



A class of fractional integral operators with multi-index Mittag-Leffler k-function and Bessel k-function of first kind



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Abstract

In this paper, we discuss the multi-index Mittag Leffler k-function and Bessel k-function of the first kind in fractional calculus. We investigate fractional integral operators (Saigo's, Erdelyi Kober, Reimann Liouvill, Weyl) and extend with the product of multi-index Mittag Leffler k-function to the Bessel k-function of the first kind. Here, we establish new theorems that provide the image of multi-index Mittag Leffler and Bessel k-functions under these k-fractional operators. These results are derived in general behave and give several well-known results in the theory of multi-index k-functions.

Keywords: Fractional integral operators, generalized Mittag-Leffler k-function, Bessel k-function, classical hypergeometric functions.

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1. Introduction

During century, Mittag Leffler function has significant topic in mathematics due to its numerous applications and generalization in different filed of mathematical analysis. Fractional calculus has closed relation with Mittag Leffler because of fractional problems. Hille and Tamarkin presented the solution of second kind Abel integral equation in term of Mittag Leffler function. In 1954, Barret considered the general solution of linear fractional differential equation with constant coefficient.

Recently, many researchers have great concentrate in the direction of Mittag Leffler type functions and improved a lot of results in the field of fractional calculus. Gösta Magnus Mittag Leffler discussed a power series function $E_\alpha(z)$ and studied its properties in five successive account ML1-ML5 at the time of 1902 – 1905. For more detailed of Mittage Leffler type functions have immense contribution in the field of integral inequalities involved different kinds of fractional operators like as differentiable exponentially convex and quasi-convex mapping, conformal fractional integral, Grüss type integral inequalities, geometrical convex functions, generalized (g, s, m, φ) -preinvex function, and some generalizations of Ostrowski type inequalities, we refer the interesting reader to [2, 6–9, 11, 15, 19, 23, 24, 32, 33].

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The classical Mittag-Leffler function [14, 20] for $z \in \mathbb{C}$ and $\alpha > 0$ is defined as

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)},$$

where $(\alpha n)! = \Gamma(\alpha n + 1)$ are power terms of the exponential series. The Mittag Leffler function also called as Queen function of fractional calculus.

Wiman studied the generalization of Mittag Leffler function and defined for $\alpha, \beta > 0$ and $z \in \mathbb{C}$, given as below [1]

$$E_{\alpha, \beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}. \quad (1.1)$$

There are many extensions, generalization or combinations for Mittag-Leffler functions which are found in the papers [16, 27–30]. Kiryakova [18] introduced and studied the multi-index Mittag-Leffler function which is the extension of generalized Mittag-Leffler function. The multi-index Mittag-Leffler function is

$$E_{(\frac{1}{\rho_j}, \mu_j)}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\prod_{j=1}^m \Gamma(\mu_j + \frac{n}{\rho_j})},$$

where $m \geq 1$, $\operatorname{Re}(\rho_1), \dots, \operatorname{Re}(\rho_m) > 0$, and μ_1, \dots, μ_m are arbitrary parameters. If we replace $m = 1$, $\rho_1 = \frac{1}{\alpha}$ and $\mu_1 = \beta$, then it will become the classical result which is given in equation (1.1).

Bessel functions [4, 31] interested from functional inequalities and geometric opinions. The first kind Bessel function of order q is defined as

$$J_q(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{z}{2})^{2n+q}}{\Gamma(q+n+1)n!}.$$

The first kind Bessel k -function [21] of order q is defined for $\operatorname{Re}(q) > -1$,

$$J_{q;k}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{z}{2})^{2n+\frac{q}{k}}}{\Gamma_k(q+nk+k)n!},$$

where k is any positive real number.

Generalized fractional integral operators [17] is defined for $y > 0$ and $\alpha', \beta', \eta' \in \mathbb{C}$, $\operatorname{Re}(\alpha') > 0$, as follows

$$(I_{0,y}^{\alpha', \beta', \eta'} f)(x) = \frac{x^{-\alpha'-\beta'}}{\Gamma(\alpha')} \int_0^x (x-t)^{\alpha'-1} {}_2F_1(\alpha' + \beta', -\eta'; \alpha'; 1 - \frac{t}{x}) f(t) dt,$$

and

$$(I_{y,\infty}^{\alpha', \beta', \eta'} f)(x) = \frac{1}{\Gamma(\alpha')} \int_x^\infty (t-x)^{\alpha'-1} t^{-\alpha'-\beta'} {}_2F_1(\alpha' + \beta', -\eta'; \alpha'; 1 - \frac{x}{t}) f(t) dt,$$

where, Gauss hypergeometric function ${}_2F_1$ used its kernel.

The Series of Gauss hypergeometric function [26] is defined as

$${}_2F_1(\alpha', \beta'; \eta'; t) = \sum_{m=0}^{\infty} \frac{(\alpha')_m (\beta')_m}{(\eta')_m} \frac{t^m}{m!},$$

where $\alpha', \beta', \eta' \in \mathbb{C}$, $\eta' \neq 0, -1, -2, \dots$, $|t| < 1$ $(\alpha')_m$, $(\beta')_m$ and $(\eta')_m$ are Pochhammer's symbols.

Generalized hypergeometric function [26] is defined as follows:

$${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; t) = \sum_{m=0}^{\infty} \frac{(a_1)_m \cdots (a_p)_m}{(b_1)_m \cdots (b_q)_m} \frac{t^m}{m!},$$

where $a_i, b_j \in \mathbb{C}$, $b_j \neq 0, -1, \dots$ ($i = 1, 2, \dots, p$; $j = 1, 2, \dots, q$).

Pochhammer's symbol [25] is defined as

$$(l)_n = \begin{cases} l(l+1)(l+2) \cdots (l+n-1), & \text{for } n \geq 1, \\ 1, & \text{for } n = 0, l \neq 0, \end{cases}$$

where $l \in \mathbb{C}$ and $n \in \mathbb{N}$.

Gamma function [22, 25] is defined for $\operatorname{Re}(u) > 0$ as follows:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt.$$

Beta function [25] is defined as

$$\beta(l, h) = \int_0^1 t^{l-1} (1-t)^{h-1} dt,$$

where $\operatorname{Re}(l) > 0$, $\operatorname{Re}(h) > 0$.

Gamma k-function and Beta k-function introduced by Diaz and Pariguan [5]. Fractional k-integral operators presented by Mubeen and Habibullah [22] in 2012.

Beta k-function [5] is defined as

$$\beta_k(l, h) = \frac{1}{k} \int_0^1 s^{\frac{l}{k}-1} (1-s)^{\frac{h}{k}-1} ds, \quad \operatorname{Re}(l) > 0, \operatorname{Re}(h) > 0. \quad (1.2)$$

For $k > 0$, $\operatorname{Re}(x)$ and $\operatorname{Re}(y)$, the relation beta gamma k-functions can be written as

$$\beta_k(x, y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)}. \quad (1.3)$$

The integral form of the generalized gamma k-function [12] is defined as

$$\Gamma_k(z) = \int_0^\infty t^{z-1} e^{-\frac{t^k}{k}} dt,$$

where $k \in \mathbb{R}^+$, $z \in \mathbb{C}$, $\operatorname{Re}(z) > 0$. Some important relations about gamma function are as follows

$$\Gamma_k(z+k) = z\Gamma_k(z), \quad \Gamma_k(\gamma) = (k)^{\frac{\gamma}{k}-1} \Gamma\left(\frac{\gamma}{k}\right).$$

The useful relation about gamma k-function is given as

$$\Gamma_k(t+nk) = (t)_{n,k} \Gamma_k(t). \quad (1.4)$$

The generalized Mittag-Leffler k-function [10] is defined as

$$E_{k;\nu,\rho}^\delta(t) = \sum_{n=0}^{\infty} \frac{(\delta)_{n,k} t^n}{\Gamma_k(\nu n + \rho) n!}, \quad (1.5)$$

where $k \in \mathbb{R}^+$, $\nu, \rho, \delta, t \in \mathbb{C}$, $\operatorname{Re}(\nu) > 0$, $\operatorname{Re}(\rho) > 0$.

The hypergeometric k-function [13] is defined for all $\alpha', \beta', \eta' \in \mathbb{C}$ and $\eta' \neq 0, -1, -2, -3, \dots$, $|t| < 1$ as follows:

$${}_2F_{1,k}((\alpha', k), (\beta', k); (\eta', k); t) = \sum_{m=0}^{\infty} \frac{(\alpha')_{m,k} (\beta')_{m,k}}{(\eta')_{m,k}} \frac{t^m}{m!}, \quad k > 0,$$

where $(\alpha')_{m,k}$, $(\beta')_{m,k}$ and $(\eta')_{m,k}$ are Pochhammer k-symbols.

The generalized hypergeometric k-function is defined as

$${}_pF_{q,k}((a_1, k), \dots, (a_p, k); (b_1, k), \dots, (b_q, k); t) = \sum_{m=0}^{\infty} \frac{(a_1)_{m,k} \cdots (a_p)_{m,k}}{(b_1)_{m,k} \cdots (b_q)_{m,k}} \frac{t^m}{m!},$$

where $a_i, b_j \in \mathbb{C}$, $b_j \neq 0, -1, \dots$ ($i = 1, 2, \dots, p$; $j = 1, 2, \dots, q$).

The Riemann Liouville fractional integral operator [3] is defined as

$$I_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad I_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt,$$

where $\operatorname{Re}(\alpha) > 0$. The above integrals also called progressive or right-sided and regressive or left-sided Riemann Liouville fractional integral operators, respectively.

The left and right sided Erdelyi-Kober fractional integrals operators [3] are defined for $\alpha, \eta \in \mathbb{C}$, $\operatorname{Re}(\alpha) > 0$ as follows

$$\begin{aligned} K_{0,u}^{\alpha,\eta} f(u) &= \frac{u^{-\alpha-\eta}}{\Gamma(\alpha)} \int_0^u (u-t)^{\alpha-1} (t)^{\eta} f(t) dt, \\ K_{u,\infty}^{\alpha,\eta} f(u) &= \frac{u^{\eta}}{\Gamma(\alpha)} \int_u^{\infty} (t-u)^{\alpha-1} (t)^{-\alpha-\eta} f(t) dt. \end{aligned}$$

The Weyl fractional integral operator [3] is defined for $\alpha \in \mathbb{C}$, $\operatorname{Re}(\alpha) > 0$ as follows

$$W_{v,\infty}^{\alpha} f(v) = \frac{1}{\Gamma(\alpha)} \int_v^{\infty} (t-v)^{\alpha-1} f(t) dt.$$

The Saigo's left, right type fractional k-integral operators are defined for $y > 0$ and $\alpha', \beta', \eta' \in \mathbb{C}$, $\operatorname{Re}(\alpha') > 0$, $k \in \mathbb{R}^+$ and Gauss hypergeometric k-function ${}_2F_{1,k}$ used its kernel, as follows

$$\begin{aligned} (I_{k;0,y}^{\alpha',\beta',\eta'} g)(y) &= \frac{y^{\frac{-\alpha'-\beta'}{k}}}{k\Gamma_k(\alpha')} \int_0^y (y-s)^{\frac{\alpha'}{k}-1} {}_2F_{1,k}(\alpha'+\beta', -\eta'; \alpha'; 1-\frac{s}{y}) g(s) ds, \\ (I_{k;y,\infty}^{\alpha',\beta',\eta'} g)(y) &= \frac{1}{k\Gamma_k(\alpha')} \int_y^{\infty} (s-y)^{\frac{\alpha'}{k}-1} s^{\frac{-\alpha'-\beta'}{k}} {}_2F_{1,k}(\alpha'+\beta', -\eta'; \alpha'; 1-\frac{y}{s}) g(s) ds. \end{aligned}$$

The Riemann Liouville fractional k-integral operators defined as follows

$$I_{k;a+}^{\alpha} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t) dt, \quad I_{k;b-}^{\alpha} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_x^b (t-x)^{\frac{\alpha}{k}-1} f(t) dt,$$

where $\operatorname{Re}(\alpha) > 0$ and k be any positive real number.

The Erdelyi-Kober fractional k-integral operators are defined as follows

$$\begin{aligned} K_{k;0,u}^{\alpha,\eta} f(u) &= \frac{u^{\frac{-\alpha-\eta}{k}}}{k\Gamma_k(\alpha)} \int_0^u (u-t)^{\frac{\alpha}{k}-1} (t)^{\frac{\eta}{k}} f(t) dt, \\ K_{k;u,\infty}^{\alpha,\eta} f(u) &= \frac{u^{\frac{\eta}{k}}}{k\Gamma_k(\alpha)} \int_u^{\infty} (t-u)^{\frac{\alpha}{k}-1} (t)^{\frac{-\alpha-\eta}{k}} f(t) dt, \end{aligned}$$

where $\alpha, \eta \in \mathbb{C}$, $\operatorname{Re}(\alpha) > 0$ and k be any positive real number.

For $\alpha \in \mathbb{C}$, $\operatorname{Re}(\alpha) > 0$ and $k \in \mathbb{R}^+$, the Weyl fractional k-integral operators are defined as

$$W_{k;v,\infty}^{\alpha} f(v) = \frac{1}{k\Gamma_k(\alpha)} \int_v^{\infty} (t-v)^{\frac{\alpha}{k}-1} f(t) dt.$$

There are special cases, we generate the some relations of Bessel k-function of first kind and Mittag-Leffler k-function.

- By Setting $j = m = 1$ and $\mu_1 = \beta$, $\frac{1}{\rho_1} = \alpha$ then, we get Wiman k-function,

$$E_{k;\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma_k(\alpha n + \beta)}.$$

- By putting $j = m = 1$ and $\mu_1 = k$, $\frac{1}{\rho_1} = \alpha$, we obtain the Mittag-Leffler k-function of one parameter,

$$E_{k;\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma_k(\alpha n + k)}.$$

- Replacing $j = m = 1$, $\mu_1 = \mu + k$, $\frac{1}{\rho_j} = k$, and $z = -\frac{z^2}{4}$ in the multiindex mittag-leffler k-function, we have Bessel k-function of first kind,

$$\left(\frac{2}{z}\right)^{\frac{p}{k}} E_{k;k,\mu+k}\left(\frac{-z^2}{4}\right) = J_{\mu,k}(z).$$

2. Saigo's generalized fractional k-integral operators with multi-index Mittag-Leffler k-function and Bessel k-function

In this section, we discuss composition of generalized fractional k-integral operators with the product of multi-index Mittag-Leffler k-function and Bessel k-function of first kind. The generalized fractional k-integral operators can be extended with these compositions which gave an important theory in the field of analysis. The following theorems can prove these consequences.

Theorem 2.1. Suppose that $m \geq 1$ is an integer and $I_{k;0,z}^{\alpha,\beta,\eta}$ be the left sided Saigo's k-integral operator and $\operatorname{Re}(\rho_j) > 0$, μ_j ($j = 1, \dots, m$) are arbitrary parameters, then the following result holds:

$$I_{k;0,z}^{\alpha,\beta,\eta} \{ z^{\frac{\lambda}{k}-1} J_{q;k}(z) E_{k;(\frac{1}{\rho_j}, \mu_j)}(az^\sigma) \} = z^{\frac{\lambda-\beta+q}{k}-1} \sum_{h=0}^{\infty} \frac{(\frac{1}{2})^q (az^\sigma)^h}{\prod_{j=1}^m \Gamma_k(\mu_j + \frac{h}{\rho_j})} \\ \times {}_2\Psi_3^k \left[\begin{matrix} (\lambda + \sigma kh + q, 2k)(\lambda + \eta - \beta + \sigma kh + q, 2k) \\ (q + k, k)(\lambda - \beta + \sigma kh + q, 2k)(\lambda + \alpha + \eta + \sigma kh + q, 2k) \end{matrix} \middle| \frac{-z^2}{4} \right],$$

where λ and σ are arbitrary parameters and k is any positive real number such that $\operatorname{Re}(\alpha) > 0$ and α, β, η and a are any complex numbers.

Proof. Consider the left sided fractional integral operators and multi-index Mittag-Leffler k-function,

$$I_{k;0,z}^{\alpha,\beta,\eta} \{ z^{\frac{\lambda}{k}-1} J_{q;k}(z) E_{k;(\frac{1}{\rho_j}, \mu_j)}(az^\sigma) \} \\ = \frac{(z)^{\frac{-\alpha-\beta}{k}}}{k\Gamma_k(\alpha)} \int_0^z (z-t)^{\frac{\alpha}{k}-1} {}_2F_{1,k}(\alpha+\beta, -\eta; \alpha; (1-\frac{t}{z})) \sum_{p,h=0}^{\infty} \frac{t^{\frac{\lambda}{k}-1} (-1)^p (\frac{t}{2})^{2p+\frac{q}{k}} (at^\sigma)^h}{\Gamma_k(q+pk+k)p! \prod_{j=1}^m \Gamma_k(\mu_j + \frac{h}{\rho_j})} dt \\ = \frac{z^{\frac{-\alpha-\beta}{k}}}{k\Gamma_k(\alpha)} \int_0^z \frac{(z-t)^{\frac{\alpha}{k}-1}}{(t)^{1-\frac{\lambda}{k}}} \sum_{n=0}^{\infty} \frac{(\alpha+\beta)_n (-\eta)_n (1-\frac{t}{z})^n}{(\alpha)_{n,k} n!} \sum_{p,h=0}^{\infty} \frac{(-1)^p (\frac{t}{2})^{2p+\frac{q}{k}} (a(t)^\sigma)^h}{\Gamma_k(q+pk+k)p! \prod_{j=1}^m \Gamma_k(\mu_j + \frac{h}{\rho_j})} dt \quad (2.1) \\ = \frac{z^{\frac{-\alpha-\beta}{k}}}{k\Gamma_k(\alpha)} \sum_{n=0}^{\infty} \frac{(\alpha+\beta)_n (-\eta)_n}{(\alpha)_{n,k} n!} \sum_{p,h=0}^{\infty} \frac{(-1)^p (\frac{1}{2})^{2p+\frac{q}{k}} a^h}{\Gamma_k(q+pk+k)p! \prod_{j=1}^m \Gamma_k(\mu_j + \frac{h}{\rho_j})} \int_0^z \frac{(z-t)^{\frac{\alpha}{k}-1} (1-\frac{t}{z})^n}{t^{1-\frac{\lambda+\sigma hk+2pk+q}{k}}} dt \\ = \frac{z^{\frac{-\beta}{k}-1}}{k\Gamma_k(\alpha)} \sum_{n=0}^{\infty} \frac{(\alpha+\beta)_n (-\eta)_n}{(\alpha)_{n,k} n!} \xi_{p,h} \int_0^z (1-\frac{t}{z})^{\frac{\alpha}{k}+n-1} t^{\frac{\lambda+\sigma hk+2pk+q}{k}-1} dt,$$

where

$$\xi_{p,h} = \sum_{p,h=0}^{\infty} \frac{(-1)^p (\frac{1}{2})^{2p+\frac{q}{k}} a^h}{\Gamma_k(q+pk+k) p! \prod_{j=1}^m \Gamma_k(\mu_j + \frac{h}{\rho_j})}.$$

By putting these values $\frac{t}{z} = u \Rightarrow t = uz \Rightarrow dt = zdu \quad t = 0 \Rightarrow u = 0 \quad t = z \Rightarrow u = 1$ in the equation (2.1), we obtain

$$\begin{aligned} I_{k;0,z}^{\alpha,\beta,\eta} \{z^{\frac{\lambda}{k}-1} J_{q;k}(z) E_{k;(\frac{1}{\rho_j},\mu_j)}(az^\sigma)\} \\ = \frac{z^{\frac{-\beta}{k}-1}}{k\Gamma_k(\alpha)} \sum_{n=0}^{\infty} \frac{(\alpha+\beta)_{n,k}(-\eta)_{n,k}}{(\alpha)_{n,k} n!} \xi_{p,h} \int_0^1 (1-u)^{\frac{\alpha}{k}+n-1} (uz)^{\frac{\lambda+\sigma hk+2pk+q}{k}-1} zdu \\ = \frac{z^{\frac{\lambda-\beta}{k}-1}}{k\Gamma_k(\alpha)} \sum_{n=0}^{\infty} \frac{(\alpha+\beta)_{n,k}(-\eta)_{n,k}}{(\alpha)_{n,k} n!} \xi_{p,h} z^{2p+\sigma h+\frac{q}{k}} \int_0^1 (1-u)^{\frac{\alpha}{k}+n-1} u^{\frac{\lambda+\sigma hk+2pk+q}{k}-1} du. \end{aligned} \quad (2.2)$$

By using equation (1.2) in the equation (2.2), we get

$$\begin{aligned} I_{k;0,z}^{\alpha,\beta,\eta} \{z^{\frac{\lambda}{k}-1} J_{q;k}(z) E_{k;(\frac{1}{\rho_j},\mu_j)}(az^\sigma)\} \\ = \frac{z^{\frac{\lambda-\beta}{k}-1}}{\Gamma_k(\alpha)} \sum_{n=0}^{\infty} \frac{(\alpha+\beta)_{n,k}(-\eta)_{n,k}}{(\alpha)_{n,k} n!} \xi_{p,h} z^{2p+\sigma h+\frac{q}{k}} \beta_k(\lambda+\sigma hk+2pk+q, \alpha+kn). \end{aligned} \quad (2.3)$$

By using equation (1.3) in the equation (2.3), we have

$$\begin{aligned} I_{k;0,z}^{\alpha,\beta,\eta} \{z^{\frac{\lambda}{k}-1} J_{q;k}(z) E_{k;(\frac{1}{\rho_j},\mu_j)}(az^\sigma)\} \\ = \frac{z^{\frac{\lambda-\beta}{k}-1}}{\Gamma_k(\alpha)} \sum_{n=0}^{\infty} \frac{(\alpha+\beta)_{n,k}(-\eta)_{n,k}}{(\alpha)_{n,k} n!} \xi_{p,h} z^{2p+\sigma h+\frac{q}{k}} \frac{\Gamma_k(\lambda+\sigma hk+2pk+q)\Gamma_k(\alpha+kn)}{\Gamma_k(\lambda+\sigma hk+2pk+q+\alpha+kn)}. \end{aligned} \quad (2.4)$$

By using the equation (1.4) in the equation (2.4), we obtain

$$\begin{aligned} I_{k;0,z}^{\alpha,\beta,\eta} \{z^{\frac{\lambda}{k}-1} J_{q;k}(z) E_{k;(\frac{1}{\rho_j},\mu_j)}(az^\sigma)\} \\ = \frac{z^{\frac{\lambda-\beta}{k}-1}}{\Gamma_k(\alpha)} \sum_{n=0}^{\infty} \frac{(\alpha+\beta)_{n,k}(-\eta)_{n,k}}{(\alpha)_{n,k} n!} \xi_{p,h} \frac{z^{2p+\sigma h+\frac{q}{k}} \Gamma_k(\lambda+\sigma hk+2pk+q)(\alpha)_{n,k} \Gamma_k(\alpha)}{\Gamma_k(\lambda+\sigma hk+2pk+q+\alpha)(\lambda+\sigma hk+2pk+q+\alpha)_{n,k}} \\ = z^{\frac{\lambda-\beta}{k}-1} \xi_{p,h} z^{2p+\sigma h+\frac{q}{k}} \sum_{n=0}^{\infty} \frac{(\alpha+\beta)_{n,k}(-\eta)_{n,k} (1)^n \Gamma_k(\lambda+\sigma hk+2pk+q)}{(n!) \Gamma_k(\lambda+\sigma hk+2pk+q+\alpha)(\lambda+\sigma hk+2pk+q+\alpha)_{n,k}}. \end{aligned} \quad (2.5)$$

Since

$${}_2F_{1,k}(a, b; c; 1) = \frac{\Gamma_k(c)\Gamma_k(c-a-b)}{\Gamma_k(c-a)\Gamma_k(c-b)}.$$

Therefore, the equation (2.5), becomes

$$\begin{aligned} I_{k;0,z}^{\alpha,\beta,\eta} \{z^{\frac{\lambda}{k}-1} J_{q;k}(z) E_{k;(\frac{1}{\rho_j},\mu_j)}(az^\sigma)\} \\ = \sum_{p,h=0}^{\infty} \frac{z^{\frac{\lambda-\beta}{k}-1} (-1)^p (\frac{z}{2})^{2p+\frac{q}{k}} (az^\sigma)^h}{\Gamma_k(q+pk+k) p! \prod_{j=1}^m \Gamma_k(\mu_j + \frac{h}{\rho_j})} \\ \times \frac{\Gamma_k(\lambda+\sigma hk+2pk+q)\Gamma_k(\lambda+\eta-\beta+\sigma hk+2pk+q)}{\Gamma_k(\lambda-\beta+\sigma hk+2pk+q)\Gamma_k(\lambda+\alpha+\eta+\sigma hk+2pk+q)} \\ = \sum_{h=0}^{\infty} \frac{z^{\frac{\lambda-\beta+q}{k}-1} (\frac{1}{2})^{\frac{q}{k}} (az^\sigma)^h}{\prod_{j=1}^m \Gamma_k(\mu_j + \frac{h}{\rho_j})} {}_2\psi_3^k \left[\begin{matrix} (\lambda+\sigma hk+q, 2k)(\lambda+\eta-\beta+\sigma hk+q, 2k) \\ (q+k, k)(\lambda-\beta+\sigma hk+q, 2k)(\lambda+\alpha+\eta+\sigma hk+q, 2k) \end{matrix} \middle| \frac{-z^2}{4} \right]. \end{aligned}$$

□

Corollary 2.2. Suppose that $m \geq 1$ is an integer and $I_{k;0,z}^{\alpha,\beta,\eta}$ be the left sided Saigo's k -integral operator and $\operatorname{Re}(\rho_j) > 0$, $u_j (j = 1, \dots, m)$ are arbitrary parameters, then the following result holds:

$$\begin{aligned} I_{k;0,z}^{\alpha,\beta,\eta} \{z^{\frac{\lambda}{k}-1} J_{q;k}(z) E_{k;(\frac{1}{\rho_j}, \mu_j)}(az^\sigma)\} &= z^{\frac{\lambda-\beta}{k}-1} \sum_{p=0}^{\infty} \frac{(-1)^p (\frac{z}{2})^{2p+\frac{q}{k}}}{\Gamma_k(q+pk+k)p!} \\ &\times {}_3\Psi_3^k \left[\begin{matrix} (\lambda+q+2kp, \sigma k)(\lambda+\eta-\beta+q+2kp, \sigma k)(k, k) \\ (\lambda-\beta+q+2kp, \sigma k)(\lambda+\alpha+\eta+q+2kp, \sigma k)(\mu_j, \frac{1}{\rho_j}) \end{matrix} \middle| \begin{matrix} az^\sigma \\ \frac{m}{k} \end{matrix} \right], \end{aligned}$$

where λ and σ are arbitrary parameters and k is any positive real number such that $\operatorname{Re}(\alpha) > 0$ and α, β, η and a are any complex numbers.

Corollary 2.3. Left sided Saigo's k -integral operator, where $\operatorname{Re}(\alpha) > 0$ and α, β, η and σ are any complex numbers then the following result holds:

$$\begin{aligned} I_{k;0,z}^{\alpha,\beta,\eta} \{J_{p,k}(z) J_{q,k}(z^\sigma)\} &= z^{\frac{q\sigma-\beta-k}{k}} \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{z}{2})^{2n+\frac{p}{k}} (\frac{1}{2})^{\frac{\sigma q}{k}}}{n! \Gamma_k(p+nk+k)} \\ &\times {}_2\Psi_3^k \left[\begin{matrix} (2nk+p+k+q\sigma k, 2\sigma)(\eta-\beta+2nk+p+k+q\sigma k, 2\sigma) \\ (2nk+p+k+q\sigma k-\beta, 2\sigma)(\alpha+\eta 2nk+p+k+q\sigma k, 2\sigma)(q+k, k) \end{matrix} \middle| \begin{matrix} (-z)^{2\sigma} \\ \frac{m}{k} \end{matrix} \right]. \end{aligned}$$

Theorem 2.4. Suppose that $m \geq 1$ and $I_{k;z,\infty}^{\alpha,\beta,\eta}$ be the right sided Saigo's k -integral operator and $\operatorname{Re}(\rho_j) > 0$, $u_j (j = 1, \dots, m)$ are arbitrary parameters then this result holds:

$$\begin{aligned} I_{k;z,\infty}^{\alpha,\beta,\eta} \{z^{\frac{\lambda}{k}} J_{q;k}(z) E_{k;(\frac{1}{\rho_j}, \mu_j)}(az^{-\sigma})\} &= z^{\frac{\lambda-\beta+q}{k}} \sum_{h=0}^{\infty} \frac{(\frac{1}{2})^{\frac{q}{k}} (az^{-\sigma})^h}{\prod_{j=1}^m \Gamma_k(\mu_j + \frac{h}{\rho_j})} \\ &\times {}_2\Psi_3^k \left[\begin{matrix} (\beta-\lambda+\sigma kh-q, -2k)(\eta-\lambda+\sigma kh-q, -2k) \\ (q+k, k)(-\lambda+\sigma kh-q, -2k)(\alpha+\beta+\eta-\lambda+\sigma kh-q, -2k) \end{matrix} \middle| \begin{matrix} -z^2 \\ \frac{m}{k} \end{matrix} \right], \end{aligned}$$

where λ and σ are arbitrary parameters and $k \in \mathbb{R}^+$ such that $\operatorname{Re}(\alpha) > 0$ and α, β, η and a are any complex numbers.

Proof. Consider the right sided fractional integral operator and multi-index Mittag-Leffler k -function

$$\begin{aligned} &I_{k;z,\infty}^{\alpha,\beta,\eta} \{z^{\frac{\lambda}{k}} J_{q;k}(z) E_{k;(\frac{1}{\rho_j}, \mu_j)}(az^{-\sigma})\} \\ &= \frac{1}{k \Gamma_k(\alpha)} \int_z^\infty \frac{(t-z)^{\frac{\alpha}{k}-1}}{t^{\frac{\alpha+\beta}{k}}} {}_2F_{1,k}(\alpha+\beta, -\eta; \alpha; (1-\frac{z}{t})) t^{\frac{\lambda}{k}} \sum_{p,h=0}^{\infty} \frac{(-1)^p (\frac{t}{2})^{2p+\frac{q}{k}} (at^{-\sigma})^h}{\Gamma_k(q+pk+k)p! \prod_{j=1}^m \Gamma_k(\mu_j + \frac{h}{\rho_j})} dt \\ &= \frac{1}{k \Gamma_k(\alpha)} \int_z^\infty \frac{(t-z)^{\frac{\alpha}{k}-1}}{t^{\frac{\alpha+\beta}{k}}} \sum_{n=0}^{\infty} \frac{(\alpha+\beta)_{n,k}(-\eta)_{n,k}}{(\alpha)_{n,k} n!} (1-\frac{z}{t})^n \sum_{p,h=0}^{\infty} \frac{t^{\frac{\lambda}{k}} (-1)^p (\frac{t}{2})^{2p+\frac{q}{k}} (at^{-\sigma})^h}{\Gamma_k(q+pk+k)p! \prod_{j=1}^m \Gamma_k(\mu_j + \frac{h}{\rho_j})} dt \quad (2.6) \\ &= \frac{1}{k \Gamma_k(\alpha)} \sum_{n=0}^{\infty} \frac{(\alpha+\beta)_{n,k}(-\eta)_{n,k}}{(\alpha)_{n,k} n!} \sum_{p,h=0}^{\infty} \frac{(-1)^p (\frac{1}{2})^{2p+\frac{q}{k}} a^h}{\Gamma_k(q+pk+k)p! \prod_{j=1}^m \Gamma_k(\mu_j + \frac{h}{\rho_j})} \int_z^\infty \frac{(t-z)^{\frac{\alpha}{k}-1} (1-\frac{z}{t})^n}{t^{\frac{-\lambda+\alpha+\beta+\sigma kh-2pk-q}{k}}} dt \\ &= \frac{1}{k \Gamma_k(\alpha)} \sum_{n=0}^{\infty} \frac{(\alpha+\beta)_{n,k}(-\eta)_{n,k}}{(\alpha)_{n,k} n!} \xi_{p,h} \int_z^\infty (1-\frac{z}{t})^{\frac{\alpha}{k}+n-1} t^{\frac{\lambda-\beta-\sigma kh+2pk+q}{k}-1} dt, \end{aligned}$$

where

$$\xi_{p,h} = \sum_{p,h=0}^{\infty} \frac{(-1)^p (\frac{1}{2})^{2p+\frac{q}{k}} a^h}{\Gamma_k(q+pk+k)p! \prod_{j=1}^m \Gamma_k(\mu_j + \frac{h}{\rho_j})}.$$

By putting the values $\frac{z}{t} = u \Rightarrow t = \frac{z}{u}$, $t = z \Rightarrow u = 1$, $t = \infty \Rightarrow u = 0$, $\frac{z}{t} = u \Rightarrow dt = -\frac{t^2}{z} du$, $dt = -\frac{z}{u^2} du$ in the equation (2.6), we obtain

$$\begin{aligned} & I_{k;z,\infty}^{\alpha,\beta,\eta} \{z^{\frac{\lambda}{k}} J_{q;k}(z) E_{k;(\frac{1}{\rho_j}, \mu_j)} az^{-\sigma}\} \\ &= \frac{1}{k \Gamma_k(\alpha)} \sum_{n=0}^{\infty} \frac{(\alpha + \beta)_{n,k} (-\eta)_{n,k}}{(\alpha)_{n,k} n!} \xi_{p,h} \int_1^0 (1-u)^{\frac{\alpha}{k}+n-1} \left(\frac{z}{u}\right)^{\frac{\lambda-\beta-\sigma hk+2pk+q}{k}-1} \left(\frac{-z}{u^2}\right) du \\ &= \frac{1}{k \Gamma_k(\alpha)} \sum_{n=0}^{\infty} \frac{(\alpha + \beta)_{n,k} (-\eta)_{n,k}}{(\alpha)_{n,k} n!} \xi_{p,h} \int_0^1 (1-u)^{\frac{\alpha}{k}+n-1} z^{\frac{\lambda-\beta}{k}} z^{-\sigma hk} z^{2p+\frac{q}{k}} u^{\frac{-\lambda+\beta+\sigma hk-2pk-q}{k}-1} du \\ &= \frac{z^{\frac{\lambda-\beta}{k}}}{\Gamma_k(\alpha)} \sum_{n=0}^{\infty} \frac{(\alpha + \beta)_{n,k} (-\eta)_{n,k}}{(\alpha)_{n,k} n!} \xi_{p,h} z^{2p-\sigma hk+\frac{q}{k}} \frac{1}{k} \int_0^1 (1-u)^{\frac{\alpha}{k}+n-1} u^{\frac{-\lambda+\beta+\sigma hk-2pk-q}{k}-1} du. \end{aligned} \quad (2.7)$$

By using the equation (1.2) in the equation (2.7), we get

$$\begin{aligned} & I_{k;z,\infty}^{\alpha,\beta,\eta} \{z^{\frac{\lambda}{k}} J_{q;k}(z) E_{k;(\frac{1}{\rho_j}, \mu_j)} az^{-\sigma}\} \\ &= \frac{z^{\frac{\lambda-\beta}{k}}}{\Gamma_k(\alpha)} \sum_{n=0}^{\infty} \frac{(\alpha + \beta)_{n,k} (-\eta)_{n,k}}{(\alpha)_{n,k} n!} \xi_{p,h} z^{2p-\sigma hk+\frac{q}{k}} \beta_k(-\lambda + \beta + \sigma hk - 2pk - q, \alpha + kn). \end{aligned} \quad (2.8)$$

By using the equation (1.3) in the equation (2.8), we have

$$\begin{aligned} & I_{k;z,\infty}^{\alpha,\beta,\eta} \{z^{\frac{\lambda}{k}} J_{q;k}(z) E_{k;(\frac{1}{\rho_j}, \mu_j)} az^{-\sigma}\} \\ &= \frac{z^{\frac{\lambda-\beta}{k}}}{\Gamma_k(\alpha)} \sum_{n=0}^{\infty} \frac{(\alpha + \beta)_{n,k} (-\eta)_{n,k}}{(\alpha)_{n,k} n!} \xi_{p,h} z^{2p-\sigma hk+\frac{q}{k}} \frac{\Gamma_k(\beta - \lambda + \sigma hk - 2pk - q) \Gamma_k(\alpha + kn)}{\Gamma_k(\beta - \lambda + \sigma hk - 2pk - q + \alpha + kn)}. \end{aligned} \quad (2.9)$$

Since

$$\Gamma_k(\alpha + kn) = (\alpha)_{n,k} \Gamma_k(\alpha). \quad (2.10)$$

Therefore, using the equation (2.9) in equation (2.10), we obtain

$$\begin{aligned} & I_{k;z,\infty}^{\alpha,\beta,\eta} \{z^{\frac{\lambda}{k}} J_{q;k}(z) E_{k;(\frac{1}{\rho_j}, \mu_j)} az^{-\sigma}\} \\ &= z^{\frac{\lambda-\beta}{k}} \xi_{p,h} \sum_{n=0}^{\infty} \frac{z^{2p-\sigma hk+\frac{q}{k}} (\alpha + \beta)_{n,k} (-\eta)_{n,k} \Gamma_k(\beta - \lambda + \sigma hk - 2pk - q) (\alpha)_{n,k} \Gamma_k(\alpha)}{\Gamma_k(\alpha) (\alpha)_{n,k} n! (\beta - \lambda + \sigma hk - 2pk - q + \alpha)_{n,k} \Gamma_k(\sigma hk - 2pk - q + \alpha + \beta - \lambda)} \\ &= z^{\frac{\lambda-\beta}{k}} \xi_{p,h} z^{2p-\sigma hk+\frac{q}{k}} \sum_{n=0}^{\infty} \frac{(\alpha + \beta)_{n,k} (-\eta)_{n,k} (1)^n}{n! (\beta - \lambda + \sigma hk - 2pk - q + \alpha)_{n,k}} \frac{\Gamma_k(\beta - \lambda + \sigma hk - 2pk - q)}{\Gamma_k(\sigma hk - 2pk - q + \alpha + \beta - \lambda)}. \end{aligned} \quad (2.11)$$

Since

$${}_2F_{1,k}(a, b; c; 1) = \frac{\Gamma_k(c) \Gamma_k(c-a-b)}{\Gamma_k(c-a) \Gamma_k(c-b)}.$$

Therefore, the equation (2.11), becomes

$$\begin{aligned} & I_{k;z,\infty}^{\alpha,\beta,\eta} \{z^{\frac{\lambda}{k}} J_{q;k}(z) E_{k;(\frac{1}{\rho_j}, \mu_j)} az^{-\sigma}\} \\ &= \sum_{p,h=0}^{\infty} \frac{z^{\frac{\lambda-\beta}{k}} (-1)^p \left(\frac{z}{2}\right)^{2p+\frac{q}{k}} (az^{-\sigma})^h}{\Gamma_k(q + pk + k) p! \prod_{j=1}^m \Gamma_k(\mu_j + \frac{h}{\rho_j})} \\ &\times \frac{\Gamma_k(\beta - \lambda + \sigma hk - 2pk - q) \Gamma_k(\eta - \lambda + \sigma hk - 2pk - q)}{\Gamma_k(-\lambda + \sigma hk - 2pk - q) \Gamma_k(\alpha + \beta + \eta - \lambda + \sigma hk - 2pk - q)} \end{aligned}$$

$$= \sum_{h=0}^{\infty} \frac{(\frac{1}{2})^{\frac{q}{k}} z^{\frac{\lambda-\beta+q}{k}} (az^{-\sigma})^h}{\prod_{j=1}^m \Gamma_k(\mu_j + \frac{h}{\rho_j})} {}_2\Psi_3^k \left[\begin{matrix} (\beta - \lambda + \sigma kh - q, -2k)(\eta - \lambda + \sigma kh - q, -2k) \\ (q+k, k)(-\lambda + \sigma kh - q, -2k)(\alpha + \beta + \eta - \lambda + \sigma kh - q, -2k) \end{matrix} \middle| \frac{-z^2}{4} \right].$$

□

Corollary 2.5. Suppose that $m \geq 1$ and $I_{k,z,\infty}^{\alpha,\beta,\eta}$ be the right sided Saigo's k -integral operator and $\operatorname{Re}(\rho_j) > 0$, $u_j(j = 1, \dots, m)$ are arbitrary parameters then this result holds:

$$\begin{aligned} I_{k,z,\infty}^{\alpha,\beta,\eta} \{ z^{\frac{\lambda}{k}} J_{q;k}(z) E_{k;(\frac{1}{\rho_j}, \mu_j)} az^{-\sigma} \} &= z^{\frac{\lambda-\beta}{k}} \sum_{p=0}^{\infty} \frac{(-1)^p (\frac{z}{2})^{2p+\frac{q}{k}}}{\Gamma_k(q+pk+k)p!} \\ &\times {}_3\Psi_3^k \left[\begin{matrix} (\beta - \lambda - q - 2kp, \sigma k)(\eta - \lambda - q - 2kp, \sigma k)(k, k) \\ (-\lambda - q - 2kp, \sigma k)(\alpha + \beta + \eta - \lambda - q - 2kp, \sigma k)(u_j, \frac{1}{\rho_j}) \end{matrix} \middle| \frac{az^{-\sigma}}{k} \right], \end{aligned}$$

where λ and σ are arbitrary parameters and $k \in \mathbb{R}^+$ such that $\operatorname{Re}(\alpha) > 0$ and α, β, η and a are any complex numbers.

Corollary 2.6. The right sided Saigo's k -integral operator for $m \geq 1$ and $\operatorname{Re}(\rho_j) > 0$, $u_j(j = 1, \dots, m)$ are arbitrary parameters then this result holds:

$$\begin{aligned} I_{k,z,\infty}^{\alpha,\beta,\eta} \{ E_{k;(\frac{1}{\rho_j}, \mu_j)}(z) E_{k;(\frac{1}{\rho_j}, \mu_j)} az^\sigma \} &= z^{\frac{-\beta}{k}} \sum_{n=0}^{\infty} \frac{(az^\sigma)^n}{\prod_{j=1}^m \Gamma_k(\mu_j + \frac{n}{\rho_j})} \\ &\times {}_3\Psi_3^k \left[\begin{matrix} (\beta - \sigma nk, -k)(\eta - \sigma nk, -k)(k, k) \\ (-nk, -k)(\alpha + \beta + \eta - \sigma nk, -k)(u_j, \frac{1}{\rho_j}) \end{matrix} \middle| \frac{z}{k} \right], \end{aligned}$$

where λ and σ are arbitrary parameters and $k \in \mathbb{R}^+$ such that $\operatorname{Re}(\alpha) > 0$ and α, β, η are complex numbers.

3. Erdelyi-Kober fractional k -integral operator with multi-index Mittag-Leffler k -function and Bessel k -function

In this section, we solve Erdelyi Kober fractional k -integrals with the product of multi-index Mittag-Leffler and Bessel k -functions in the form of theorems.

Theorem 3.1. Suppose that $m \geq 1$ is an integer and $K_{k,0,z}^{\alpha,\eta}$ be the Erdelyi-Kober integral operator and $\operatorname{Re}(\rho_j) > 0$, $u_j(j = 1, \dots, m)$ are arbitrary parameters, then this result holds:

$$\begin{aligned} K_{k,0,z}^{\alpha,\eta} \{ z^{\frac{\lambda}{k}-1} J_{q;k}(z) E_{k;(\frac{1}{\rho_j}, \mu_j)} az^\sigma \} \\ = z^{\frac{\lambda+q}{k}-1} \sum_{h=0}^{\infty} \frac{(\frac{1}{2})^{\frac{q}{k}} (az^\sigma)^h}{\prod_{j=1}^m \Gamma_k(\mu_j + \frac{h}{\rho_j})} {}_1\Psi_2^k \left[\begin{matrix} (\lambda + \eta + \sigma kh + q, 2k) \\ (q+k, k)(\lambda + \alpha + \eta + \sigma kh + q, 2k) \end{matrix} \middle| \frac{-z^2}{4} \right], \end{aligned}$$

where η , α and a are any complex numbers and $k \in \mathbb{R}^+$.

Proof. Consider the Erdelyi-kober fractional k -integral and multi-index Mittag-Leffler k -function:

$$\begin{aligned} K_{k,0,z}^{\alpha,\eta} \{ z^{\frac{\lambda}{k}-1} J_{q;k}(z) E_{k;(\frac{1}{\rho_j}, \mu_j)} az^\sigma \} \\ = \frac{z^{\frac{-\alpha-\eta}{k}}}{k\Gamma_k(\alpha)} \int_0^z (z-t)^{\frac{\alpha}{k}-1} t^{\frac{\eta}{k}} t^{\frac{\lambda}{k}-1} \sum_{p,h=0}^{\infty} \frac{(-1)^p (\frac{t}{2})^{2p+\frac{q}{k}} (at^\sigma)^h}{\Gamma_k(q+pk+k)p!\prod_{j=1}^m \Gamma_k(\mu_j + \frac{h}{\rho_j})} dt \\ = \frac{z^{\frac{-\alpha-\eta}{k}}}{k\Gamma_k(\alpha)} \sum_{p,h=0}^{\infty} \frac{(-1)^p (\frac{1}{2})^{2p+\frac{q}{k}} a^h}{\Gamma_k(q+pk+k)p!\prod_{j=1}^m \Gamma_k(\mu_j + \frac{h}{\rho_j})} \int_0^z (z-t)^{\frac{\alpha}{k}-1} t^{\frac{\lambda+\eta+\sigma hk+2pk+q}{k}-1} dt \\ = \frac{z^{\frac{-\eta}{k}-1}}{k\Gamma_k(\alpha)} \xi_{p,h} \int_0^z (1-\frac{t}{z})^{\frac{\alpha}{k}-1} t^{\frac{\lambda+\eta+\sigma hk+2pk+q}{k}-1} dt, \end{aligned} \tag{3.1}$$

where

$$\xi_{p,h} = \sum_{p,h=0}^{\infty} \frac{(-1)^p (\frac{1}{2})^{2p+\frac{q}{k}} a^h}{\Gamma_k(q+pk+k)p! \prod_{j=1}^m \Gamma_k(\mu_j + \frac{h}{\rho_j})}.$$

By putting these values $\frac{t}{z} = u \Rightarrow t = uz \Rightarrow dt = zdu$ $t = 0 \Rightarrow u = 0$ $t = z \Rightarrow u = 1$ in the equation (3.1), we obtain

$$\begin{aligned} K_{k;0,z}^{\alpha,\eta} \{z^{\frac{\lambda}{k}-1} J_{q;k}(z) E_{k;(\frac{1}{\rho_j},\mu_j)} az^\sigma\} &= \frac{z^{\frac{-\eta}{k}-1}}{k \Gamma_k(\alpha)} \xi_{p,h} \int_0^1 (1-u)^{\frac{\alpha}{k}-1} (uz)^{\frac{\lambda+\eta+\sigma hk+2pk+q}{k}-1} z du \\ &= \frac{z^{\frac{\lambda}{k}-1}}{k \Gamma_k(\alpha)} \xi_{p,h} z^{2p+\sigma h+\frac{q}{k}} \int_0^1 (1-u)^{\frac{\alpha}{k}-1} u^{\frac{\lambda+\eta+\sigma hk+2pk+q}{k}-1} du. \end{aligned} \quad (3.2)$$

By using the equation (1.2) in the equation (3.2), we get

$$K_{k;0,z}^{\alpha,\eta} \{z^{\frac{\lambda}{k}-1} J_{q;k}(z) E_{k;(\frac{1}{\rho_j},\mu_j)} az^\sigma\} = \frac{z^{\frac{\lambda}{k}-1}}{\Gamma_k(\alpha)} \xi_{p,h} z^{2p+\sigma h+\frac{q}{k}} \beta_k(\eta + \lambda + \sigma kh + 2pk + q, \alpha). \quad (3.3)$$

By using the equation (1.3) in the equation (3.3), we have

$$\begin{aligned} K_{k;0,z}^{\alpha,\eta} \{z^{\frac{\lambda}{k}-1} J_{q;k} E_{k;(\frac{1}{\rho_j},\mu_j)} az^\sigma\} &= \frac{z^{\frac{\lambda}{k}-1}}{\Gamma_k(\alpha)} \xi_{p,h} z^{2p+\sigma h+\frac{q}{k}} \frac{\Gamma_k(\lambda + \eta + \sigma kh + 2pk + q) \Gamma_k(\alpha)}{\Gamma_k(\alpha + \lambda + \eta + \sigma kh + 2pk + q)} \\ &= z^{\frac{\lambda}{k}-1} \sum_{p,h=0}^{\infty} \frac{(-1)^p (\frac{z}{2})^{2p+\frac{q}{k}} (az^\sigma)^h}{\Gamma_k(q+pk+k)p! \prod_{j=1}^m \Gamma_k(\mu_j + \frac{h}{\rho_j})} \frac{\Gamma_k(\lambda + \eta + \sigma kh + 2pk + q)}{\Gamma_k(\alpha + \lambda + \eta + \sigma kh + 2pk + q)} \\ &= z^{\frac{\lambda+q}{k}-1} \sum_{h=0}^{\infty} \frac{(\frac{1}{2})^{\frac{q}{k}} (az^\sigma)^h}{\prod_{j=1}^m \Gamma_k(\mu_j + \frac{h}{\rho_j})} {}_1\Psi_2^k \left[\begin{matrix} (\lambda + \eta + \sigma kh + q, 2k) \\ (q+k, k)(\lambda + \alpha + \eta + \sigma kh + q, 2k) \end{matrix} \middle| \frac{-z^2}{4} \right]. \end{aligned} \quad \square$$

Corollary 3.2. Suppose that $m \geq 1$ is an integer and $K_{k;0,z}^{\alpha,\eta}$ be the Erdelyi-Kober integral operator and $\operatorname{Re}(\rho_j) > 0$, $u_j (j = 1, \dots, m)$ are arbitrary parameters, then this result holds:

$$\begin{aligned} K_{k;0,z}^{\alpha,\eta} \{z^{\frac{\lambda}{k}-1} J_{q;k}(z) E_{k;(\frac{1}{\rho_j},\mu_j)} az^\sigma\} &= z^{\frac{\lambda}{k}-1} \sum_{p=0}^{\infty} \frac{(-1)^p (\frac{z}{2})^{2p+\frac{q}{k}}}{\Gamma_k(q+pk+k)p!} {}_2\Psi_2^k \left[\begin{matrix} (\lambda + \eta + q + 2kp, \sigma k)(k, k) \\ (\lambda + \alpha + \eta + q + 2kp, \sigma k)(\mu_j, \frac{1}{\rho_j}) \end{matrix} \middle| \frac{az^\sigma}{k} \right], \end{aligned}$$

where η , α and a are any complex numbers and $k \in \mathbb{R}^+$.

Theorem 3.3. Suppose that $m \geq 1$ and $K_{k;z,\infty}^{\alpha,\eta}$ be the Erdelyi-Kober fractional k -integral operator and $\operatorname{Re}(\rho_j) > 0$, $u_j (j = 1, \dots, m)$ are arbitrary parameters then this result holds:

$$\begin{aligned} K_{k;z,\infty}^{\alpha,\eta} \{z^{\frac{\lambda}{k}} J_{q;k}(z) E_{k;(\frac{1}{\rho_j},\mu_j)} az^{-\sigma}\} &= z^{\frac{\lambda+q}{k}} \sum_{h=0}^{\infty} \frac{(\frac{1}{2})^{\frac{q}{k}} (az^{-\sigma})^h}{\prod_{j=1}^m \Gamma_k(\mu_j + \frac{h}{\rho_j})} {}_1\Psi_2^k \left[\begin{matrix} (\eta - \lambda + \sigma kh - q, -2k) \\ (q+k, k)(\alpha + \eta - \lambda + \sigma kh - q, -2k) \end{matrix} \middle| \frac{-z^2}{4} \right], \end{aligned}$$

where α , η , a are complex numbers, k is any positive real number and $\operatorname{Re}(\alpha) > 0$.

Proof. Consider Erdelyi-Kober fractional k-integral and multi-index Miitag-Leffler k-function

$$\begin{aligned}
 & K_{k;z,\infty}^{\alpha,n} \{z^{\frac{\lambda}{k}} J_{q;k}(z) E_{k;(\frac{1}{\rho_j}, \mu_j)} az^{-\sigma}\} \\
 &= \frac{z^{\frac{\eta}{k}}}{k \Gamma_k(\alpha)} \int_z^\infty (t-z)^{\frac{\alpha}{k}-1} t^{-\frac{\alpha-\eta}{k}} t^{\frac{\lambda}{k}} \sum_{p,h=0}^\infty \frac{(-1)^p (\frac{t}{2})^{2p+\frac{q}{k}} (at^{-\sigma})^h}{\Gamma_k(q+pk+k) p! \prod_{j=1}^m \Gamma_k(\mu_j + \frac{h}{\rho_j})} dt \\
 &= \frac{z^{\frac{\eta}{k}}}{k \Gamma_k(\alpha)} \sum_{p,h=0}^\infty \frac{(-1)^p (\frac{1}{2})^{2p+\frac{q}{k}} a^h}{\Gamma_k(q+pk+k) p! \prod_{j=1}^m \Gamma_k(\mu_j + \frac{h}{\rho_j})} \int_z^\infty (t-z)^{\frac{\alpha}{k}-1} t^{\frac{\lambda-\alpha-\eta}{k}} t^{-\sigma h + 2p + \frac{q}{k}} dt \\
 &= \frac{z^{\frac{\eta}{k}}}{k \Gamma_k(\alpha)} \xi_{p,h} \int_z^\infty (1 - \frac{z}{t})^{\frac{\alpha}{k}-1} t^{\frac{\lambda-\eta-\sigma hk+2pk+q}{k}-1} dt,
 \end{aligned} \tag{3.4}$$

where

$$\xi_{p,h} = \sum_{p,h=0}^\infty \frac{(-1)^p (\frac{1}{2})^{2p+\frac{q}{k}} a^h}{\Gamma_k(q+pk+k) p! \prod_{j=1}^m \Gamma_k(\mu_j + \frac{h}{\rho_j})}.$$

By substituting $\frac{z}{t} = u \Rightarrow t = \frac{z}{u} \quad t = z \Rightarrow u = 1 \quad t = \infty \Rightarrow u = 0 \quad \frac{z}{t} = u \Rightarrow dt = \frac{-t^2}{z} du \quad dt = -\frac{z}{u^2} du$ in the equation (3.4), we obtain

$$\begin{aligned}
 K_{k;z,\infty}^{\alpha,n} \{z^{\frac{\lambda}{k}} J_{q;k}(z) E_{k;(\frac{1}{\rho_j}, \mu_j)} az^{-\sigma}\} &= \frac{z^{\frac{\eta}{k}}}{k \Gamma_k(\alpha)} \xi_{p,h} \int_1^0 (1-u)^{\frac{\alpha}{k}-1} (\frac{z}{u})^{\frac{\lambda-\eta-\sigma hk+2pk+q}{k}-1} (-\frac{z}{u^2}) du \\
 &= \frac{z^{\frac{\eta}{k}}}{k \Gamma_k(\alpha)} \xi_{p,h} \int_0^1 (1-u)^{\frac{\alpha}{k}-1} z^{\frac{\lambda-\eta-\sigma hk+2pk+q}{k}} u^{\frac{-\lambda+\eta+\sigma hk-2pk-q}{k}-1} du \\
 &= \frac{z^{\frac{\lambda}{k}}}{\Gamma_k(\alpha)} \xi_{p,h} z^{2p-\sigma h + \frac{q}{k}} \frac{1}{k} \int_0^1 (1-u)^{\frac{\alpha}{k}-1} u^{\frac{-\lambda+\eta+\sigma hk-2pk-q}{k}-1} du.
 \end{aligned} \tag{3.5}$$

By using the equation (1.2) in the equation (3.5), we get

$$K_{k;z,\infty}^{\alpha,n} \{z^{\frac{\lambda}{k}} J_{q;k}(z) E_{k;(\frac{1}{\rho_j}, \mu_j)} az^{-\sigma}\} = \frac{z^{\frac{\lambda}{k}}}{\Gamma_k(\alpha)} \xi_{p,h} z^{2p-\sigma h + \frac{q}{k}} \beta_k(\eta - \lambda + \sigma kh - 2pk - q, \alpha). \tag{3.6}$$

By using the equation (1.3) in the equation (3.6), we have

$$\begin{aligned}
 & K_{k;z,\infty}^{\alpha,n} \{z^{\frac{\lambda}{k}} J_{q;k}(z) E_{k;(\frac{1}{\rho_j}, \mu_j)} az^{-\sigma}\} \\
 &= \frac{z^{\frac{\lambda}{k}}}{\Gamma_k(\alpha)} \xi_{p,h} z^{2p-\sigma h + \frac{q}{k}} \frac{\Gamma_k(\eta - \lambda + \sigma kh - 2pk - q) \Gamma_k(\alpha)}{\Gamma_k(\eta - \lambda + \sigma kh - 2pk - q + \alpha)} \\
 &= z^{\frac{\lambda}{k}} \sum_{p,h=0}^\infty \frac{(-1)^p (\frac{z}{2})^{2p+\frac{q}{k}} (az^{-\sigma})^h}{\Gamma_k(q+pk+k) p! \prod_{j=1}^m \Gamma_k(\mu_j + \frac{h}{\rho_j})} \frac{\Gamma_k(\eta - \lambda + \sigma kh - 2pk - q)}{\Gamma_k(\eta - \lambda + \sigma kh - 2pk - q + \alpha)} \\
 &= z^{\frac{\lambda+q}{k}} \sum_{h=0}^\infty \frac{(\frac{1}{2})^{\frac{q}{k}} (az^{-\sigma})^h}{\prod_{j=1}^m \Gamma_k(\mu_j + \frac{h}{\rho_j})} {}_1\Psi_2^k \left[\begin{matrix} (\eta - \lambda + \sigma kh - q, -2k) \\ (q+k, k)(\alpha + \eta - \lambda + \sigma kh - q, -2k) \end{matrix} \middle| \frac{-z^2}{4} \right]. \quad \square
 \end{aligned}$$

Corollary 3.4. Suppose that $m \geq 1$ and $K_{k;z,\infty}^{\alpha,n}$ be the Erdelyi-Kober fractional k-integral operator and $\operatorname{Re}(\rho_j) > 0$, $u_j (j = 1, \dots, m)$ are arbitrary parameters then this result holds:

$$\begin{aligned}
 & K_{k;z,\infty}^{\alpha,n} \{z^{\frac{\lambda}{k}} J_{q;k}(z) E_{k;(\frac{1}{\rho_j}, \mu_j)} az^{-\sigma}\} \\
 &= z^{\frac{\lambda}{k}} \sum_{p=0}^\infty \frac{(-1)^p (\frac{z}{2})^{2p+\frac{q}{k}}}{\Gamma_k(q+pk+k) p!} {}_2\Psi_2^k \left[\begin{matrix} (\eta - \lambda - q - 2kp, \sigma k)(k, k) \\ (\alpha + \eta - \lambda - q - 2kp, \sigma k)(\mu_j, \frac{1}{\rho_j}) \end{matrix} \middle| \frac{az^{-\sigma}}{k} \right],
 \end{aligned}$$

where α, η, a are complex numbers, k is any positive real number and $\operatorname{Re}(\alpha) > 0$.

4. Riemann-Liouville fractional k-integral operator and multi-index Mittag-Leffler k-function and Bessel k-function

In this section, we solved the results in which elaborated the relation between the multi-index Mittag-Leffler k-function with Bessel k-function and Riemann-Liouville fractional k-integral operator.

Theorem 4.1. Suppose that $m \geq 1$ is an integer and $R_{k,0,z}^{\alpha}$ be the Riemann-Liouville fractional k-integral operator and $\operatorname{Re}(\rho_j) > 0$, $u_j (j = 1, \dots, m)$ are arbitrary parameters then this result holds:

$$\begin{aligned} & R_{k,0,z}^{\alpha} \{ z^{\frac{\lambda}{k}-1} J_{q;k}(z) E_{k;(\frac{1}{\rho_j}, \mu_j)} az^{\sigma} \} \\ &= z^{\frac{\lambda+\alpha+q}{k}-1} \sum_{h=0}^{\infty} \frac{(\frac{1}{2})^{\frac{q}{k}} (az^{\sigma})^h}{\prod_{j=1}^m \Gamma_k(\mu_j + \frac{h}{\rho_j})} {}_1\Psi_2^k \left[\begin{matrix} (\lambda + \sigma kh + q, 2k) \\ (q+k, k)(\lambda + \alpha + \sigma kh + q, 2k) \end{matrix} \middle| \frac{-z^2}{4} \right], \end{aligned}$$

where α, a are complex numbers, $k \in \mathbb{R}^+$ and $\operatorname{Re}(\alpha) > 0$.

Proof. Consider Riemann-Liouville fractional k-integral and multi-index Mittag-Leffler kfunction

$$\begin{aligned} & R_{k,0,z}^{\alpha} \{ z^{\frac{\lambda}{k}-1} J_{q;k}(z) E_{k;(\frac{1}{\rho_j}, \mu_j)} az^{\sigma} \} \\ &= \frac{1}{k \Gamma_k(\alpha)} \int_0^z (z-t)^{\frac{\alpha}{k}-1} t^{\frac{\lambda}{k}-1} \sum_{p,h=0}^{\infty} \frac{(-1)^p (\frac{t}{2})^{2p+\frac{q}{k}} (at^{\sigma})^h}{\Gamma_k(q+pk+k)p! \prod_{j=1}^m \Gamma_k(\mu_j + \frac{h}{\rho_j})} dt \\ &= \frac{1}{k \Gamma_k(\alpha)} \sum_{p,h=0}^{\infty} \frac{(-1)^p (\frac{1}{2})^{2p+\frac{q}{k}} a^h}{\Gamma_k(q+pk+k)p! \prod_{j=1}^m \Gamma_k(\mu_j + \frac{h}{\rho_j})} \int_0^z (z-t)^{\frac{\alpha}{k}-1} t^{\frac{\lambda}{k}-1} t^{\sigma h+2p+\frac{q}{k}} dt \\ &= \frac{1}{k \Gamma_k(\alpha)} \xi_{p,h} \int_0^z z^{\frac{\alpha}{k}-1} (1-\frac{t}{z})^{\frac{\alpha}{k}-1} t^{\frac{\lambda}{k}-1+\sigma h+2p+\frac{q}{k}} dt, \end{aligned} \tag{4.1}$$

where

$$\xi_{p,h} = \sum_{p,h=0}^{\infty} \frac{(-1)^p (\frac{1}{2})^{2p+\frac{q}{k}} a^h}{\Gamma_k(q+pk+k)p! \prod_{j=1}^m \Gamma_k(\mu_j + \frac{h}{\rho_j})}.$$

By putting these values $\frac{t}{z} = u \Rightarrow t = uz \Rightarrow dt = zd\mu \quad t = 0 \Rightarrow u = 0, \quad t = z \Rightarrow u = 1$ in the equation (4.1), we get

$$\begin{aligned} R_{k,0,z}^{\alpha} \{ z^{\frac{\lambda}{k}-1} J_{q;k}(z) E_{k;(\frac{1}{\rho_j}, \mu_j)} az^{\sigma} \} &= \frac{1}{k \Gamma_k(\alpha)} \xi_{p,h} \int_0^1 z^{\frac{\alpha}{k}-1} (1-u)^{\frac{\alpha}{k}-1} (zu)^{\frac{\lambda+q}{k}-1+\sigma h+2p} z du \\ &= \frac{1}{k \Gamma_k(\alpha)} \xi_{p,h} \int_0^1 z^{\frac{\alpha+\lambda+q}{k}-1+\sigma h+2p} (1-u)^{\frac{\alpha}{k}-1} u^{\frac{\lambda+q}{k}-1+\sigma h+2p} du \\ &= \frac{z^{\frac{\alpha+\lambda}{k}-1}}{\Gamma_k(\alpha)} \xi_{p,h} z^{2p+\sigma h+\frac{q}{k}} \frac{1}{k} \int_0^1 (1-u)^{\frac{\alpha}{k}-1} u^{\frac{\lambda+q+\sigma kh+2pk}{k}-1} du. \end{aligned} \tag{4.2}$$

By using the equation (1.2) in the equation (4.2), we have

$$R_{k,0,z}^{\alpha} \{ z^{\frac{\lambda}{k}-1} J_{q;k}(z) E_{k;(\frac{1}{\rho_j}, \mu_j)} az^{\sigma} \} = \frac{z^{\frac{\alpha+\lambda}{k}-1}}{\Gamma_k(\alpha)} \xi_{p,h} z^{2p+\sigma h+\frac{q}{k}} \beta_k(\lambda + \sigma kh + 2pk + q, \alpha). \tag{4.3}$$

By using the equation (1.3) in the equation (4.3), we obtain

$$R_{k,0,z}^{\alpha} \{ z^{\frac{\lambda}{k}-1} J_{q;k}(z) E_{k;(\frac{1}{\rho_j}, \mu_j)} az^{\sigma} \}$$

$$\begin{aligned}
&= \frac{z^{\frac{\alpha+\lambda}{k}-1}}{\Gamma_k(\alpha)} \xi_{p,h} z^{2p+\sigma h+\frac{q}{k}} \frac{\Gamma_k(\lambda + \sigma kh + 2pk + q) \Gamma_k(\alpha)}{\Gamma_k(\lambda + \sigma kh + 2pk + q + \alpha)} \\
&= z^{\frac{\alpha+\lambda}{k}-1} \sum_{p,h=0}^{\infty} \frac{(-1)^p (\frac{z}{2})^{2p+\frac{q}{k}} (az^\sigma)^h}{\Gamma_k(q + pk + k)p! \prod_{j=1}^m \Gamma_k(\mu_j + \frac{h}{\rho_j})} \frac{\Gamma_k(\lambda + \sigma kh + 2pk + q)}{\Gamma_k(\lambda + \sigma kh + 2pk + q + \alpha)} \\
&= z^{\frac{\lambda+\alpha+q}{k}-1} \sum_{h=0}^{\infty} \frac{(\frac{1}{2})^{\frac{q}{k}} (az^\sigma)^h}{\prod_{j=1}^m \Gamma_k(\mu_j + \frac{h}{\rho_j})} {}_1\Psi_2^k \left[\begin{matrix} (\lambda + \sigma kh + q, 2k) \\ (q + k, k)(\lambda + \alpha + \sigma kh + q, 2k) \end{matrix} \middle| \frac{-z^2}{4} \right].
\end{aligned}$$

□

Corollary 4.2. Suppose that $m \geq 1$ is an integer and $R_{0,z}^\alpha$ be the Riemann-Liouville fractional k -integral operator and $\operatorname{Re}(\rho_j) > 0$, μ_j ($j = 1, \dots, m$) are arbitrary parameters then this result holds:

$$\begin{aligned}
&R_{k,0,z}^\alpha \{ z^{\frac{\lambda}{k}-1} J_{q;k}(z) E_{k;(\frac{1}{\rho_j}, \mu_j)}(az^\sigma) \} \\
&= z^{\frac{\lambda+\alpha}{k}-1} \sum_{p=0}^{\infty} \frac{(-1)^p (\frac{z}{2})^{2p+\frac{q}{k}}}{\Gamma_k(q + pk + k)p!} {}_2\Psi_1^k \left[\begin{matrix} (\lambda + q + 2kp, \sigma k)(k, k) \\ (\lambda + \alpha + q + 2kp, \sigma k)(\mu_j, \frac{1}{\rho_j}) \end{matrix} \middle| \frac{az^\sigma}{k} \right],
\end{aligned}$$

where α, a are complex numbers, $k \in \mathbb{R}^+$ and $\operatorname{Re}(\alpha) > 0$.

Corollary 4.3. The Riemann-Liouville fractional k -integral operator and $\operatorname{Re}(\mu) > 0$, $\operatorname{Re}(\nu) > 0$ and $k \in \mathbb{R}^+$, $\operatorname{Re}(q) > -1$ then this result holds:

$$R_{k,0,z}^\alpha \{ J_{q;k}(z^{\nu k}) E_{k;\mu,\nu} z^{q/k} \} = z^{\frac{\alpha}{k}-1} \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{z^{\nu k}}{2})^{2n+\frac{q}{k}}}{n! \Gamma_k(q + nk + k)} {}_2\Psi_1^k \left[\begin{matrix} (k(2n\nu k + q\nu + 1), q)(k, k) \\ (\alpha + k(2n\nu k + q\nu + 1), q) \end{matrix} \middle| \frac{z^{q/k}}{k} \right].$$

5. Weyl fractional k -integral operator with multi-index Mittag-Leffler k -function Bessel k -function

In this section, we discussed product of multi-index Mittag-Leffler and Bessel k -functions with Weyl fractional k -integral operator in the form of theorem.

Theorem 5.1. Suppose that $m \geq 1$ is an integer and $W_{k,z,\infty}^{\alpha,n}$ be the Weyl fractional k -integral operator and $\operatorname{Re}(\rho_j) > 0$, μ_j ($j = 1, \dots, m$) are arbitrary parameters then this result holds:

$$\begin{aligned}
&W_{k,z,\infty}^\alpha \{ z^{\frac{\lambda}{k}} J_{q;k}(z) E_{k;(\frac{1}{\rho_j}, \mu_j)}(az^{-\sigma}) \} \\
&= z^{\frac{\alpha+\lambda+q}{k}} \sum_{h=0}^{\infty} \frac{(\frac{1}{2})^{\frac{q}{k}} (az^{-\sigma})^h}{\prod_{j=1}^m \Gamma_k(\mu_j + \frac{h}{\rho_j})} {}_1\Psi_2^k \left[\begin{matrix} (-\alpha - \lambda + \sigma kh - q, -2k) \\ (q + k, k)(\sigma kh - q - \lambda, -2k) \end{matrix} \middle| \frac{-z^2}{4} \right],
\end{aligned}$$

where α and a be any complex number, $k \in \mathbb{R}^+$.

Proof. Consider Weyl fractional k -integrals and multi-index Mittag-leffler k -function

$$\begin{aligned}
&W_{k,z,\infty}^\alpha \{ z^{\frac{\lambda}{k}} J_{q;k}(z) E_{k;(\frac{1}{\rho_j}, \mu_j)}(az^{-\sigma}) \} \\
&= \frac{1}{k \Gamma_k(\alpha)} \int_z^\infty (t-z)^{\frac{\alpha}{k}-1} t^{\frac{\lambda}{k}} \sum_{p,h=0}^{\infty} \frac{(-1)^p (\frac{t}{2})^{2p+\frac{q}{k}} (at^{-\sigma})^h}{\Gamma_k(q + pk + k)p! \prod_{j=1}^m \Gamma_k(\mu_j + \frac{h}{\rho_j})} dt \\
&= \frac{1}{k \Gamma_k(\alpha)} \sum_{p,h=0}^{\infty} \frac{(-1)^p (\frac{1}{2})^{2p+\frac{q}{k}} a^h}{\Gamma_k(q + pk + k)p! \prod_{j=1}^m \Gamma_k(\mu_j + \frac{h}{\rho_j})} \int_z^\infty (t-z)^{\frac{\alpha}{k}-1} t^{\frac{\lambda}{k}} t^{-\sigma h + 2p + \frac{q}{k}} dt \\
&= \frac{1}{k \Gamma_k(\alpha)} \xi_{p,h} \int_z^\infty (1 - \frac{z}{t})^{\frac{\alpha}{k}-1} t^{\frac{\lambda+\alpha-\sigma h k + 2pk + q}{k}-1} dt,
\end{aligned} \tag{5.1}$$

where

$$\xi_{p,h} = \sum_{p,h=0}^{\infty} \frac{(-1)^p (\frac{1}{2})^{2p+\frac{q}{k}} a^h}{\Gamma_k(q+pk+k) p! \prod_{j=1}^m \Gamma_k(\mu_j + \frac{h}{\rho_j})}.$$

By substituting $\frac{z}{t} = u \Rightarrow dt = -\frac{t^2}{z} du$, $dt = -\frac{z}{u^2} du$, $\frac{z}{t} = u \Rightarrow t = \frac{z}{u}$, $t = z \Rightarrow u = 1$, $t = \infty \Rightarrow u = 0$ in the equation (5.1), we have

$$\begin{aligned} W_{k;z,\infty}^{\alpha} \{z^{\frac{\lambda}{k}} J_{q;k}(z) E_{k;(\frac{1}{\rho_j}, \mu_j)}(az^{-\sigma})\} &= \frac{1}{k \Gamma_k(\alpha)} \xi_{p,h} \int_1^0 (1-u)^{\frac{\alpha}{k}-1} \left(\frac{z}{u}\right)^{\frac{\lambda+\alpha-\sigma hk+2pk+q}{k}-1} \left(-\frac{z}{u^2}\right) du \\ &= \frac{z^{\frac{\alpha+\lambda}{k}}}{\Gamma_k(\alpha)} \xi_{p,h} z^{2p-\sigma h+\frac{q}{k}} \frac{1}{k} \int_0^1 (1-u)^{\frac{\alpha}{k}-1} u^{\frac{-\alpha-\lambda+\sigma hk-2pk-q}{k}-1} du. \end{aligned} \quad (5.2)$$

By using the equation (1.2) in the equation (5.2), we get

$$W_{k;z,\infty}^{\alpha} \{z^{\frac{\lambda}{k}} J_{q;k}(z) E_{k;(\frac{1}{\rho_j}, \mu_j)}(az^{-\sigma})\} = \frac{z^{\frac{\alpha+\lambda}{k}}}{\Gamma_k(\alpha)} \xi_{p,h} z^{2p-\sigma h+\frac{q}{k}} \beta_k(-\alpha-\lambda+\sigma kh-2pk-q, \alpha). \quad (5.3)$$

Therefore, the equation (5.3), becomes

$$\begin{aligned} &W_{k;z,\infty}^{\alpha} \{z^{\frac{\lambda}{k}} J_{q;k}(z) E_{k;(\frac{1}{\rho_j}, \mu_j)}(az^{-\sigma})\} \\ &= \frac{z^{\frac{\alpha+\lambda}{k}}}{\Gamma_k(\alpha)} \xi_{p,h} z^{2p-\sigma h+\frac{q}{k}} \frac{\Gamma_k(-\alpha-\lambda+\sigma kh-2pk-q) \Gamma_k(\alpha)}{\Gamma_k(-\alpha-\lambda+\sigma kh-2pk-q+\alpha)} \\ &= z^{\frac{\alpha+\lambda}{k}} \sum_{p,h=0}^{\infty} \frac{(-1)^p (\frac{z}{2})^{2p+\frac{q}{k}} (az^{-\sigma})^h}{\Gamma_k(q+pk+k) p! \prod_{j=1}^m \Gamma_k(\mu_j + \frac{h}{\rho_j})} \frac{\Gamma_k(-\alpha-\lambda+\sigma kh-2pk-q)}{\Gamma_k(\sigma kh-2pk-q-\lambda)} \\ &= z^{\frac{\alpha+\lambda+q}{k}} \sum_{h=0}^{\infty} \frac{(\frac{1}{2})^{\frac{q}{k}} (az^{-\sigma})^h}{\prod_{j=1}^m \Gamma_k(\mu_j + \frac{h}{\rho_j})} {}_1\Psi_2^k \left[\begin{matrix} (-\alpha-\lambda+\sigma kh-q, -2k) \\ (q+k, k)(\sigma kh-q-\lambda, -2k) \end{matrix} \middle| \frac{-z^2}{4} \right]. \end{aligned}$$

□

Corollary 5.2. Suppose that $m \geq 1$ is an integer and $W_{k;z,\infty}^{\alpha,n}$ be the Weyl fractional k -integral operator and $\operatorname{Re}(\rho_j) > 0$, $u_j (j = 1, \dots, m)$ are arbitrary parameters then this result holds:

$$\begin{aligned} &W_{k;z,\infty}^{\alpha} \{z^{\frac{\lambda}{k}} J_{q;k}(z) E_{k;(\frac{1}{\rho_j}, \mu_j)}(az^{-\sigma})\} \\ &= z^{\frac{\alpha+\lambda}{k}} \sum_{p=0}^{\infty} \frac{(-1)^p (\frac{z}{2})^{2p+\frac{q}{k}}}{\Gamma_k(q+pk+k) p!} {}_2\Psi_2^k \left[\begin{matrix} (-\alpha-\lambda-q-2kp, \sigma k)(k, k) \\ (-q-\lambda-2kp, \sigma k)(\mu_j, \frac{1}{\rho_j}) \end{matrix} \middle| \frac{az^{-\sigma}}{k} \right], \end{aligned}$$

where α and a be any complex number, $k \in \mathbb{R}^+$.

Corollary 5.3. Suppose the Weyl fractional k -integral operator for $\operatorname{Re}(\beta) > 0$, $\operatorname{Re}(\alpha) > 0$ and q, β are arbitrary complex numbers and $k \in \mathbb{R}^+$ then this result holds:

$$W_{k;z,\infty}^{\alpha} \{J_{q-k;k}(z) E_{k;\beta} z^{q-k}\} = \frac{z^{\frac{\alpha+q}{k}}}{2^{\frac{q-1}{k}}} \sum_{n=0}^{\infty} \frac{z^{(q-k)n}}{\Gamma_k(\beta n+k)} {}_1\Psi_2^k \left[\begin{matrix} (-\alpha+(k-q)(1+kn), -2k) \\ ((k-q)(1+kn), -2k)(q, k) \end{matrix} \middle| \frac{z^2}{4} \right].$$

6. Conclusion

In this article, we discuss the product of multi-index Mittag Leffler k -function with Bessel k -function of first kind in fractional integral k -operators, and all results are developed in the form of Wright k -functions. This article is devoted to further research in the field of fractional calculus and special functions.

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