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$(\overline{\in}, \overline{\in \wedge q_k})$-Fuzzy Subalgebras in BCK/BCI-Algebras

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Abstract

In this paper, the notion of not quasi-coincidence $(\overline{q})$ of a fuzzy point with a fuzzy set is considered. We introduce the notion of $(\overline{\in}, \overline{\in \wedge q_k})$-fuzzy $((\overline{\in}, \overline{q_k})$-fuzzy) subalgebra in a BCK/BCI-algebra X and several properties are investigated. Specially, we show that under certain conditions an $(\overline{\in}, \overline{\in \wedge q_k})$-fuzzy subalgebra can be expressed such that consist of a union of two proper non-equivalent $(\overline{\in}, \overline{\in \wedge q_k})$-fuzzy subalgebras.

Keywords: BCK/BCI-algebra, $(\overline{\in}, \overline{q_k})$-fuzzy subalgebra, $(\overline{\in}, \overline{\in \wedge q_k})$-fuzzy subalgebra, $(\overline{\in \wedge q_k})$-level subalgebra.

1. Introduction

It is well known, BCK and BCI-algebras are two classes of algebras of logic. They were introduced by Imai and Iseki (e.g. [6], [9]-[11]) and have been extensively investigated by many researchers, see (e.g. [3], [17]-[19], [23], [25]). BCI-algebras are generalizations of BCK-algebras. Iorgulescu (e.g. [7], [8]) showed that pocrims and BCK-algebras with condition (S) are

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categorically isomorphic, and residuated lattices and bounded BCK-lattices with condition (S) are categorically isomorphic. Iseki and Tanaka [11] proved that Boolean algebras are equivalent to the bounded implicative BCK-algebras. Mundici [19] proved that MV-algebras are equivalent to the bounded commutative BCK-algebras, and so on.

The theory of fuzzy sets, proposed by Zadeh [24] in 1965, has provided a useful mathematical tool for describing the behavior of systems that are too complex or illdefined to admit precise mathematical analysis by classical methods and tools. Murali [20] proposed a definition of a fuzzy point belonging to fuzzy subset under a natural equivalence on fuzzy subset. A new type of fuzzy subgroup, that is, the $(\in, \in\vee\, q)$-fuzzy subgroup, was introduced by Bhakat and Das in (e.g. [1], [2]) by using the combined notions of "belongingness" and "quasi-coincidence" of fuzzy points and fuzzy sets, which was introduced by Pu and Liu [21]. In fact, the $(\in, \in\vee\, q)$-fuzzy subgroup is an important generalization of Rosenfeld's fuzzy subgroup [22]. It is now natural to investigate similar type of generalizations of the existing fuzzy subsystems with other algebraic structures. With this objective in view, Jun [13] introduced the concept of (α, β)-fuzzy subalgebras of a BCK/BCI-algebra and investigated related results.

In this paper, we consider more general form of the $\overline{q}$ (not quasi-coincidence) of a fuzzy point with a fuzzy set. As a generalization of $\left(\overline{\in}, \overline{\in\wedge\, q}\right)$-fuzzy subalgebras, we introduce the notions of $\left(\overline{\in}, \overline{q_k}\right)$-fuzzy subalgebras and $\left(\overline{\in}, \overline{\in\wedge\, q_k}\right)$-fuzzy subalgebras in a BCK/BCI-algebra X, and several properties are investigated. Finally, we consider $\left(\overline{\in\wedge\, q_k}\right)$-level subalgebra of a fuzzy set, and some related results are proved.

2. Preliminaries

By a BCI-algebra, we mean an algebra $(X,*,0)$ of type $(2,0)$ satisfying the axioms:

(i) $(\forall x, y, z \in X)$ $\Big(\big((x*y)*(x*z)\big)*(z*y)=0\Big)$;
(ii) $(\forall x, y \in X)$ $\big(\big(x*(x*y)\big)*y=0\big)$;
(iii) $(\forall x \in X)$ $(x*x=0)$;
(iv) $(\forall x, y \in X)$ $(x*y=y*x=0 \implies x=y)$.

We can define a partial ordering $\leq$ by $x \leq y$ if and only if $x*y=0$. If a BCI-algebra X satisfies $0*x=0$ for all $x \in X$, then we say that X is a BCK-algebra. Hung and Jun [5] studied ideals and subalgebras in BCI-algebras. In what follows, X is a BCK/BCI-algebra unless otherwise specified. A nonempty subset S of X is called a subalgebra of X if $x*y \in S$ for all $x, y \in S$. We refer the reader to the books (e.g. [3], [17]) for further information regarding BCK/BCI-algebras.

A fuzzy set μ in a set X of the form

$$\mu(y)=\begin{cases} t \in (0,1] & if \;\; y=x \\ 0 & if \;\; y \neq x \end{cases}$$

is said to be a fuzzy point with support x and value t and is denoted by $(x)_t$. For a fuzzy point $(x)_t$ and a fuzzy set μ in a set X, Pu and Liu [21] introduced the symbol $(x)_t \alpha \mu$, where $\alpha \in \{\in, q, \in\vee\, q, \in\wedge\, q\}$. A fuzzy point $(x)_t$ is said to "belong to" (resp. be quasi-coincident with) a fuzzy set μ, written as $(x)_t \in \mu$ (resp. $(x)_t q \mu$) if $\mu(x) \geq t$ (resp. $\mu(x)+t>1$). If $(x)_t \in \mu$ or

$(x)_t q\mu$, then we write $(x)_t \in\vee q\mu$. If $(x)_t \in \mu$ and $(x)_t q\mu$, then we write $(x)_t \in\wedge q\mu$. To say that $(x)_t \overline{\alpha}\mu$, we mean $(x)_t \alpha\mu$ does not hold [14], and the symbol $\overline{\in\wedge q}$ means $\overline{\in}\vee \overline{q}$.

Let k denote an arbitrary element of $[0,1)$ unless otherwise specified. To say that $(x)_t q_k \mu$, we mean $\mu(x) + t + k > 1$. To say that $(x)_t \in\vee q_k \mu$, we mean $(x)_t \in \mu$ or $(x)_t q_k \mu$ [14].

3. Generalization of $(\overline{\in}, \overline{\in\wedge q})$-fuzzy subalgebras

Let X denote a BCK/BCI-algebras unless otherwise specified.

Definition 3.1. A fuzzy set μ in X is called an $(\overline{\in}, \overline{\in\wedge q_k})$-*fuzzy subalgebra* of X if, for all $t_1, t_2 \in (0,1]$ and $x, y \in X$

$$(x)_{t_1} \overline{\in} \mu,\ (y)_{t_2} \overline{\in} \mu \implies (x * y)_{\max(t_1, t_2)} \overline{\in\wedge q_k} \mu. \tag{1}$$

Theorem 3.2. A fuzzy set μ in X is called an $(\overline{\in}, \overline{\in\wedge q_k})$-fuzzy subalgebra of X if only if, for all $x, y \in X$

$$\mu(x * y) \leq max\{\mu(x), \mu(y), \tfrac{1-k}{2}\}. \tag{2}$$

Proof. Let μ be an $(\overline{\in}, \overline{\in\wedge q_k})$-fuzzy subalgebra of X. Assume that (2) is not valid. Then there exist $a, b \in X$ such that

$$\mu(a * b) > max\{\mu(a), \mu(b), \tfrac{1-k}{2}\}.$$

Hence we can take $t \in (0,1)$ such that

$$\mu(a * b) \geq t > max\{\mu(a), \mu(b), \tfrac{1-k}{2}\}.$$

It follows that $(a)_t \overline{\in} \mu$ and $(b)_t \overline{\in} \mu$, then $(a * b)_t \overline{\in\wedge q_k} \mu$. Since $\mu(a * b) \geq t$, $(a * b)_t \in \mu$ and so $(a * b)_t \overline{q_k} \mu$.. Hence $\mu(a * b) + t \leq 1 - k$. Thus $2t \leq \mu(a * b) + t \leq 1 - k$, then $t \leq \frac{1-k}{2}$,which is a contradiction.

Conversely, suppose that μ satisfies (2). Let $x, y \in X$ and $t_1, t_2 \in (0,1]$ be such that $(x)_{t_1} \overline{\in} \mu$ and $(y)_{t_2} \overline{\in} \mu$. Then

$$\mu(x * y) \leq max\left\{\mu(x), \mu(y), \tfrac{1-k}{2}\right\} < max\left\{t_1, t_2, \tfrac{1-k}{2}\right\}.$$

Assume that $t_1 \geq \frac{1-k}{2}$ or $t_2 \geq \frac{1-k}{2}$. Then $\mu(x * y) < max(t_1, t_2)$, which implies that $(x * y)_{\max(t_1, t_2)} \overline{\in} \mu$. Now, suppose that $t_1 < \frac{1-k}{2}$ and $t_2 < \frac{1-k}{2}$. Then $\mu(x * y) < \frac{1-k}{2}$, and thus

$$\mu(x * y) + max(t_1, t_2) < \frac{1-k}{2} + \frac{1-k}{2} = 1 - k,$$

i.e., $(x * y)_{\max(t_1, t_2)} \overline{q_k} \mu$. Hence $(x * y)_{\max(t_1, t_2)} \overline{\in\wedge q_k} \mu$, and consequently, μ is an $(\overline{\in}, \overline{\in\wedge q_k})$-fuzzy subalgebra of X. ■

Corollary 3.3. A fuzzy set μ in X is called an $(\overline{\in}, \overline{\in\wedge q})$-fuzzy subalgebra of X if only if, for all $x, y \in X$, $\mu(x * y) \leq max\{\mu(x), \mu(y), 0.5\}$.

Proof. It follows taking $k = 0$ in Theorem 3.2.

Theorem 3.4. Let μ be a fuzzy set of X. Then μ is an $(\overline{\in}, \overline{\in \wedge\, q_k})$-fuzzy subalgebra of X if only if the set $\overline{\mu_t} = \{x \in X |\, \mu(x) < t\}$ is a subalgebra of X for all $t \in (\frac{1-k}{2}, 1]$.

Proof. Assume that μ be an $(\overline{\in}, \overline{\in \wedge\, q_k})$-fuzzy subalgebra of X. Let $t \in (\frac{1-k}{2}, 1]$. and $x, y \in \overline{\mu_t}$. Then $\mu(x) < t$ and $\mu(y) < t$. It follows that

$$\mu(x * y) \leq max\left\{\mu(x), \mu(y), \tfrac{1-k}{2}\right\} < max\left\{t, \tfrac{1-k}{2}\right\} = t.$$

so that $x * y \in \overline{\mu_t}$. Therefore $\overline{\mu_t}$ is a subalgebra of X.

Conversely, suppose that $\overline{\mu_t}$ is a subalgebra of X for all $t \in (\frac{1-k}{2}, 1]$. Let (2) is not valid, then there exist $a, b \in X$ such that

$$\mu(a * b) > max\{\mu(x), \mu(y), \tfrac{1-k}{2}\}.$$

*	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Hence we can take $t \in (0,1)$ such that

$$\mu(a * b) \geq t > max\{\mu(a), \mu(b), \tfrac{1-k}{2}\}.$$

Then $t \in (\frac{1-k}{2}, 1]$ and $a, b \in \overline{\mu_t}$. Since $\overline{\mu_t}$ is a subalgebra of X, it follows that $a * b \in \overline{\mu_t}$, so that $\mu(a * b) < t$. This is a contradiction. Therefore (2) is valid. Consequently, μ is an $(\overline{\in}, \overline{\in \wedge\, q_k})$-fuzzy subalgebra of X by Theorem 3.2. ■

Corollary 3.5. Let μ be a fuzzy set of X. Then μ is an $(\overline{\in}, \overline{\in \wedge\, q})$-fuzzy subalgebra of X if only if the set $\overline{\mu_t} = \{x \in X |\, \mu(x) < t\}$ is a subalgebra of X for all $t \in (0.5, 1]$.

Proof. In Theorem 3.4, taking $k = 0$.

Example 3.6. Consider a BCI-algebra $X = \{0, a, b, c\}$ with the following table:

Let μ be a fuzzy set in X defined by $\mu(0) = 0.37$, $\mu(a) = 0.3$ and $\mu(b) = \mu(c) = 0.42$.

(1) If $k = 0.1$, then $\overline{\mu_t} = X$. for all $t \in (0.45, 1]$. Hence μ is an $(\overline{\in}, \overline{\in \wedge\, q_{0.1}})$-fuzzy subalgebra of X by Theorem 3.4.

(2) If $k = 0.2$, then

$$\overline{\mu_t} = \begin{cases} \{0, a\} & if\ t \in (0.4, 0.42] \\ X & if\ t \in (0.42, 1]. \end{cases}$$

Since X and $\{0, a\}$ are subalgebras of X, μ is an $(\overline{\in}, \overline{\in \wedge\, q_{0.2}})$-fuzzy subalgebra of X by Theorem 3.4.

Example 3.7. Let X be the BCI-algebra given in Example 3.6. Let μ be a fuzzy set in X defined by $\mu(0) = 0.47, \mu(a) = \mu(b) = 0.49$, and $\mu(c) = 0.4$. If $k = 0.12$, then

$$\overline{\mu_t} = \begin{cases} \{c\} & if\ \ t \in (0.44, 0.47] \\ \{0, c\} & if\ \ t \in (0.47, 0.49] \\ X & if\ \ t \in (0.49, 1]. \end{cases}$$

Note that $\overline{\mu_t}$ is not a subalgebra for $t \in (0.44, 0.47]$. Hence μ is not an $(\overline{\in}, \overline{\in \wedge\, q_{0.12}})$-fuzzy subalgebra of X by Theorem 3.4.

Theorem 3.8. Every $(\in, \in)$-fuzzy subalgebra of X is an $(\overline{\in}, \overline{\in \wedge\, q_k})$-fuzzy subalgebra of X.

Proof. Straightforward. ∎

The next corollary immediately follow from Theorem 3.8, by taking $k = 0$.

Corollary 3.9. Every $(\overline{\in}, \overline{\in})$-fuzzy subalgebra of X is an $(\overline{\in}, \overline{\in \wedge\, q_k})$-fuzzy subalgebra of X.

The converse of Theorem 3.8 is not true as seen in the following example.

Example 3.10. Consider the $(\overline{\in}, \overline{\in \wedge\, q_{0.1}})$-fuzzy subalgebra of X given in Example 3.6. Then μ is not an $(\overline{\in}, \overline{\in})$-fuzzy subalgebra of X since $(a)_{0.32}\ \overline{\in}\ \mu$ and $(a)_{0.36}\ \overline{\in}\ \mu$, but $(0)_{0.36} = (a * a)_{\max(0.32, 0.36)} \in \mu$, because $\mu(0) = 0.37 \geq 36$.

Definition 3.11. A fuzzy set μ in X is called an $(\overline{\in}, \overline{q_k})$-*fuzzy subalgebra* of X if, for all $t_1, t_2 \in (0,1]$ and $x, y \in X$

$$(x)_{t_1}\ \overline{\in}\ \mu,\ \ (y)_{t_2}\ \overline{\in}\ \mu\ \Longrightarrow (x * y)_{\max(t_1, t_2)} \overline{q_k}\ \mu. \qquad (3)$$

Theorem 3.12. Every $(\overline{\in}, \overline{q_k})$-fuzzy subalgebra of X is an $(\overline{\in}, \overline{\in \wedge\, q_k})$-fuzzy subalgebra of X.

Proof. Straightforward. ∎

Taking $k = 0$ in Theorem 3.12, we have the following corollary.

Corollary 3.13. Every $(\overline{\in}, \overline{q_k})$-fuzzy subalgebra of X is an $(\overline{\in}, \overline{\in \wedge\, q_k})$-fuzzy subalgebra of X.

The next example shows that the converse of Theorem 3.12 does not hold.

Example 3.14. Consider the $(\overline{\in}, \overline{\in \wedge\, q_{0.2}})$-fuzzy subalgebra of X given in Example 3.6. Note that $(a)_{0.36}\ \overline{\in}\ \mu$ and $(b)_{0.43}\ \overline{\in}\ \mu$, but $(a * b)_{\max(0.36, 0.43)} = (c)_{0.43} q_{0.2} \mu$, since $\mu(c) + 0.43 + 0.2 > 1$. Therefore μ is not an $(\overline{\in}, \overline{q_{0.2}})$-fuzzy subalgebra of X.

Theorem 3.15. Let X be BCK/BCI-algebra. If $0 \leq r < k < 1$, then every $(\overline{\in}, \overline{\in \wedge\, q_k})$-fuzzy subalgebra of X is an $(\overline{\in}, \overline{\in \wedge\, q_r})$-fuzzy subalgebra of X.

Proof. Straightforward. ∎

The following example shows that if $0 \leq r < k < 1$, then an $(\overline{\in}, \overline{\in\wedge q_r})$-fuzzy subalgebra of
X may not be an $(\overline{\in}, \overline{\in\wedge q_k})$-fuzzy subalgebra of X.

Example 3.16. Let X and μ be as in Example 3.7. If $r = 0.06$ and $k = 0.12$, then

$$\overline{\mu_t} = \begin{cases} \{0, c\} & if\ t \in (0.47, 0.49] \\ X & if\ t \in (0.49, 1]. \end{cases}$$

Since X and $\{0, c\}$ are subalgebras of X, then μ is an $(\overline{\in}, \overline{\in\wedge q_{0.06}})$-fuzzy subalgebra of X by Theorem 3.4. But μ is not an $(\overline{\in}, \overline{\in\wedge q_{0.12}})$-fuzzy subalgebra of X (see Example 3.7).

Let S be a subset of X. Consider a fuzzy set μ_s in X where for all $x \in X$ defined by

$$\mu_s(x) = \begin{cases} 0 & if\ x \in S \\ 1 & otherwise. \end{cases}$$

Theorem 3.17. A non-empty subset S of X is a subalgebra of X if and only if the fuzzy set μ_s in X is an $(\overline{\in}, \overline{\in\wedge q_k})$-fuzzy subalgebra of X.

Proof. Let S be a subalgebra of X. Then $\overline{(\mu_s)_t}$ is clearly a subalgebra of X for all $t \in (\frac{1-k}{2}, 1]$. Hence μ_s is an $(\overline{\in}, \overline{\in\wedge q_k})$-fuzzy subalgebra of X by Theorem 3.4.

Conversely, assume that μ_s is an $(\overline{\in}, \overline{\in\wedge q_k})$-fuzzy subalgebra of X. Let $x, y \in S$. Then

$$\mu_s(x * y) \leq max\left\{\mu_s(x), \mu_s(y), \tfrac{1-k}{2}\right\} = max\left\{0, \tfrac{1-k}{2}\right\} = \frac{1-k}{2} < 1$$

for all $k \in [0, 1)$. Then $\mu_s(x * y) = 0$ and so $x * y \in S$. Therefore S is a subalgebra of X. ■

Theorem 3.18. Let S be a subalgebra of X. Then for every $t \in (\frac{1-k}{2}, 1]$, there exists an $(\overline{\in}, \overline{\in\wedge q_k})$-fuzzy subalgebra μ of X such that $\overline{\mu_t} = S$.

Proof. Let μ be a fuzzy set of X defined by

$$\mu(x) = \begin{cases} 0 & if\ x \in S \\ t & otherwise \end{cases}$$

for all $x \in X$, where $t \in (\frac{1-k}{2}, 1]$. Obviously, $\overline{\mu_t} = S$. Assume that (2) of Theorem 3.2 is not valid, then there exist $a, b \in X$ such that

$$\mu(a * b) > max\{\mu(a), \mu(b), \tfrac{1-k}{2}\}.$$

Hence we can take $t \in (0,1)$ such that

$$\mu(a * b) \geq t > max\{\mu(a), \mu(b), \tfrac{1-k}{2}\}.$$

Hence $\mu(a) < t$ and $\mu(b) < t$, and so $a, b \in \overline{\mu_t} = S$. Since S is subalgebra of X, $a * b \in S$. Thus $\mu(a * b) = 0 < t$ for all $t \in (0, 1)$, which is a contradiction. Therefore

$$\mu(x*y) \leq max\{\mu(x), \mu(y), \frac{1-k}{2}\}$$

for all $x, y \in X$. Using Theorem 3.2, we know that μ is an $(\overline{\in}, \overline{\in\wedge\, q_k})$-fuzzy subalgebra of X. ■

Taking $k = 0$ in Theorem 3.18, we have the following corollary.

Corollary 3.19. Let S be a subalgebra of X. Then for every $t \in (0.5, 1]$, there exists an $(\overline{\in}, \overline{\in\wedge\, q})$-fuzzy subalgebra μ of X such that $\overline{\mu_t} = S$.

Theorem 3.20. Let μ be an $(\overline{\in}, \overline{\in\wedge\, q_k})$-fuzzy subalgebra of X such that $\mu(x) \geq \frac{1-k}{2}$, for all $x \in X$. Then μ is an $(\overline{\in}, \overline{\in})$-fuzzy subalgebra of X.

Proof. Straightforward. ■

Taking $k = 0$ in Theorem 3.20, we have the following corollary.

Corollary 3.21. Let μ be an $(\overline{\in}, \overline{\in\wedge\, q})$-fuzzy subalgebra of X such that $\mu(x) \geq 0.5$, for all $x \in X$. Then μ is an $(\overline{\in}, \overline{\in})$-fuzzy subalgebra of X.

Theorem 3.22. Let $\{ \mu_i \mid i \in A \}$ be a family of $(\overline{\in}, \overline{\in\wedge\, q_k})$-fuzzy subalgebra of X. Then $\mu = \cup_{i\in A} \mu_i$ is an $(\overline{\in}, \overline{\in\wedge\, q_k})$-fuzzy subalgebra of X.

Proof. Let $x, y \in X$ and $t_1, t_2 \in (0, 1]$ be such that $(x)_{t_1} \overline{\in} \mu$ and $(y)_{t_2} \overline{\in} \mu$. Assume that $(x*y)_{\max(t_1,t_2)} \in\wedge\, q_k\mu$. Then $\mu(x*y) \geq max(t_1, t_2)$ and $\mu(x*y) + max(t_1, t_2) > 1-k$, which imply that $\mu(x*y) \geq \frac{1-k}{2}$. (4)

Let $\phi_1 = \{ i \in A \mid (x*y)_{max\,(t_1,t_2)} \overline{\in} \mu_i \}$ and

$\phi_2 = \{ i \in A \mid (x*y)_{max\,(t_1,t_2)} \overline{q_k} \mu_i \} \cap \{ j \in A \mid (x*y)_{max\,(t_1,t_2)} \in \mu_j \}$.

Then $A = \phi_1 \cup \phi_2$ and $\phi_1 \cap \phi_2 = \emptyset$. If $\phi_2 = \emptyset$, then $(x*y)_{max\,(t_1,t_2)} \overline{\in} \mu_i$ for all $i \in A$, that is, $\mu_i(x*y) < max(t_1, t_2)$ for all $i \in A$, which yields $\mu(x*y) < max(t_1, t_2)$. This is a contradiction. Hence $\phi_2 \neq \emptyset$, and so for every $i \in \phi_2$ we have $\mu_i(x*y) \geq max(t_1, t_2)$ and $\mu_i(x*y) + max(t_1, t_2) \leq 1-k$. It follows that $max(t_1, t_2) \leq \frac{1-k}{2}$. Now, $(x)_{t_1} \overline{\in} \mu$ implies $\mu(x) < t_1$ and thus $\mu_i(x) < \mu(x) < t_1 < max(t_1, t_2) \leq \frac{1-k}{2}$ for all $i \in A$. Similarly $\mu_i(y) < \frac{1-k}{2}$ for all $i \in A$. Next suppose that $t = \mu_i(x*y) \geq \frac{1-k}{2}$. Taking $t > r > \frac{1-k}{2}$, we get $(x)_r \overline{\in} \mu_i$ and $(y)_r \overline{\in} \mu_i$, but $(x*y)_r \in\wedge\, q_k\mu_i$. This contradicts that μ_i is an $(\overline{\in}, \overline{\in\wedge\, q_k})$-fuzzy subalgebra of X. Hence $\mu_i(x*y) < \frac{1-k}{2}$ for all $i \in A$, and so $\mu(x*y) < \frac{1-k}{2}$, which contradicts (4). Therefore $(x*y)_{\max(t_1,t_2)} \overline{\in\wedge\, q_k}\mu$.

Consequently μ is an $(\overline{\in}, \overline{\in\wedge\, q_k})$-fuzzy subalgebra of X. ■

Taking $k = 0$ in Theorem 3.22, we have the following corollary.

Corollary 3.23. Let $\{ \mu_i \mid i \in A \}$ be a family of $(\overline{\in}, \overline{\in\wedge\, q})$-fuzzy subalgebra of X. Then $\mu = \cup_{i\in A} \mu_i$ is an $(\overline{\in}, \overline{\in\wedge\, q})$-fuzzy subalgebra of X.

The following example shows that there exists $k \in [0, 1)$ such that the intersection of two $(\overline{\in}, \overline{\in\wedge\, q_k})$-fuzzy subalgebras of X may not be an $(\overline{\in}, \overline{\in\wedge\, q_k})$-fuzzy subalgebra of X.

Example 3.24. Let $X = \{0, a, b, c\}$ be a BCI-algebras given in Example 3.6 and μ an $(\overline{\in}, \overline{\in\wedge q_{0.2}})$-fuzzy subalgebra of X described in Example 3.6 (2). Let ν be a fuzzy set in X defined by $\nu(0) = 0.33$, $\nu(a) = \nu(c) = 0.42$, and $\nu(b) = 0.4$. Then

$$\overline{\nu_t} = \begin{cases} \{0, b\} & if\ t \in (0.4, 0.42] \\ X & if\ t \in (0.42, 1]. \end{cases}$$

Since X and $\{0, b\}$ are subalgebras of X, so ν is an $(\overline{\in}, \overline{\in\wedge q_{0.2}})$-fuzzy subalgebra of X by Theorem 3.4. The intersection $\mu \cap \nu$ of μ and ν is given by $\mu \cap \nu(0) = 0.33$, $\mu \cap \nu(a) = 0.3$, $\mu \cap \nu(b) = 0.4$, and $\mu \cap \nu(c) = 0.42$. Hence

$$\overline{(\mu \cap \nu)_t} = \begin{cases} \{0, a, b\} & if\ t \in (0.4, 0.42] \\ X & if\ t \in (0.42, 1]. \end{cases}$$

Since $\{0, a, b\}$ is not a subalgebra of X, it follows that $\mu \cap \nu$ is not an $(\overline{\in}, \overline{\in\wedge q_{0.2}})$-fuzzy subalgebra of X by Theorem 3.4.

For any fuzzy set μ in X and $t \in (0, 1]$, we denote

$$< \overline{\mu} >_t = \{ x \in X \mid (x)_t\ \overline{q_k}\mu \} \ \text{ and } \ \overline{[\mu]}_t = \{ x \in X \mid (x)_t\ \overline{\in\wedge q_k}\mu \}.$$

Obviously, $\overline{[\mu]}_t = \overline{\mu_t} \cup < \overline{\mu} >_t$.

Theorem 3.25. Let μ be a fuzzy set in X. Then μ is an $(\overline{\in}, \overline{\in\wedge q_k})$-fuzzy subalgebra of X if and only if $\overline{[\mu]}_t$ is a subalgebra of X for all $t \in (0, 1]$.

We call $\overline{[\mu]}_t$ an $(\overline{\in\wedge q_k})$-level subalgebra of μ.

Proof. Assume that hen μ is an $(\overline{\in}, \overline{\in\wedge q_k})$-fuzzy subalgebra of X and let $x, y \in \overline{[\mu]}_t$ for $t \in (0, 1]$. Then $(x)_t\ \overline{\in\wedge q_k}\mu$ and $(y)_t\ \overline{\in\wedge q_k}\mu$, that is, $\mu(x) < t$ or $\mu(x) + t \leq 1 - k$, and $\mu(y) < t$ or $\mu(y) + t \leq 1 - k$. Using Theorem 3.2, we have $\mu(x * y) \leq max\{\mu(x), \mu(y), \frac{1-k}{2}\}$.

Case 1. $\mu(x) < t$ and $\mu(y) < t$. If $t \leq \frac{1-k}{2}$, then

$$\mu(x * y) \leq max\{\mu(x), \mu(y), \tfrac{1-k}{2}\} < \boldsymbol{max}\left\{\boldsymbol{t}, \tfrac{\boldsymbol{1-k}}{\boldsymbol{2}}\right\} = \tfrac{\boldsymbol{1-k}}{\boldsymbol{2}}.$$

Hence $\mu(x * y) + t < \frac{\boldsymbol{1-k}}{\boldsymbol{2}} + \frac{\boldsymbol{1-k}}{\boldsymbol{2}} = 1 - k$, and so $(x * y)_t \overline{q_k}\mu$. If $t > \frac{1-k}{2}$, then

$$\mu(x * y) \leq max\{\mu(x), \mu(y), \tfrac{1-k}{2}\} < \boldsymbol{max}\left\{\boldsymbol{t}, \tfrac{\boldsymbol{1-k}}{\boldsymbol{2}}\right\} = \boldsymbol{t}.$$

and thus $(x * y)_t\ \overline{\in}\ \mu$. Therefore $(x * y)_t\ \overline{\in\wedge q_k}\mu$, i.e., $x * y \in \overline{[\mu]}_t$.

Case 2. $\mu(x) < t$ and $\mu(y) + t \leq 1 - k$. If $t \leq \frac{1-k}{2}$, then

$$\mu(x * y) \leq max\left\{\mu(x), \mu(y), \tfrac{1-k}{2}\right\} < \boldsymbol{max}\left\{\boldsymbol{\mu(y)}, \tfrac{\boldsymbol{1-k}}{\boldsymbol{2}}\right\}$$

$$\leq \boldsymbol{max}\left(\mathbf{1}-\boldsymbol{k}-\boldsymbol{t},\tfrac{\mathbf{1}-\boldsymbol{k}}{\mathbf{2}}\right)=\mathbf{1}-\boldsymbol{k}-\boldsymbol{t}$$

and so $(x*y)_t\overline{q_k}\mu$. If $t>\frac{1-k}{2}$, then

$$\mu(x*y)\leq max\{\mu(x),\mu(y),\tfrac{1-k}{2}\}<\boldsymbol{max}\{\boldsymbol{t},\mathbf{1}-\boldsymbol{k}-\boldsymbol{t}\}=\boldsymbol{t}.$$

Hence $(x*y)_t\ \overline{\in}\ \mu$. Therefore $(x*y)_t\ \overline{\in\wedge\, q_k}\mu$, i.e., $x*y\in\overline{[\mu]}_t$.

Case 3. $\mu(x)+t\leq 1-k$ and $\mu(y)<t$. Similar to the case 2.

Case 4. $\mu(x)+t\leq 1-k$ and $\mu(y)+t\leq 1-k$. If $t\leq\frac{1-k}{2}$, then

$$\mu(x*y)\leq max\{\mu(x),\mu(y),\tfrac{1-k}{2}\}\leq\boldsymbol{max}\left(\mathbf{1}-\boldsymbol{k}-\boldsymbol{t},\tfrac{\mathbf{1}-\boldsymbol{k}}{\mathbf{2}}\right)=\mathbf{1}-\boldsymbol{k}-\boldsymbol{t}.$$

Thus $(x*y)_t\overline{q_k}\mu$. If $t>\frac{1-k}{2}$, then

$$\mu(x*y)\leq max\{\mu(x),\mu(y),\tfrac{1-k}{2}\}\leq\boldsymbol{max}\left(\mathbf{1}-\boldsymbol{k}-\boldsymbol{t},\tfrac{\mathbf{1}-\boldsymbol{k}}{\mathbf{2}}\right)=\frac{\mathbf{1}-\boldsymbol{k}}{\mathbf{2}}<\boldsymbol{t},$$

and so $(x*y)_t\ \overline{\in}\ \mu$. Therefore $(x*y)_t\ \overline{\in\wedge\, q_k}\mu$, i.e., $x*y\in\overline{[\mu]}_t$. Consequently, $\overline{[\mu]}_t$ is a subalgebra of X.

Conversely, let μ be a fuzzy set in X and $t\in(0,1]$ be such that $\overline{[\mu]}_t$ is a subalgebra of X. Let there exists $a,b\in X$ such that $\mu(a*b)\geq t>max\{\mu(a),\mu(b),\frac{1-k}{2}\}$ for some $t\in(0,1]$. Then $a,b\in\overline{\mu_t}\subseteq\overline{[\mu]}_t$, which implies that $a*b\in\overline{[\mu]}_t$. Hence $\mu(a*b)<t$ or $\mu(a*b)+t+k\leq 1$, a contradiction. Thus $\mu(x*y)\leq max\{\mu(x),\mu(y),\frac{1-k}{2}\}$ for all $x,y\in X$. Using Theorem 3.2, we conclude that μ is an $(\overline{\in},\overline{\in\wedge\, q_k})$-fuzzy subalgebra of X. ■

A fuzzy set μ in X is said to be proper if $Im(\mu)$ has at least two elements. Two fuzzy sets are said to be equivalent if they have same family of level subsets. Otherwise, they are said to be non-equivalent.

Theorem 3.26. Let μ be an $(\overline{\in},\overline{\in\wedge\, q_k})$-fuzzy subalgebra of X such that $\#\{\ \mu(x)\mid\mu(x)>\frac{1-k}{2}\}\geq 2$. Then there exist two proper non-equivalent $(\overline{\in},\overline{\in\wedge\, q_k})$-fuzzy subalgebra of X such that μ can be expressed such that consist of a union of them.

Proof. Let $\{\ \mu(x)|\mu(x)>\frac{1-k}{2}\}=\{t_1,t_2,\dots,t_r\}$, where $t_1<t_2<\ \dots<t_r$ and $r\geq 2$. Then the chain of $(\overline{\in\wedge\, q_k})$-level subalgebras of μ is

$$\overline{[\mu]}_{\frac{1-k}{2}}\subseteq\overline{[\mu]}_{t_1}\subseteq\overline{[\mu]}_{t_2}\subseteq\cdots\subseteq\overline{[\mu]}_{t_r}\subseteq X.$$

Define two fuzzy sets ν and γ of X by

$$\nu(x)=\begin{cases} t_1, & if\ \ x\in\overline{[\mu]}_{t_1} \\ t_2, & if\ \ x\in\overline{[\mu]}_{t_2}\backslash\overline{[\mu]}_{t_1} \\ \dots & \\ t_r, & if\ \ x\in\overline{[\mu]}_{t_r}\backslash\overline{[\mu]}_{t_{r-1}} \end{cases} \qquad \gamma(x)=\begin{cases} \mu(x), & if\ \ x\in\overline{[\mu]}_{\frac{1-k}{2}} \\ k, & if\ \ x\in\overline{[\mu]}_{t_2}\backslash\overline{[\mu]}_{\frac{1-k}{2}} \\ t_3, & if\ \ x\in\overline{[\mu]}_{t_3}\backslash\overline{[\mu]}_{t_2} \\ \dots & \\ t_r, & if\ \ x\in\overline{[\mu]}_{t_r}\backslash\overline{[\mu]}_{t_{r-1}} \end{cases}$$

respectively, where $t_2 < k < t_3$. Then ν and γ are $(\overline{\in},\overline{\in\wedge q_k})$-fuzzy subalgebras of X, and $\nu,\gamma\leq\mu$. The chain of $(\overline{\in\wedge q_k})$-level subalgebras of ν and γ are, respectively, given by

$$\overline{[\mu]}_{t_1}\subseteq\overline{[\mu]}_{t_2}\subseteq\cdots\subseteq\overline{[\mu]}_{t_{r-1}} \quad\text{and}\quad \overline{[\mu]}_{\frac{1-k}{2}}\subseteq\overline{[\mu]}_{t_2}\subseteq\cdots\subseteq\overline{[\mu]}_{t_{r-1}}$$

Therefore ν and γ are non-equivalent and clearly $\mu\geq\nu\cup\gamma$. This completes the proof. ■

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