(x)$, (q-Euler polynomials (q-EP) (see [9, 22]));
(ii) in the case when $m = 2, C_{\kappa,m,q}(x) = T_{\kappa,q}(x)$, (q-tangent polynomials);
(iii) $C_{\kappa,m,q}(0) := C_{\kappa,m,q}$, (q-generalized tangent numbers).

Next, we define the 2-variable q-generalized tangent polynomials (2Vq-GTP) as:

Definition 2.2. The 2-variable q-analogue of the generalized tangent polynomials is defined as:

$$\left(\frac{2}{e_q(m\tau)+1}\right)e_q(x\tau)E_q(y\tau) = \sum_{\kappa=0}^{\infty} C_{\kappa,m,q}(x,y)\frac{\tau^\kappa}{[\kappa]_q!}, \quad (|m\tau| < \pi, \ m \in \mathbb{R}^+) \tag{2.2}$$

and

(a) in the case when $m = 1, C_{\kappa,m,q}(x,y) = \mathcal{E}_{\kappa,q}(x,y)$, (2-variable q-Euler polynomials (2Vq-EP)[24]);
(b) in the case when $m = 2, C_{\kappa,m,q}(x,y) = T_{\kappa,q}(x,y)$, (2-variable q-tangent polynomials);
(c) $C_{\kappa,m,q}(0,0) := C_{\kappa,m,q}$, (q-generalized tangent numbers).

Now, in view of replacement technique and using generating relations (1.6) and (2.1), we define the 2-variable q-generalized tangent-Apostol-type polynomials as:

Definition 2.3. The 2-variable q-generalized tangent-Apostol-type polynomials of order α are defined by means of the generating function:

$$\left(\frac{2^{1-\delta}\,\tau^\delta}{\lambda^b e_q(\tau) - a^b}\right)^\alpha \left(\frac{2}{e_q(m\tau)+1}\right)e_q(x\tau)E_q(y\tau) = \sum_{\kappa=0}^{\infty} {}_cP_{\kappa,\lambda,q}^{(\alpha,m)}(x,y;\delta,a,b)\frac{\tau^\kappa}{[\kappa]_q!}, \tag{2.3}$$

$$\left(|\tau| < 2\pi \text{ when } \lambda = a; |\tau| < \left|b\log\left(\frac{\lambda}{a}\right)\right| \text{ when } \lambda \neq a; |m\tau| < \pi; \alpha, \delta \in \mathbb{N}_0, \ a, b, m \in \mathbb{R}^+; \ \lambda \in \mathbb{C}\right).$$

For $x = y = 0$, ${}_cP_{\kappa,\lambda,q}^{(\alpha,m)}(0,0;\delta,a,b) := {}_cP_{\kappa,\lambda,q}^{(\alpha,m)}(\delta,a,b)$ denotes the q-generalized tangent-Apostol-type numbers of order α and are defined by

$$\left(\frac{2^{1-\delta}\,\tau^\delta}{\lambda^b e_q(\tau) - a^b}\right)^\alpha \left(\frac{2}{e_q(m\tau)+1}\right) = \sum_{\kappa=0}^{\infty} {}_cP_{\kappa,\lambda,q}^{(\alpha,m)}(\delta,a,b)\frac{\tau^\kappa}{[\kappa]_q!}. \tag{2.4}$$

Utilizing generating relations (2.1) and (1.6) in generating relation (2.3) and making use of the Cauchy product rule in the resultant equation, thereafter, comparing the coefficients of the like powers of τ in both sides of the resultant expression, we obtain the following series definition of 2Vq-GTATP ${}_cP_{\kappa,\lambda,q}^{(\alpha,m)}(x,y;\delta,a,b)$.

Definition 2.4. The 2-variable q-generalized tangent-Apostol-type polynomials of order α are defined by the series:

$$_cP_{\kappa,\lambda,q}^{(\alpha,m)}(x,y;\delta,a,b) = \sum_{l=0}^{\kappa} \binom{\kappa}{l}_q C_{l,m,q} P_{\kappa-l,\lambda,q}^{(\alpha,m)}(x,y;\delta,a,b). \tag{2.5}$$

Suitably utilizing equations (1.1), (1.2), (1.3) and (2.4) in generating relation (2.3) and then making use of the Cauchy product rule in the resultant equations, we get three different expressions. Further, comparing the identical powers of τ in both sides of the resultant expressions, we get the following formulas

$$_C P_{\kappa,\lambda,q}^{(\alpha,m)}(x,y;\delta,a,b) = \sum_{l=0}^{\kappa} \binom{\kappa}{l}_q {}_C P_{l,\lambda,q}^{(\alpha,m)}(\delta,a,b)(x+y)_q^{\kappa-l},$$

$$_C P_{\kappa,\lambda,q}^{(\alpha,m)}(x,y;\delta,a,b) = \sum_{l=0}^{\kappa} \binom{\kappa}{l}_q x^l {}_C P_{\kappa-l,\lambda,q}^{(\alpha,m)}(0,y;\delta,a,b),$$

$$_C P_{\kappa,\lambda,q}^{(\alpha,m)}(x,y;\delta,a,b) = \sum_{l=0}^{\kappa} \binom{\kappa}{l}_q q^{\frac{l(l-1)}{2}} y^l {}_C P_{\kappa-l,\lambda,q}^{(\alpha,m)}(x,0;\delta,a,b).$$

Furthermore, we have

$$_C P_{\kappa,\lambda,q}^{(0,m)}(x,y;\delta,a,b) = C_{\kappa,m,q}(x,y).$$

Based on suitable selections for the parameters, several members belonging to the family of 2-variable q-generalized tangent-Apostol-type polynomials $_C P_{\kappa,\lambda,q}^{(\alpha,m)}(x,y;\delta,a,b)$ can be obtained. These members are mentioned in Table 1.

Table 1: Some special cases of the 2Vq-GTATP $_C P_{\kappa,\lambda,q}^{(\alpha,m)}(x,y;\delta,a,b)$.

S. No.	Values of the parameters	Relation between the 2Vq-GTATP $_C P_{\kappa,\lambda,q}^{(\alpha,m)}(x,y;\delta,a,b)$ and its special case	Name of the resultant special polynomials	Generating functions of the resultant special polynomials
I.	$\delta=a=b=1$	$_C P_{\kappa,\lambda,q}^{(\alpha,m)}(x,y;1,1,1)={}_C\mathfrak{B}_{\kappa,q}^{(\alpha,m)}(x,y;\lambda)$	2-variable q-generalized tangent-Apostol -Bernoulli polynomials (2Vq-GTABP)	$\left(\frac{\tau}{\lambda e_q(\tau)-1}\right)^\alpha\left(\frac{2}{e_q(m\tau)+1}\right)e_q(x\tau)E_q(y\tau)$ $=\sum_{\kappa=0}^\infty {}_C\mathfrak{B}_{\kappa,q}^{(\alpha,m)}(x,y;\lambda)\frac{\tau^\kappa}{[\kappa]_q!}$
II.	$\delta=0,\, a=-1,\, b=1$	$_C P_{\kappa,\lambda,q}^{(\alpha,m)}(x,y;0,-1,1)={}_C\mathcal{E}_{\kappa,q}^{(\alpha,m)}(x,y;\lambda)$	2-variable q-generalized tangent -Apostol-Euler polynomials (2Vq-GTAEP)	$\left(\frac{2}{\lambda e_q(\tau)+1}\right)^\alpha\left(\frac{2}{e_q(m\tau)+1}\right)e_q(x\tau)E_q(y\tau)$ $=\sum_{\kappa=0}^\infty {}_C\mathcal{E}_{\kappa,q}^{(\alpha,m)}(x,y;\lambda)\frac{\tau^\kappa}{[\kappa]_q!}$
III.	$\delta=1,\, a=-\frac{1}{2},\, b=1,$ $\lambda\to\frac{\lambda}{2}$	$_C P_{\kappa,\frac{\lambda}{2},q}^{(\alpha,m)}(x,y;1,-\frac{1}{2},1)={}_C\mathcal{G}_{\kappa,q}^{(\alpha,m)}(x,y;\lambda)$	2-variable q-generalized tangent -Apostol-Genocchi polynomials (2Vq-GTAGP)	$\left(\frac{2\tau}{\lambda e_q(\tau)+1}\right)^\alpha\left(\frac{2}{e_q(m\tau)+1}\right)e_q(x\tau)E_q(y\tau)$ $=\sum_{\kappa=0}^\infty {}_C\mathcal{G}_{\kappa,q}^{(\alpha,m)}(x,y;\lambda)\frac{\tau^\kappa}{[\kappa]_q!}$
IV.	$\delta=a=b=1,$ $m=2$	$_C P_{\kappa,\lambda,q}^{(\alpha,2)}(x,y;1,1,1)={}_T\mathfrak{B}_{\kappa,q}^{(\alpha)}(x,y;\lambda)$	2-variable q-tangent-Apostol -Bernoulli polynomials (2Vq-TABP)	$\left(\frac{\tau}{\lambda e_q(\tau)-1}\right)^\alpha\left(\frac{2}{e_q(2\tau)+1}\right)e_q(x\tau)E_q(y\tau)$ $=\sum_{\kappa=0}^\infty {}_T\mathfrak{B}_{\kappa,q}^{(\alpha)}(x,y;\lambda)\frac{\tau^\kappa}{[\kappa]_q!}$
V.	$\delta=0,\, a=-1,\, b=1,$ $m=2$	$_C P_{\kappa,\lambda,q}^{(\alpha,2)}(x,y;0,-1,1)={}_T\mathcal{E}_{\kappa,q}^{(\alpha)}(x,y;\lambda)$	2-variable q-tangent-Apostol -Euler polynomials (2Vq-TAEP)	$\left(\frac{2}{\lambda e_q(\tau)+1}\right)^\alpha\left(\frac{2}{e_q(2\tau)+1}\right)e_q(x\tau)E_q(y\tau)$ $=\sum_{\kappa=0}^\infty {}_T\mathcal{E}_{\kappa,q}^{(\alpha)}(x,y;\lambda)\frac{\tau^\kappa}{[\kappa]_q!}$
VI.	$\delta=1,\, a=-\frac{1}{2},\, b=1,$ $\lambda\to\frac{\lambda}{2},\, m=2$	$_C P_{\kappa,\frac{\lambda}{2},q}^{(\alpha,2)}(x,y;1,-\frac{1}{2},1)={}_T\mathcal{G}_{\kappa,q}^{(\alpha)}(x,y;\lambda)$	2-variable q-tangent-Apostol -Genocchi polynomials (2Vq-TAGP)	$\left(\frac{2\tau}{\lambda e_q(\tau)+1}\right)^\alpha\left(\frac{2}{e_q(2\tau)+1}\right)e_q(x\tau)E_q(y\tau)$ $=\sum_{\kappa=0}^\infty {}_T\mathcal{G}_{\kappa,q}^{(\alpha)}(x,y;\lambda)\frac{\tau^\kappa}{[\kappa]_q!}$
VII.	$\delta=a=b=1,$ $m=1$	$_C P_{\kappa,\lambda,q}^{(\alpha,1)}(x,y;1,1,1)={}_\mathcal{E}\mathfrak{B}_{\kappa,q}^{(\alpha)}(x,y;\lambda)$	2-variable q-Euler-Apostol -Bernoulli polynomials (2Vq-EABP)	$\left(\frac{\tau}{\lambda e_q(\tau)-1}\right)^\alpha\left(\frac{2}{e_q(\tau)+1}\right)e_q(x\tau)E_q(y\tau)$ $=\sum_{\kappa=0}^\infty {}_\mathcal{E}\mathfrak{B}_{\kappa,q}^{(\alpha)}(x,y;\lambda)\frac{\tau^\kappa}{[\kappa]_q!}$
VIII.	$\delta=0,\, a=-1,\, b=1,$ $m=1$	$_C P_{\kappa,\lambda,q}^{(\alpha,1)}(x,y;0,-1,1)={}_\mathcal{E}\mathcal{E}_{\kappa,q}^{(\alpha)}(x,y;\lambda)$	2-variable q-Euler-Apostol -Euler polynomials (2Vq-EAEP)	$\left(\frac{2}{\lambda e_q(\tau)+1}\right)^\alpha\left(\frac{2}{e_q(\tau)+1}\right)e_q(x\tau)E_q(y\tau)$ $=\sum_{\kappa=0}^\infty {}_\mathcal{E}\mathcal{E}_{\kappa,q}^{(\alpha)}(x,y;\lambda)\frac{\tau^\kappa}{[\kappa]_q!}$
IX.	$\delta=1,\, a=-\frac{1}{2},\, b=1,$ $\lambda\to\frac{\lambda}{2},\, m=1$	$_C P_{\kappa,\frac{\lambda}{2},q}^{(\alpha,1)}(x,y;1,-\frac{1}{2},1)={}_\mathcal{E}\mathcal{G}_{\kappa,q}^{(\alpha)}(x,y;\lambda)$	2-variable q-Euler-Apostol -Genocchi polynomials (2Vq-EAGP)	$\left(\frac{2\tau}{\lambda e_q(\tau)+1}\right)^\alpha\left(\frac{2}{e_q(\tau)+1}\right)e_q(x\tau)E_q(y\tau)$ $=\sum_{\kappa=0}^\infty {}_\mathcal{E}\mathcal{G}_{\kappa,q}^{(\alpha)}(x,y;\lambda)\frac{\tau^\kappa}{[\kappa]_q!}$
X.	$\delta=a=b=1,$ $m=2,\, \alpha=\lambda=1$	$_C P_{\kappa,1,q}^{(1,2)}(x,y;1,1,1)={}_T\mathfrak{B}_{\kappa,q}(x,y)$	2-variable q-tangent -Bernoulli polynomials (2Vq-TBP)	$\left(\frac{\tau}{e_q(\tau)-1}\right)\left(\frac{2}{e_q(2\tau)+1}\right)e_q(x\tau)E_q(y\tau)$ $=\sum_{\kappa=0}^\infty {}_T\mathfrak{B}_{\kappa,q}(x,y)\frac{\tau^\kappa}{[\kappa]_q!}$
XI.	$\delta=0,\, a=-1,\, b=1,$ $m=2,\, \alpha=\lambda=1$	$_C P_{\kappa,1,q}^{(1,2)}(x,y;0,-1,1)={}_T\mathcal{E}_{\kappa,q}(x,y)$	2-variable q-tangent -Euler polynomials (2Vq-TEP)	$\left(\frac{2}{e_q(\tau)+1}\right)\left(\frac{2}{e_q(2\tau)+1}\right)e_q(x\tau)E_q(y\tau)$ $=\sum_{\kappa=0}^\infty {}_T\mathcal{E}_{\kappa,q}(x,y)\frac{\tau^\kappa}{[\kappa]_q!}$

XII.	$\delta=1,\ a=-\frac{1}{2},\ b=1,$ $\lambda=\frac{1}{2},\ m=2,\ \alpha=1$	${}_C P^{(1,2)}_{\kappa,\frac{1}{2},q}(x,y;1,-\frac{1}{2},1)={}_T\mathcal{G}_{\kappa,q}(x,y)$	2-variable q-tangent -Genocchi polynomials (2Vq-TGP)	$\left(\frac{2\tau}{e_q(\tau)+1}\right)\left(\frac{2}{e_q(2\tau)+1}\right)e_q(x\tau)E_q(y\tau)$ $=\sum_{\kappa=0}^{\infty}{}_T\mathcal{G}_{\kappa,q}(x,y)\frac{\tau^{\kappa}}{[\kappa]_q!}$
XIII.	$\delta=a=b=1,$ $m=2,\ y=0$	${}_C P^{(\alpha,2)}_{\kappa,\lambda,q}(x,0;1,1,1)={}_T\mathfrak{B}^{(\alpha)}_{\kappa,q}(x;\lambda)$	q-tangent-Apostol -Bernoulli polynomials (q-TABP)	$\left(\frac{\tau}{\lambda e_q(\tau)-1}\right)^{\alpha}\left(\frac{2}{e_q(2\tau)+1}\right)e_q(x\tau)$ $=\sum_{\kappa=0}^{\infty}{}_T\mathfrak{B}^{(\alpha)}_{\kappa,q}(x;\lambda)\frac{\tau^{\kappa}}{[\kappa]_q!}$
XIV.	$\delta=0,\ a=-1,\ b=1,$ $m=2,\ y=0$	${}_C P^{(\alpha,2)}_{\kappa,\lambda,q}(x,0;0,-1,1)={}_T\mathcal{E}^{(\alpha)}_{\kappa,q}(x;\lambda)$	q-tangent-Apostol -Euler polynomials (q-TAEP)	$\left(\frac{2}{\lambda e_q(\tau)+1}\right)^{\alpha}\left(\frac{2}{e_q(2\tau)+1}\right)e_q(x\tau)$ $=\sum_{\kappa=0}^{\infty}{}_T\mathcal{E}^{(\alpha)}_{\kappa,q}(x;\lambda)\frac{\tau^{\kappa}}{[\kappa]_q!}$
XV.	$\delta=1,\ a=-\frac{1}{2},\ b=1,$ $\lambda\to\frac{\lambda}{2},\ m=2,\ y=0$	${}_C P^{(\alpha,2)}_{\kappa,\frac{\lambda}{2},q}(x,0;1,-\frac{1}{2},1)={}_T\mathcal{G}^{(\alpha)}_{\kappa,q}(x;\lambda)$	q-tangent-Apostol -Genocchi polynomials (q-TAGP)	$\left(\frac{2\tau}{\lambda e_q(\tau)+1}\right)^{\alpha}\left(\frac{2}{e_q(2\tau)+1}\right)e_q(x\tau)$ $=\sum_{\kappa=0}^{\infty}{}_T\mathcal{G}^{(\alpha)}_{\kappa,q}(x;\lambda)\frac{\tau^{\kappa}}{[\kappa]_q!}$
XVI.	$\delta=a=b=1,\ \alpha=1,$ $m=2,\ y=0,\ \lambda=1$	${}_C P^{(1,2)}_{\kappa,1,q}(x,0;1,1,1)={}_T\mathfrak{B}_{\kappa,q}(x)$	q-tangent-Bernoulli polynomials (q-TBP)	$\left(\frac{\tau}{e_q(\tau)-1}\right)\left(\frac{2}{e_q(2\tau)+1}\right)e_q(x\tau)$ $=\sum_{\kappa=0}^{\infty}{}_T\mathfrak{B}_{\kappa,q}(x)\frac{\tau^{\kappa}}{[\kappa]_q!}$
XVII.	$\delta=0,\ a=-1,\ b=1,\ \alpha=1,$ $m=2,\ y=0,\ \lambda=1$	${}_C P^{(1,2)}_{\kappa,1,q}(x,0;0,-1,1)={}_T\mathcal{E}_{\kappa,q}(x)$	q-tangent-Euler polynomials (q-TEP)	$\left(\frac{2}{e_q(\tau)+1}\right)\left(\frac{2}{e_q(2\tau)+1}\right)e_q(x\tau)$ $=\sum_{\kappa=0}^{\infty}{}_T\mathcal{E}_{\kappa,q}(x)\frac{\tau^{\kappa}}{[\kappa]_q!}$
XVIII.	$\delta=1,\ a=-\frac{1}{2},\ b=1,\ \alpha=1,$ $\lambda=\frac{1}{2},\ m=2,\ y=0$	${}_C P^{(\alpha,2)}_{\kappa,\frac{1}{2},q}(x,0;1,-\frac{1}{2},1)={}_T\mathcal{G}_{\kappa,q}(x)$	q-tangent-Genocchi polynomials (q-TGP)	$\left(\frac{2\tau}{e_q(\tau)+1}\right)\left(\frac{2}{e_q(2\tau)+1}\right)e_q(x\tau)$ $=\sum_{\kappa=0}^{\infty}{}_T\mathcal{G}_{\kappa,q}(x)\frac{\tau^{\kappa}}{[\kappa]_q!}$

Lemma 2.5. *The following recurrence relations hold true:*

$$D_{q,x}\ {}_C P^{(\alpha,m)}_{\kappa,\lambda,q}(x,y;\delta,a,b)=[\kappa]_q\ {}_C P^{(\alpha,m)}_{\kappa-1,\lambda,q}(x,y;\delta,a,b),$$

$$D^{(\xi)}_{q,x}\ {}_C P^{(\alpha,m)}_{\kappa,\lambda,q}(x,y;\delta,a,b)=\frac{[\kappa]_q!}{[\kappa-\xi]_q!}\ {}_C P^{(\alpha,m)}_{\kappa-\xi,\lambda,q}(x,y;\delta,a,b),$$

$$D_{q,y}\ {}_C P^{(\alpha,m)}_{\kappa,\lambda,q}(x,y;\delta,a,b)=[\kappa]_q\ {}_C P^{(\alpha,m)}_{\kappa-1,\lambda,q}(x,qy;\delta,a,b),$$

$$D^{(\xi)}_{q,y}\ {}_C P^{(\alpha,m)}_{\kappa,\lambda,q}(x,y;\delta,a,b)=q^{\frac{\xi(\xi-1)}{2}}\frac{[\kappa]_q!}{[\kappa-\xi]_q!}\ {}_C P^{(\alpha,m)}_{\kappa-\xi,\lambda,q}(x,q^{\xi}y;\delta,a,b).$$

Theorem 2.6. *The following relation involving the 2Vq-GTATP* ${}_C P^{(\alpha,m)}_{\kappa,\lambda,q}(x,y;\delta,a,b)$ *holds true:*

$${}_C P^{(\alpha,m)}_{\kappa,\lambda,q}(x,y;\delta,a,b)=2\,P^{(\alpha)}_{\kappa,\lambda,q}(x,y;\delta,a,b)-\sum_{l=0}^{\kappa}\binom{\kappa}{l}_q m^l\ {}_C P^{(\alpha,m)}_{\kappa-l,\lambda,q}(x,y;\delta,a,b). \qquad (2.6)$$

Proof. Consider the following identity

$$\frac{2}{e_q(m\tau)(e_q(m\tau)+1)}=\frac{2}{e_q(m\tau)}-\frac{2}{e_q(m\tau)+1}.$$

In view of the above identity, we can write

$$\left(\frac{2^{1-\delta}\tau^{\delta}}{\lambda^b e_q(\tau)-a^b}\right)^{\alpha}\left(\frac{2}{e_q(m\tau)(e_q(m\tau)+1)}\right)e_q(x\tau)E_q(y\tau)$$
$$=\left(\frac{2^{1-\delta}\,\tau^{\delta}}{\lambda^b e_q(\tau)-a^b}\right)^{\alpha}\left(\frac{2}{e_q(m\tau)}\right)e_q(x\tau)E_q(y\tau)-\left(\frac{2^{1-\delta}\,\tau^{\delta}}{\lambda^b e_q(\tau)-a^b}\right)^{\alpha}\left(\frac{2}{e_q(m\tau)+1}\right)e_q(x\tau)E_q(y\tau).$$

Multiplying both sides of the above equation by $e_q(m\tau)$ then utilizing equations (1.6) and (2.3) in the resultant relation, we get

$$\sum_{\kappa=0}^{\infty}{}_C P^{(\alpha,m)}_{\kappa,\lambda,q}(x,y;\delta,a,b)\frac{\tau^{\kappa}}{[\kappa]_q!}=2\sum_{\kappa=0}^{\infty}P^{(\alpha)}_{\kappa,\lambda,q}(x,y;\delta,a,b)\frac{\tau^{\kappa}}{[\kappa]_q!}-e_q(m\tau)\sum_{\kappa=0}^{\infty}{}_C P^{(\alpha,m)}_{\kappa,\lambda,q}(x,y;\delta,a,b)\frac{\tau^{\kappa}}{[\kappa]_q!}, \qquad (2.7)$$

which in view of equation (1.2) and making use of the Cauchy product rule in the right hand side then after comparing the coefficients of $\frac{\tau^{\kappa}}{[\kappa]_q!}$ on both sides of the resultant relation yields the assertion in equation (2.6). $\qquad\square$

Taking $\delta = a = b = 1$ and $m = 2$ in Theorem 2.6, we get the following corollary.

Corollary 2.7. *The following relation involving the 2Vq-TABP* $_T\mathfrak{B}_{\kappa,q}^{(\alpha)}(x,y;\lambda)$ *holds true:*

$$_T\mathfrak{B}_{\kappa,q}^{(\alpha)}(x,y;\lambda) = 2\,\mathfrak{B}_{\kappa,q}^{(\alpha)}(x,y;\lambda) - \sum_{l=0}^{\kappa} \binom{\kappa}{l}_q 2^l\,_T\mathfrak{B}_{\kappa-l,q}^{(\alpha)}(x,y;\lambda). \tag{2.8}$$

Theorem 2.8. *The following relationship between 2Vq-GTATP* $_C\mathsf{P}_{\kappa,\lambda,q}^{(1,m)}(x,y;\delta,a,b)$ *and 2Vq-GTP* $\mathsf{C}_{\kappa,m,q}(x,y)$ *holds true:*

$$_C\mathsf{P}_{\kappa,\lambda,q}^{(m)}(x,y;\delta,a,b) = -\frac{[\kappa]_q!\,2^{1-\delta}}{a^b[\kappa-\delta]_q!}\mathsf{C}_{\kappa-\delta,m,q}(x,y) + \left(\frac{\lambda}{a}\right)^b \sum_{l=0}^{\kappa} \binom{\kappa}{l}_q {}_C\mathsf{P}_{\kappa-l,\lambda,q}^{(m)}(x,y;\delta,a,b). \tag{2.9}$$

Proof. Take $\alpha = 1$ and consider the following identify

$$\frac{1}{(\lambda^b e_q(\tau) - a^b)\lambda^b e_q(\tau)} = \frac{1}{a^b(\lambda^b e_q(\tau) - a^b)} - \frac{1}{a^b\lambda^b e_q(\tau)}. \tag{2.10}$$

In view of the above identity, we have

$$\left(\frac{2^{1-\delta}\tau^\delta}{(\lambda^b e_q(\tau) - a^b)\lambda^b e_q(\tau)}\right)\left(\frac{2}{e_q(m\tau)+1}\right)e_q(x\tau)E_q(y\tau)$$
$$= \frac{1}{a^b}\left(\frac{2^{1-\delta}\tau^\delta}{\lambda^b e_q(\tau) - a^b}\right)\left(\frac{2}{e_q(m\tau)+1}\right)e_q(x\tau)E_q(y\tau) - \left(\frac{2^{1-\delta}\tau^\delta}{a^b\lambda^b e_q(\tau)}\right)\left(\frac{2}{e_q(m\tau)+1}\right)e_q(x\tau)E_q(y\tau).$$

Now, utilizing equations (2.2) and (2.3) in the above equation, then multiplying both side of the resultant relation by $\lambda^b e_q(\tau)$, we get

$$\sum_{\kappa=0}^{\infty} {}_C\mathsf{P}_{\kappa,\lambda,q}^{(m)}(x,y;\delta,a,b)\frac{\tau^\kappa}{[\kappa]_q!} = \left(\frac{\lambda}{a}\right)^b \sum_{l=0}^{\infty}\frac{\tau^l}{[l]_q!} \sum_{\kappa=0}^{\infty}\mathsf{P}_{\kappa,\lambda,q}^{(\alpha)}(x,y;\delta,a,b)\frac{\tau^\kappa}{[\kappa]_q!} - \frac{2^{1-\delta}}{a^b}\sum_{\kappa=0}^{\infty}\mathsf{C}_{\kappa,m,q}(x,y)\frac{\tau^{\kappa+\delta}}{[\kappa]_q!}.$$

Finally, using Cauchy product rule in the right hand side of the above equation and simplifying then after comparing the coefficients of $\frac{\tau^\kappa}{[\kappa]_q!}$ on both sides of the resultant relation, we get the assertion in equation (2.9). □

Taking $\delta = 0$, $a = -1$, $b = 1$ and $m = 2$ in Theorem 2.8, we get the following corollary.

Corollary 2.9. *The following relationship between 2Vq-TAEP* $_T\mathcal{E}_{\kappa,q}(x,y;\lambda)$ *and 2Vq-TP* $\mathsf{T}_{\kappa,q}(x,y)$ *holds true:*

$$_T\mathcal{E}_{\kappa,q}(x,y;\lambda) = 2\,\mathsf{T}_{\kappa,q}(x,y) - \lambda \sum_{l=0}^{\kappa} \binom{\kappa}{l}_q {}_T\mathcal{E}_{\kappa-l,q}(x,y;\lambda).$$

Theorem 2.10. *The following relationship between 2Vq-GTATP* $_C\mathsf{P}_{\kappa,\lambda,q}^{(\alpha,m)}(x,y;\delta,a,b)$ *and 2Vq-EP* $\mathcal{E}_{\kappa,q}(x,y)$ *holds true:*

$$2[\kappa]_q {}_C\mathsf{P}_{\kappa-1,\lambda,q}^{(\alpha,m)}(x,y;\delta,a,b)$$
$$= \sum_{\xi=0}^{\kappa}\frac{1}{\omega^{\xi-1}}\binom{\kappa}{\xi}_q \sum_{l=0}^{\kappa-\xi}\left\{(\frac{1}{\omega^l}\binom{\kappa-\xi}{l}_q {}_C\mathsf{P}_{\kappa-l-\xi,\lambda,q}^{(\alpha,m)}(x,0;\delta,a,b) + {}_C\mathsf{P}_{\kappa-\xi,\lambda,q}^{(\alpha,m)}(x,0;\delta,a,b)\right\}\mathcal{E}_{\xi,q}(0,\omega y). \tag{2.11}$$

Proof. Making use of generating relation (2.3), we have

$$\sum_{\kappa=0}^{\infty} {}_{C}P_{\kappa,\lambda,q}^{(\alpha,m)}(x,y;\delta,a,b)\frac{\tau^{\kappa}}{[\kappa]_q!}$$

$$= \left(\frac{2^{1-\delta}\,\tau^{\delta}}{\lambda^b e_q(\tau)-a^b}\right)^{\alpha}\left(\frac{2}{e_q(m\tau)+1}\right)e_q(x\tau)\left(\frac{e_q(\tau/\omega)+1}{2\tau}\right)\left(\frac{2\tau}{e_q(\tau/\omega)+1}\right)E_q\left(\omega y\frac{\tau}{\omega}\right)$$

$$= \frac{\omega}{2\tau}\left\{\sum_{\kappa=0}^{\infty}\sum_{l=0}^{\infty}{}_{C}P_{\kappa,\lambda,q}^{(\alpha,m)}(x,0;\delta,a,b)\frac{\tau^{\kappa+l}}{\omega^l[l]_q![\kappa]_q!}+\sum_{\kappa=0}^{\infty}{}_{C}P_{\kappa,\lambda,q}^{(\alpha,m)}(x,0;\delta,a,b)\frac{\tau^{\kappa}}{[\kappa]_q!}\right\}\sum_{\kappa=0}^{\infty}\mathcal{E}_{\kappa,q}(0,\omega y)\frac{\tau^{\kappa}}{\omega^{\kappa}[\kappa]_q!},$$

for which, upon using Cauchy product rule in the right hand side and after simplification, it becomes

$$\sum_{\kappa=0}^{\infty}[\kappa]_q{}_{C}P_{\kappa-1,\lambda,q}^{(\alpha,m)}(x,y;\delta,a,b)\frac{\tau^{\kappa}}{[\kappa]_q!}$$

$$= \frac{\omega}{2}\sum_{\kappa=0}^{\infty}\sum_{\xi=0}^{\kappa}\frac{1}{\omega^{\xi}}\binom{\kappa}{\xi}_q\sum_{l=0}^{\kappa-\xi}\left(\frac{1}{\omega^l}\binom{\kappa-\xi}{l}_q{}_{C}P_{\kappa-l-\xi,\lambda,q}^{(\alpha,m)}(x,0;\delta,a,b)+{}_{C}P_{\kappa-\xi,\lambda,q}^{(\alpha,m)}(x,0;\delta,a,b)\right)\mathcal{E}_{\xi,q}(0,\omega y)\frac{\tau^{\kappa}}{[\kappa]_q!}.$$

Finally, comparing the coefficients of identical powers of τ in both sides of above equations, we get the assertion in equation (2.11). □

Taking $\delta=b=1$, $a=-\frac{1}{2}$, $m=2$ and $\lambda\to\frac{\lambda}{2}$, the following corollary of Theorem 2.10 is obtained.

Corollary 2.11. *The following relationship between 2Vq-TAGP* ${}_T\mathcal{G}_{\kappa,q}^{(\alpha)}(x,y;\lambda)$, *q-TAGP* ${}_T\mathcal{G}_{\kappa,q}^{(\alpha)}(x;\lambda)$ *and 2Vq-EP* $\mathcal{E}_{\kappa,q}(x,y)$ *holds true:*

$$2[\kappa]_q{}_T\mathcal{G}_{\kappa-1,q}^{(\alpha)}(x,y;\lambda) = \sum_{\xi=0}^{\kappa}\frac{1}{\omega^{\xi-1}}\binom{\kappa}{\xi}_q\sum_{l=0}^{\kappa-\xi}\left(\frac{1}{\omega^l}\binom{\kappa-\xi}{l}_q{}_T\mathcal{G}_{\kappa-l-\xi,q}^{(\alpha)}(x;\lambda)+{}_T\mathcal{G}_{\kappa-\xi,q}^{(\alpha)}(x;\lambda)\right)\mathcal{E}_{\xi,q}(0,\omega y).$$

Next, we derive two connections between the 2Vq-GTATP ${}_{C}P_{\kappa,\lambda,q}^{(\alpha,m)}(x,y;\delta,a,b)$ and the family of numbers denoted by $S_q(\kappa,\alpha;a,b,\lambda)$ which are defined by means of the generating function [17]:

$$\sum_{\kappa=0}^{\infty}S_q(\kappa,\alpha;a,b,\lambda)\frac{\tau^{\kappa}}{[\kappa]_q!} = \frac{\left(\lambda^b e_q(\tau)-a^b\right)^{\alpha}}{[\alpha]_q!}. \tag{2.12}$$

Theorem 2.12. *The following relationships between 2Vq-GTATP* ${}_{C}P_{\kappa,\lambda,q}^{(\alpha,m)}(x,y;\delta,a,b)$ *and the numbers* $S_q(\kappa,\alpha;a,b,\lambda)$ *hold true:*

$$_{C}P_{\kappa-\sigma\delta,\lambda,q}^{(\alpha-\sigma,m)}(x,y;\delta,a,b) = \frac{[\kappa-\sigma\delta]_q![\sigma]_q!}{2^{\sigma(1-\delta)}[\kappa]_q!}\sum_{l=0}^{\kappa}\binom{\kappa}{l}_q{}_{C}P_{\kappa-l,\lambda,q}^{(\alpha,m)}(x,y;\delta,a,b)S_q(l,\sigma;a,b,\lambda), \tag{2.13}$$

$$C_{\kappa-\alpha\delta,m,q}(x,y) = \frac{[\kappa-\alpha\delta]_q![\alpha]_q!}{2^{\alpha(1-\delta)}[\kappa]_q!}\sum_{l=0}^{\kappa}\binom{\kappa}{l}_q{}_{C}P_{\kappa-l,\lambda,q}^{(\alpha,m)}(x,y;\delta,a,b)S_q(l,\alpha;a,b,\lambda). \tag{2.14}$$

Proof. Rearrange generating function (2.3) as:

$$\sum_{\kappa=0}^{\infty}{}_{C}P_{\kappa,\lambda,q}^{(\alpha-\sigma,m)}(x,y;\delta,a,b)\frac{\tau^{\kappa}}{[\kappa]_q!} = \left(\frac{2^{1-\delta}\,\tau^{\delta}}{\lambda^b e_q(\tau)-a^b}\right)^{\alpha}\left(\frac{2}{e_q(m\tau)+1}\right)e_q(x\tau)E_q(y\tau)\left(\frac{\lambda^b e_q(\tau)-a^b}{2^{1-\delta}\,\tau^{\delta}}\right)^{\sigma},$$

which on utilizing relation (2.12), becomes

$$\sum_{\kappa=0}^{\infty}{}_{C}P_{\kappa,\lambda,q}^{(\alpha-\sigma,m)}(x,y;\delta,a,b)\frac{\tau^{\kappa}}{[\kappa]_q!} = \frac{[\sigma]_q!}{(2^{1-\delta}\tau^{\delta})^{\sigma}}\sum_{\kappa=0}^{\infty}{}_{C}P_{\kappa,\lambda,q}^{(\alpha,m)}(x,y;\delta,a,b)\frac{\tau^{\kappa}}{[\kappa]_q!}\sum_{l=0}^{\infty}S_q(l,\sigma;a,b,\lambda)\frac{\tau^l}{[l]_q!}.$$

Now, simplifying the above relation and then comparing the coefficients of $\frac{\tau^\kappa}{[\kappa]_q!}$ on both sides of the resultant relation, we get the assertion in equation (2.13). Using a similar approach, with help of relations (2.1), (2.3) and (2.12), we can get the assertion in equation (2.14). □

Setting $\delta = a = b = m = 1$ in Theorem 2.12, the following corollary is obtained.

Corollary 2.13. *The following relationships between 2Vq-EABT $_\mathcal{E}\mathfrak{B}^{(\alpha)}_{\kappa,q}(x,y;\lambda)$ and the numbers $S_q(\kappa,\alpha;\lambda)$ hold true:*

$$_\mathcal{E}\mathfrak{B}^{(\alpha-\sigma)}_{\kappa-\sigma,q}(x,y;\lambda) = \frac{[\kappa-\sigma]_q!\,[\sigma]_q!}{[\kappa]_q!} \sum_{l=0}^{\kappa} \binom{\kappa}{l}_q {}_\mathcal{E}\mathfrak{B}^{(\alpha)}_{\kappa-l,q}(x,y;\lambda)S_q(l,\sigma;\lambda),$$

$$\mathcal{E}_{\kappa-\alpha\,q}(x,y) = \frac{[\kappa-\alpha]_q!\,[\alpha]_q!}{[\kappa]_q!} \sum_{l=0}^{\kappa} \binom{\kappa}{l}_q {}_\mathcal{E}\mathfrak{B}^{(\alpha)}_{\kappa-l,q}(x,y;\lambda)S_q(l,\alpha;\lambda).$$

Theorem 2.14. *The 2Vq-GTATP $_\mathsf{c}\mathsf{P}^{(\alpha,m)}_{\kappa,\lambda,q}(x,y;\delta,a,b)$ satisfy the following relation:*

$$\lambda^b {}_\mathsf{c}\mathsf{P}^{(\alpha,m)}_{\kappa,\lambda,q}(x,0;\delta,a,b) - a^b {}_\mathsf{c}\mathsf{P}^{(\alpha,m)}_{\kappa,\lambda,q}(x,-1;\delta,a,b)$$

$$= \frac{[\kappa]_q!\,2^{1-\delta}}{[\kappa-\delta]_q!} \sum_{l=0}^{\kappa-\delta}(-1)^l \binom{\kappa-\delta}{l}_q q^{\frac{l(l-1)}{2}} {}_\mathsf{c}\mathsf{P}^{(\alpha-1,m)}_{\kappa-\delta-l,\lambda,q}(x,0;\delta,a,b). \tag{2.15}$$

Proof. In view of generating function (2.3) and equation (1.4), we have

$$\sum_{\kappa=0}^{\infty} \lambda^b {}_\mathsf{c}\mathsf{P}^{(\alpha,m)}_{\kappa,\lambda,q}(x,0;\delta,a,b)\frac{\tau^\kappa}{[\kappa]_q!} - \sum_{\kappa=0}^{\infty} a^b {}_\mathsf{c}\mathsf{P}^{(\alpha,m)}_{\kappa,\lambda,q}(x,-1;\delta,a,b)\frac{\tau^\kappa}{[\kappa]_q!}$$

$$= \lambda^b \left(\frac{2^{1-\delta}\,\tau^\delta}{\lambda^b e_q(\tau)-a^b}\right)^\alpha \left(\frac{2}{e_q(m\tau)+1}\right)e_q(x\tau) - a^b \left(\frac{2^{1-\delta}\,\tau^\delta}{\lambda^b e_q(\tau)-a^b}\right)^\alpha \left(\frac{2}{e_q(m\tau)+1}\right)e_q(x\tau)E_q(-\tau)$$

$$= \left(\frac{2^{1-\delta}\,\tau^\delta}{\lambda^b e_q(\tau)-a^b}\right)^\alpha \left(\frac{2}{e_q(m\tau)+1}\right)e_q(x\tau)\left(\frac{\lambda^b e_q(\tau)-a^b}{e_q(\tau)}\right)$$

$$= 2^{1-\delta}\tau^\delta E_q(-\tau) \sum_{\kappa=0}^{\infty} {}_\mathsf{c}\mathsf{P}^{(\alpha-1,m)}_{\kappa,\lambda,q}(x,0;\delta,a,b)\frac{\tau^\kappa}{[\kappa]_q!}.$$

Now, using (1.3) and Cauchy product rule in the left hand side and then after some simplifications comparing the identical powers of τ of the resultant equation, we get the assertion in equation (2.15). □

Setting $\delta = 0$, $a = -1$, $b = m = 1$ in Theorem 2.14, the following corollary is obtained.

Corollary 2.15. *The following formula involving q-EAEP $_\mathcal{E}\mathcal{E}^{(\alpha)}_{\kappa,q}(x;\lambda)$ hold true:*

$$\lambda\, {}_\mathcal{E}\mathcal{E}^{(\alpha)}_{\kappa,q}(x;\lambda) + {}_\mathcal{E}\mathcal{E}^{(\alpha)}_{\kappa,q}(x,-1;\lambda) = 2\sum_{l=0}^{\kappa}(-1)^l\binom{\kappa}{l}_q q^{\frac{l(l-1)}{2}} {}_\mathcal{E}\mathcal{E}^{(\alpha-1)}_{\kappa-l,q}(x;\lambda).$$

Theorem 2.16 (Addition relation). *Let $\alpha,\beta \in \mathbb{N}_0$, then the 2-variable q-generalized tangent-Apostol-type polynomials satisfy the following relation:*

$$_\mathsf{c}\mathsf{P}^{(\alpha+\beta,m)}_{\kappa,\lambda,q}(x,y;\delta,a,b) = \sum_{l=0}^{\kappa}\binom{\kappa}{l}_q {}_\mathsf{c}\mathsf{P}^{(\alpha,m)}_{\kappa-l,\lambda,q}(x,0;\delta,a,b)\,\mathsf{P}^{(\beta)}_{l,\lambda,q}(0,y;\delta,a,b). \tag{2.16}$$

Proof. In view of generating relation (2.3), we can write

$$\sum_{\kappa=0}^{\infty} {}_\mathsf{c}\mathsf{P}^{(\alpha+\beta,m)}_{\kappa,\lambda,q}(x,0;\delta,a,b)\frac{\tau^\kappa}{[\kappa]_q!} = \left(\frac{2^{1-\delta}\,\tau^\delta}{\lambda^b e_q(\tau)-a^b}\right)^\alpha \left(\frac{2}{e_q(m\tau)+1}\right)e_q(x\tau)\left(\frac{2^{1-\delta}\,\tau^\delta}{\lambda^b e_q(\tau)-a^b}\right)^\beta E_q(y\tau),$$

which on using relations (1.6) and (2.3) then after some simplifications, it yields assertion in Eq. (2.16). □

Similarly, we can prove the following theorem.

Theorem 2.17 (Difference relation). *Let* $\alpha, \beta \in \mathbb{N}_0$, *then the 2-variable* q-*generalized tangent-Apostol-type polynomials satisfy the following relation:*

$$_C\mathsf{P}_{\kappa,\lambda,q}^{(\alpha-\beta,m)}(x,y;\delta,a,b) = \sum_{l=0}^{\kappa} \binom{\kappa}{l}_q \mathsf{P}_{\kappa-l,\lambda,q}^{(\alpha)}(x,0;\delta,a,b) \; _C\mathsf{P}_{l,\lambda,q}^{(-\beta,m)}(0,y;\delta,a,b).$$

Taking $a = -\frac{1}{2}$, $b = \delta = 1$, $m = 1$ and $\lambda \to \frac{\lambda}{2}$ in Theorem 2.16, we get the following corollary.

Corollary 2.18. *Let* $\alpha, \beta \in \mathbb{N}_0$, *then the* q-*EAGP* $_{\mathcal{E}}\mathcal{G}_{\kappa,q}^{(\alpha)}(x,y;\lambda)$ *satisfies the following relation:*

$$_{\mathcal{E}}\mathcal{G}_{\kappa,q}^{(\alpha+\beta)}(x,y;\lambda) = \sum_{l=0}^{\kappa} \binom{\kappa}{l}_q \; _{\mathcal{E}}\mathcal{G}_{\kappa-l,q}^{(\alpha)}(x;\lambda) \; \mathcal{G}_{\kappa,q}^{(\beta)}(0,y;\lambda).$$

Taking $\delta = a = b = 1$, $m = 2$, $y = 0$ in Theorem 2.17, we get the following corollary

Corollary 2.19. *Let* $\alpha, \beta \in \mathbb{N}_0$, *then the* q-*TABP* $_T\mathfrak{B}_{\kappa,q}^{(\alpha)}(x;\lambda)$ *satisfy the following relation:*

$$_T\mathfrak{B}_{\kappa,q}^{(\alpha-\beta)}(x;\lambda) = \sum_{l=0}^{\kappa} \binom{\kappa}{l}_q \mathfrak{B}_{\kappa-l,q}^{(\alpha)}(x;\lambda) \; _T\mathfrak{B}_{\kappa,q}^{(-\beta)}(\lambda).$$

3. Zeros and graphical representation

In this section, we display the shapes of some members belonging to the 2-variable q-generalized tangent-Apostol-type family such as q-tangent-Bernoulli polynomials $_T\mathfrak{B}_{\kappa,q}(x)$, q-tangent-Euler polynomials $_T\mathcal{E}_{\kappa,q}(x)$ and q-tangent-Genocchi polynomials $_T\mathcal{G}_{\kappa,q}(x)$. Further, the beautiful zeros of these members are investigated and their behavior is also shown.

Here, we first calculate some values of the q-generalized tangent numbers $C_{\kappa,m,q}$. The first five values are obtained as Table 2.

Table 2: Values of the first five q-GTN $C_{\kappa,m,q}$.

κ	0	1	2	3	4
$C_{\kappa,m,q}$	1	$-\frac{m}{2}$	$\frac{m^2}{4}\left([2]_q - 2\right)$	$\frac{m^3}{8}\left(4[3]_q - [2]_q[3]_q - 4\right)$	$\frac{m^4}{16[2]_q}\left(8[2]_q[4]_q + [2]_q([4]_q!) - 6([4]_q!) + 4[3]_q[4]_q - 8[2]_q\right)$

Since for $m = 2$, the q-GTN $C_{\kappa,m,q}$ reduces to q-TN $T_{\kappa,q}$ and for $m = 1$, the q-GTN $C_{\kappa,m,q}$ reduces to q-Euler numbers (q-EN) $\mathcal{E}_{\kappa,q}$, therefore, setting $m = 1, 2$ in Table 2, we get the following values for q-TN $T_{\kappa,q}$ and q-EN $\mathcal{E}_{\kappa,q}$ (Table 3).

Table 3: Values of the first five q-TN $T_{\kappa,q}$ and q-EN $\mathcal{E}_{\kappa,q}$.

κ	0	1	2	3	4
$T_{\kappa,q}$	1	-1	$[2]_q - 2$	$4[3]_q - [2]_q[3]_q - 4$	$\frac{1}{[2]_q}\left(8[2]_q[4]_q + [2]_q([4]_q!) - 6([4]_q!) + 4[3]_q[4]_q - 8[2]_q\right)$
$\mathcal{E}_{\kappa,q}$	1	$-\frac{1}{2}$	$\frac{1}{4}\left([2]_q - 2\right)$	$\frac{1}{8}\left(4[3]_q - [2]_q[3]_q - 4\right)$	$\frac{1}{16[2]_q}\left(8[2]_q[4]_q + [2]_q([4]_q!) - 6([4]_q! + 4[3]_q[4]_q - 8[2]_q\right)$

In view of equation (1.2) and generating relation (2.1), the q-GTP $C_{\kappa,m,q}(x)$ can be defined by the following series

$$C_{\kappa,m,q}(x) = \sum_{l=0}^{\kappa} \binom{\kappa}{l}_q C_{\kappa-l,m,q}x^l. \tag{3.1}$$

Making use of the values of $C_{\kappa,m,q}$ from Table 2 in equation (3.1), we get the expressions of the first five $C_{\kappa,m,q}(x)$. These expressions are listed in Table 4.

Table 4: Expressions of first five q-GTP $C_{\kappa,m,q}(x)$.

κ	0	1	2	3	4
$C_{\kappa,m,q}(x)$	1	$x - \frac{m}{2}$	$x^2 - \frac{m}{2}[2]_q x$ $+\frac{m^2}{4}([2]_q - 2)$	$x^3 - \frac{m}{2}[3]_q x^2 + \frac{m^2}{4}[3]_q([2]_q - 2)x$ $+\frac{m^3}{8}(4[3]_q - [2]_q[3]_q - 4)$	$x^4 - \frac{m}{2}[4]_q x^3 + \frac{m^2[4]_q[3]_q}{4[2]_q}([2]_q - 2)x^2 + \frac{m^3}{8}[4]_q(4[3]_q - [2]_q[3]_q - 4)x$ $+\frac{m^4}{16[2]_q}(8[2]_q[4]_q + [2]_q([4]_q!) - 6([4]_q!) + 4[3]_q[4]_q - 8[2]_q)$

Taking $m = 2, 1$ in Table 4, we get the expressions of first five $T_{\kappa,q}(x)$ and $\mathcal{E}_{\kappa,q}(x)$, respectively. These expressions are listed in Table 5.

Table 5: Expressions of first five q-TP $T_{\kappa,q}(x)$ and q-EP $\mathcal{E}_{\kappa,q}(x)$.

κ	0	1	2	3	4
$T_{\kappa,q}(x)$	1	$x - 1$	$x^2 - [2]_q x$ $+([2]_q - 2)$	$x^3 - [3]_q x^2 + [3]_q([2]_q - 2)x$ $+(4[3]_q - [2]_q[3]_q - 4)$	$x^4 - [4]_q x^3 + \frac{[4]_q[3]_q}{[2]_q}([2]_q - 2)x^2 + [4]_q(4[3]_q - [2]_q[3]_q - 4)x$ $+\frac{1}{[2]_q}(8[2]_q[4]_q + [2]_q([4]_q!) - 6([4]_q!) + 4[3]_q[4]_q - 8[2]_q)$
$\mathcal{E}_{\kappa,q}(x)$	1	$x - \frac{1}{2}$	$x^2 - \frac{[2]_q}{2}x$ $+\frac{1}{4}([2]_q - 2)$	$x^3 - \frac{[3]_q}{2}x^2 + \frac{[3]_q}{4}([2]_q - 2)x$ $+\frac{1}{8}(4[3]_q - [2]_q[3]_q - 4)$	$x^4 - \frac{[4]_q}{2}x^3 + \frac{[4]_q[3]_q}{4[2]_q}([2]_q - 2)x^2 + \frac{[4]_q}{8}(4[3]_q - [2]_q[3]_q - 4)x$ $+\frac{1}{16[2]_q}(8[2]_q[4]_q + [2]_q([4]_q!) - 6([4]_q!) + 4[3]_q[4]_q - 8[2]_q)$

Further, we obtain a list of few q-Bernoulli numbers (q-BN) $\mathfrak{B}_{\kappa,q}$ and q-Genocchi numbers (q-GN) $\mathcal{G}_{\kappa,q}$. These numbers are given in Table 6.

Table 6: Values of the first five q-BN $\mathfrak{B}_{\kappa,q}$ and q-GN $\mathcal{G}_{\kappa,q}$.

κ	0	1	2	3	4
$\mathfrak{B}_{\kappa,q}$	1	$-\frac{1}{[2]_q}$	$\frac{1}{[2]_q} - \frac{1}{[3]_q}$	$\frac{2}{[2]_q} - \frac{[3]_q}{([2]_q)^2} - \frac{1}{[4]_q}$	$\frac{[3]_q[4]_q}{([2]_q)^3} - \frac{3[4]_q}{([2]_q)^2} + \frac{2}{[2]_q} + \frac{[4]_q}{[2]_q[3]_q} - \frac{1}{[5]_q}$
$\mathcal{G}_{\kappa,q}$	0	1	$-\frac{[2]_q}{2}$	$\frac{[3]_q!}{4} - \frac{[3]_q}{2}$	$\frac{1}{8}\left(-[4]_q! + 4[4]_q[3]_q - 4[4]_q\right)$

Now, in view of the values mentioned in Table 6, we obtain a list of few q-Bernoulli polynomials (q-BP) $\mathfrak{B}_{\kappa,q}(x)$ and q-Genocchi polynomials (q-GP) $\mathcal{G}_{\kappa,q}(x)$. These polynomials are mentioned in Table 7.

Table 7: Values of the first five q-BP $\mathfrak{B}_{\kappa,q}(x)$ and q-GP $\mathcal{G}_{\kappa,q}(x)$.

κ	0	1	2	3	4
$\mathfrak{B}_{\kappa,q}(x)$	1	$x - \frac{1}{[2]_q}$	$x^2 - x + \frac{1}{[2]_q} - \frac{1}{[3]_q}$	$x^3 - \frac{[3]_q}{[2]_q}x^2 + \left(\frac{[3]_q}{[2]_q} - 1\right)x$ $+\frac{2}{[2]_q} - \frac{[3]_q}{([2]_q)^2} - \frac{1}{[4]_q}$	$x^4 - \frac{[4]_q}{[2]_q}x^3 + \frac{[4]_q[3]_q}{[2]_q}\left(\frac{1}{[2]_q} - \frac{1}{[3]_q}\right)x^2 + \left(\frac{2[4]_q}{[2]_q} - \frac{[3]_q[4]_q}{([2]_q)^2} - 1\right)x$ $+\frac{[3]_q[4]_q}{([2]_q)^3} - \frac{3[4]_q}{([2]_q)^2} + \frac{2}{[2]_q} + \frac{[4]_q}{[2]_q[3]_q} - \frac{1}{[5]_q}$
$\mathcal{G}_{\kappa,q}(x)$	0	1	$[2]_q x - \frac{[2]_q}{2}$	$[3]_q x^2 - \frac{[2]_q[3]_q}{2}x - \frac{[3]_q}{2} + \frac{[3]_q!}{4}$	$[4]_q x^3 - \frac{[4]_q!}{2[2]_q}x^2 + \left(\frac{[4]_q([3]_q!)}{4} - \frac{[4]_q[3]_q}{2}\right)x$ $+\frac{1}{8}\left(-[4]_q! + 4[4]_q[3]_q - 4[4]_q\right)$

In view of equation (2.5) and Table 1 (XVI-XVIII), the series definitions of the q-TBP $_T\mathfrak{B}_{\kappa,q}(x)$, q-TEP $_T\mathcal{E}_{\kappa,q}(x)$ and q-TGP $_T\mathcal{G}_{\kappa,q}(x)$ are obtained as:

$$_T\mathfrak{B}_{\kappa,q}(x) = \sum_{l=0}^{\kappa} \binom{\kappa}{l}_q T_{l,q}\mathfrak{B}_{\kappa-l,q}(x), \tag{3.2}$$

$$_T\mathcal{E}_{\kappa,q}(x) = \sum_{l=0}^{\kappa} \binom{\kappa}{l}_q T_{l,q}\mathcal{E}_{\kappa-l,q}(x), \tag{3.3}$$

$$_T\mathcal{G}_{\kappa,q}(x) = \sum_{l=0}^{\kappa} \binom{\kappa}{l}_q T_{l,q}\mathcal{G}_{\kappa-l,q}(x).$$

Setting $q = \frac{1}{2}$ and making suitable substitutions of the values of $T_{\kappa,q}$, $\mathfrak{B}_{\kappa,q}(x)$, $\mathcal{E}_{\kappa,q}(x)$, and $\mathcal{G}_{\kappa,q}(x)$ from Tables 3, 5, and 7 in series definitions (3.2)-(3.3), we obtain the expressions of q-TBP $_T\mathfrak{B}_{\kappa,q}(x)$, q-TEP $_T\mathcal{E}_{\kappa,q}(x)$ and q-TGP $_T\mathcal{G}_{\kappa,q}(x)$ for $q = \frac{1}{2}$ and $\kappa = 0,1,2,3,4$. These expressions are given in Table 8.

Table 8: Expressions of first five $_T\mathfrak{B}_{\kappa,\frac{1}{2}}(x)$, $_T\mathcal{E}_{\kappa,\frac{1}{2}}(x)$ and $_T\mathcal{G}_{\kappa,\frac{1}{2}}(x)$.

κ	0	1	2	3	4
$_T\mathfrak{B}_{\kappa,\frac{1}{2}}(x)$	1	$x - \frac{5}{3}$	$x^2 - \frac{5}{2}x + \frac{25}{42}$	$x^3 - \frac{35}{12}x^2 + \frac{25}{24}x + \frac{293}{360}$	$x^4 - \frac{25}{8}x^3 + \frac{125}{96}x^2 + \frac{293}{192}x + 0.373503905$
$_T\mathcal{E}_{\kappa,\frac{1}{2}}(x)$	1	$x - \frac{3}{2}$	$x^2 - \frac{9}{4}x + \frac{1}{8}$	$x^3 - \frac{21}{8}x^2 + \frac{7}{32}x + \frac{69}{64}$	$x^4 - \frac{45}{16}x^3 + \frac{35}{128}x^2 + \frac{1035}{512}x + \frac{761}{1024}$
$_T\mathcal{G}_{\kappa,\frac{1}{2}}(x)$	0	1	$\frac{3}{2}x - \frac{9}{4}$	$\frac{7}{4}x^2 - \frac{63}{16}x + \frac{7}{32}$	$\frac{15}{8}x^3 - \frac{315}{64}x^2 + \frac{105}{256}x + \frac{1035}{512}$

Further, with the help of Matlab, we compute the real and complex zeros of $_T\mathfrak{B}_{\kappa,\frac{1}{2}}(x)$, $_T\mathcal{E}_{\kappa,\frac{1}{2}}(x)$ and $_T\mathcal{G}_{\kappa,\frac{1}{2}}(x)$ for $\kappa = 1,2,3,4$ and $x \in \mathbb{C}$. These zeros are mentioned in Tables 9 and 10.

Table 9: Real zeros of $_T\mathfrak{B}_{\kappa,\frac{1}{2}}(x)$, $_T\mathcal{E}_{\kappa,\frac{1}{2}}(x)$, and $_T\mathcal{G}_{\kappa,\frac{1}{2}}(x)$.

κ	$_T\mathfrak{B}_{\kappa,\frac{1}{2}}(x)$	$_T\mathcal{E}_{\kappa,\frac{1}{2}}(x)$	$_T\mathcal{G}_{\kappa,\frac{1}{2}}(x)$
1	1.6667	1.5000	0
2	0.2665, 2.2335	0.0570, 2.1930	1.5000
3	$-0.3640, 0.9659, 2.3148$	$-0.5493, 0.8411, 2.3332$	0.0570, 2.1930
4	1.5510, 2.1546	1.4853, 2.2016	$-0.5493, 0.8411, 2.3332$

Table 10: Complex zeros of $_T\mathfrak{B}_{\kappa,\frac{1}{2}}(x)$, $_T\mathcal{E}_{\kappa,\frac{1}{2}}(x)$, and $_T\mathcal{G}_{\kappa,\frac{1}{2}}(x)$.

κ	$_T\mathfrak{B}_{\kappa,\frac{1}{2}}(x)$	$_T\mathcal{E}_{\kappa,\frac{1}{2}}(x)$	$_T\mathcal{G}_{\kappa,\frac{1}{2}}(x)$
1	—	—	—
2	—	—	—
3	—	—	—
4	$-0.2903 - 0.1658i, -0.2903 + 0.1658i$	$-0.4372 - 0.1900i, -0.4372 + 0.1900i$	—

With the help of Matlab and using the expressions of $_T\mathfrak{B}_{\kappa,\frac{1}{2}}(x)$, $_T\mathcal{E}_{\kappa,\frac{1}{2}}(x)$ and $_T\mathcal{G}_{\kappa,\frac{1}{2}}(x)$ given in Table 8, we get the Figures 1-3.
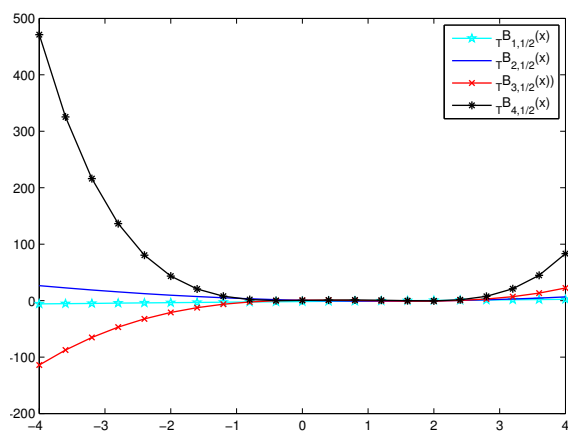

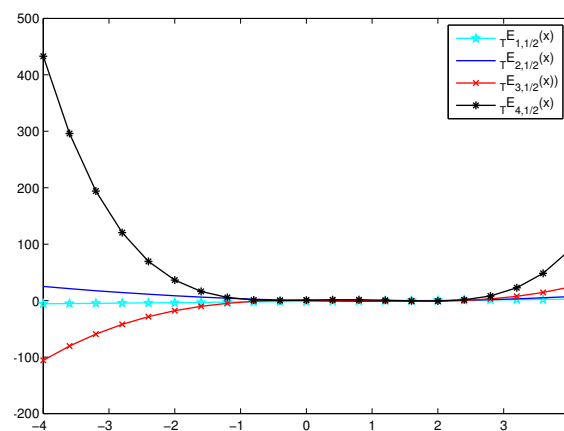Figure 1: Graph of $_T\mathfrak{B}_{\kappa,\frac{1}{2}}(x)$.

Figure 2: Graph of $_T\mathcal{E}_{\kappa,\frac{1}{2}}(x)$.

Also, with the help of Matlab, the zeros mentioned in Tables 9 and 10 are shown in the Figures 4-6.
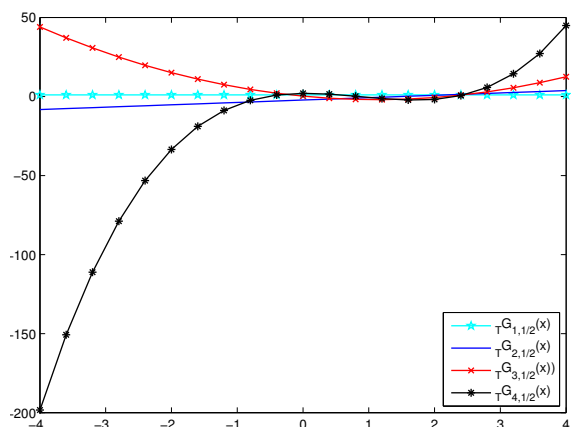
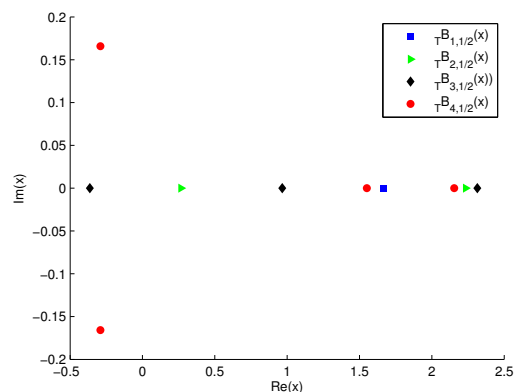Figure 3: Graph of $_T\mathcal{G}_{\kappa,\frac{1}{2}}(x)$.

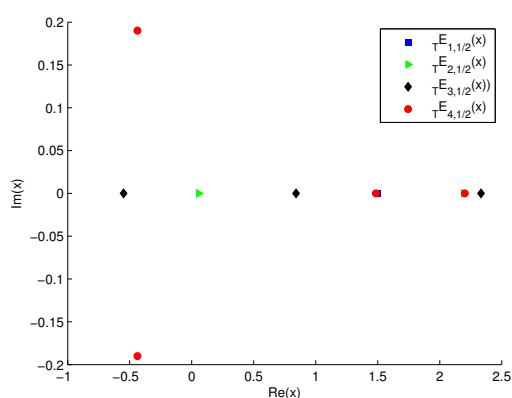Figure 4: Zeros of $_T\mathfrak{B}_{\kappa,\frac{1}{2}}(x)$.

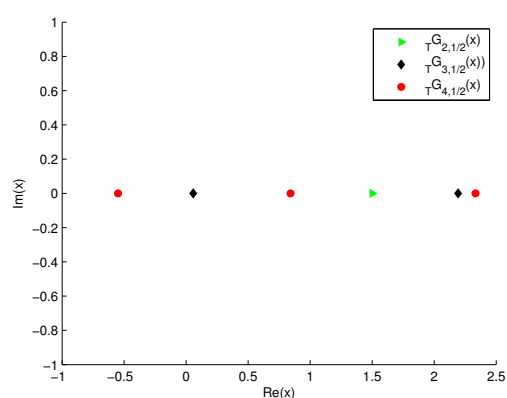Figure 5: Zeros of $_T\mathcal{E}_{\kappa,\frac{1}{2}}(x)$.

Figure 6: Zeros of $_T\mathcal{G}_{\kappa,\frac{1}{2}}(x)$.

Acknowledgment

References

[1] M. Acikgoz, S. Araci, U. Duran, *New extensions of some known special polynomials under the theory of multiple* q-*calculus*, Turkish J. Anal. Number Theory, **3** (2015), 128–139. 1

[2] W. A. Al-Salam, q-*Appell polynomials*, Ann. Mat. Pura Appl., **77** (1967), 31–45. 1

[3] G. E. Andrews, R. Askey, R. Roy, *Special Functions*, Cambridge University Press, Cambridge, (1999). 1

[4] T. M. Apostol, *On the Lerch Zeta function*, Pacific. J. Math., **1** (1951), 161–167. 1

[5] S. Araci, M. Acikgoz, T. Diagana, H. M. Srivastava, *A novel approach for obtaining new identities for the* λ *extension of* q-*Euler polynomials arising from the* q-*umbral calculus*, J. Nonlinear Sci. Appl., **10** (2007), 1316–1325. 1, 1

[6] A. Aral, V. Gupta, R. P. Agarwal, *Applications of* q-*Calculus in Operator Theory*, Springer, New York, (2013). 1

[7] C. Bildirici, M. Acikgoz, S. Araci, *A note on analogues of tangent polynomials*, J. Algebra Number Theory Acad., **4** (2014), 21 pages. 1, 1

[8] J. Choi, P. J. Anderson, H. M. Srivastava, *Some* q-*extensions of the Apostol–Bernoulli and the Apostol–Euler polynomials of order* n, *and the multiple Hurwitz Zeta function*, Appl. Math. Comput., **199** (2008), 723–737. 1

[9] T. Ernst, q-*Bernoulli and* q-*Euler polynomials, an umbral approach*, Int. J. Diff. Equ., **1** (2006), 31–80. i

[10] T. Ernst, *A comprehensive treatment of* q-*calculus*, Springer Science & Business Media, Basel, (2012). 1

[11] T. Kim, *A note on the* q-*Genocchi numbers and polynomials*, J. Inequal. Appl., **2007** (2007), 8 pages. 1, 1

[12] T. Kim, q-*Euler numbers and polynomials associated with* p-*adic* q-*integral*, J. Nonlinear Math. Phys., **14** (2007), 15–27.

[13] T. Kim, q-*Bernoulli numbers and polynomials associated with Gaussian binomial coefficients*, Russ. J. Math. Phys., **15** (2008), 51–57. 1

[14] D. S. Kim, T. Kim, T. Komatsu, J. J. Seo, *An umbral calculus approach to poly-Cauchy polynomials with a* q *parameter*, J. Comput. Anal. Appl., **18** (2015), 762–792. 1

[15] T. Kim, D. S. Kim, H. I. Kwon, J. J. Seo, D. V. Dolgy, *Some identities of* q-*Euler polynomials under the symmetric group of degree* n, J. Nonlinear Sci. Appl., **9** (2016), 1077–1082. 1

[16] D. S. Kim, T. Kim, H. Y. Lee, p-*adic* q-*integral on* Z_p *associated with Frobenius-type Eulerian polynomials and umbral calculus*, Adv. Stud Contemp. Math. (Kyungshang), **23** (2013), 243–251. 1

[17] B. Kurt, *Notes on unified* q-*Apostol-type polynomials*, Filomat, **30** (2016), 921–927. 1, 1, 2

[18] Q.-M. Luo, *Apostol-Euler polynomials of higher order and the Gaussian hypergeometric function*, Taiwanese J. Math., **10** (2006), 917–925. 1

[19] Q.-M. Luo, *Extension for the Genocchi polynomials and its Fourier expansions and integral representations*, Osaka J. Math., **48** (2011), 291–309. 1

[20] Q.-M. Luo, H. M. Srivastava, *Some generalizations of the Apostol-Bernoulli and Apostol-Euler polynomials*, J. Math. Anal. Appl., **308** (2005), 290–302. 1

[21] N. I. Mahmudov, q-*Analogues of the Bernoulli and Genocchi polynomials and the Srivastava-Pintér addition theorems*, Discrete Dyn. Nat. Soc., **2012** (2012), 8 pages. 1

[22] N. I. Mahmudov, *On a class of* q-*Bernoulli and* q-*Euler polynomials*, Adv. Difference Equ., **2013** (2013), 11 pages. i

[23] N. I. Mahmudov, M. E. Keleshteri, q-*Extensions for the Apostol type polynomials*, J. Appl. Math., **2014** (2014), 7 pages. 1, 1

[24] N. I. Mahmudov, M. Momenzadeh, *On a class of* q-*Bernoulli,* q-*Euler, and* q-*Genocchi polynomials*, Abstr. Appl. Anal., **2014** (2014), 10 pages. a

[25] C. S. Ryoo, *A note on the tangent numbers and polynomials*, Adv. Studies Theor. Phys., **7** (2013), 447–454. 1

[26] C. S. Ryoo, *Generalized tangent numbers and polynomials associated with* p-*adic integral on* z_p, Appl. Math. Sci., **7** (2013), 4929–4934. 1

[27] C. S. Ryoo, *Some properties of two dimensional* q-*tangent numbers and polynomials*, Global J. Pur. Appl. Math., **12** (2016), 2999–3007. 1

[28] G. Yasmin, A. Muhyi, S. Araci, *Certain Results of* q-*Sheffer–Appell Polynomials*, Symmetry, **11** (2019), 19 pages. 1