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How to obtain Lie point symmetries of PDEs

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Abstract

In this research, we have studied how to obtain Lie point symmetries of a partial differential equation (PDEs) of second order. We have also studied some PDEs' applications as one-dimensional and two-dimensional heat equations. We have used Manale's formula for solving second-order ordinary differential equations to determine new symmetries. Burgers equation has been studied, and Lie point symmetries have been obtained for these equations.

Keywords: Symmetries, infinitesimal, invariant condition.

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1. Introduction

More than a century ago, the Norwegian mathematician Sophus Lie put forward many of the fundamental ideas behind symmetry methods. The Lie symmetry method is a modern approximated method of solving of DEs and The Lie symmetry method (LSM) is efficient technique to find the approximate solution for ODEs and PDEs which describe different fields or science, physical phenomena, engineering,mechanics and so on. Sophus Lie (1872): develop DE invariant of continuous groups transformation. Hermann (1928) coined the term "Lie group", Bluman and Kumei (1989): dealt with symmetries and DEs, Hydone (2005): solved some DEs using Lie symmetry, Laheeb Muhsen (2015): develop the Lie symmetry analysis, especially Lie group analysis method to classify higher-order DDEs to solvable Lie algebra. So Lie symmetry method is necessary to obtain exact solutions or numerical solutions for partial differential equations (PDEs). During last few decades several analytical numerical and semi-analytical methods have been used for solving PDEs [4–6, 14–18]. In this paper we consider the symmetry analysis as presented in the books of Bluman, Kumei, Ibragimov and Anco [8, 9, 13]. We present the Lie symmetries of first-order PDEs and we find the Lie symmetry of Burgers equation [12] and finally we find the Lie symmetries of one-dimensional and two-dimensional heat equations.

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2. Scalar PDEs with two dependent variables

Definition 2.1 (Point symmetry of PDE [12]). Consider PDEs with one dependent variable u and two independent variables x and t. A point transformation is called diffeomorphism

$$X = (x,t,u) \mapsto \big(\hat{x}(x,t,u), \hat{t}(x,t,u), \hat{u}(x,t,u)\big),$$

this transformation maps the surface $u = u(x,t)$ to the following surface

$$\hat{x} = \hat{x}\,(x,t,u(x,t)), \qquad \hat{t} = \hat{t}\,(x,t,u(x,t)), \qquad \hat{u} = \hat{u}\,(x,t,u(x,t)), \tag{2.1}$$

to calculate the extend of the generator of a given transformation, we should find the derivatives of (2.1) with respect to each x and t. We introduce the following total derivatives

$$D_x = \partial_x + u_x\partial_u + u_{xx}\partial_{u_x} + u_{xt}\partial_{u_t} + \cdots, \qquad D_t = \partial_t + u_t\partial_u + u_{xt}\partial_{u_x} + u_{tt}\partial_{u_t} + \cdots.$$

Now if the jacobian

$$\mathbf{J} \equiv \begin{vmatrix} D_x\hat{x} & D_x\hat{t} \\ D_t\hat{x} & D_t\hat{t} \end{vmatrix} \neq 0, \quad \text{when} \quad u = u(x,t), \tag{2.2}$$

then the first two equations of (2.1) may be inverted to give x and t in terms of $\hat{x}$ and $\hat{t}$. If Eq. (2.2) holds then the last equation of (2.1) becomes

$$\hat{u} = \hat{u}(\hat{x}, \hat{t}), \tag{2.3}$$

now by using chain rule to (2.3), we get

$$\begin{bmatrix} D_x\hat{u} \\ D_t\hat{u} \end{bmatrix} = \begin{bmatrix} D_x\hat{x} & D_x\hat{t} \\ D_t\hat{x} & D_t\hat{t} \end{bmatrix} \begin{bmatrix} \hat{u}_{\hat{x}} \\ \hat{u}_{\hat{t}} \end{bmatrix},$$

now by Cramer's rule we find the transformation of the first derivatives

$$\hat{u}_{\hat{x}} = \frac{1}{\mathbf{J}} \begin{vmatrix} D_x\hat{u} & D_x\hat{t} \\ D_t\hat{u} & D_t\hat{t} \end{vmatrix}, \quad \hat{u}_{\hat{t}} = \frac{1}{\mathbf{J}} \begin{vmatrix} D_x\hat{x} & D_x\hat{u} \\ D_t\hat{x} & D_t\hat{u} \end{vmatrix}, \tag{2.4}$$

now if $\hat{u}_h$ any derivative of $\hat{u}$ with respect to $\hat{x}$ and $\hat{t}$, then

$$\hat{u}_{h\hat{x}} \equiv \frac{\partial \hat{u}_h}{\partial \hat{x}} = \frac{1}{\mathbf{J}} \begin{vmatrix} D_x\hat{u}_h & D_x\hat{t} \\ D_t u\hat{u}_h & D_t\hat{t} \end{vmatrix}, \quad \hat{u}_{h\hat{t}} \equiv \frac{\partial \hat{u}_h}{\partial \hat{t}} = \frac{1}{\mathbf{J}} \begin{vmatrix} D_x\hat{x} & D_x\hat{u}_h \\ D_t\hat{x} & D_t\hat{u}_h \end{vmatrix}. \tag{2.5}$$

Now we find the transformation of second derivatives

$$\hat{u}_{\hat{x}\hat{x}} = \frac{1}{\mathbf{J}} \begin{vmatrix} D_x\hat{u}_{\hat{x}} & D_x\hat{t} \\ D_t\hat{u}_{\hat{x}} & D_t\hat{t} \end{vmatrix}, \ \hat{u}_{\hat{t}\hat{t}} = \frac{1}{\mathbf{J}} \begin{vmatrix} D_x\hat{x} & D_x\hat{u}_{\hat{t}} \\ D_t\hat{x} & D_t\hat{u}_{\hat{t}} \end{vmatrix}, \ \hat{u}_{\hat{x}\hat{t}} = \frac{1}{\mathbf{J}} \begin{vmatrix} D_x\hat{u}_{\hat{t}} & D_x\hat{t} \\ D_t\hat{u}_{\hat{t}} & D_t\hat{t} \end{vmatrix} = \frac{1}{\mathbf{J}} \begin{vmatrix} D_x\hat{x} & D_x\hat{u}_{\hat{x}} \\ D_t\hat{x} & D_t\hat{u}_{\hat{x}} \end{vmatrix}.$$

Since

$$\hat{u}_{\hat{t}} + \hat{u}\hat{u}_{\hat{x}} = 8t^3(u_t + uu_x),$$

so the point transformation holds the symmetry condition

$$\hat{u}_{\hat{t}} + \hat{u}\hat{u}_{\hat{x}} = \hat{u}_{\hat{x}\hat{x}} \quad \text{when} \quad u_t + uu_x = u_{xx}.$$

Definition 2.2 (Point symmetries Of n^{th} order PDE [12]). Consider

$$f(x,t,u,u_x,u_t,\ldots) = 0, \tag{2.6}$$

the point transformation X is a point symmetry of (2.6) if

$$f(\hat{x}, \hat{t}, \hat{u}, \hat{u}_{\hat{x}}, \hat{u}_{\hat{t}}, \ldots) = 0, \quad \text{when (2.6) holds.}$$

We seek point symmetries of the form

$$\hat{x} = x + \lambda\epsilon(x,t,u) + O(\lambda^2), \qquad \hat{t} = t + \lambda\mu(x,t,u) + O(\lambda^2), \qquad \hat{u} = u + \lambda\zeta(x,t,u) + O(\lambda^2).$$

In analogy with ODEs we define $\hat{u}_{\hat{x}}$ and $\hat{u}_{\hat{t}}$ as

$$\hat{u}_{\hat{x}} = u_x + \lambda\zeta^{(x)}(x,t,u,u_x,u_t) + O(\lambda^2), \qquad \hat{u}_{\hat{t}} = u_t + \lambda\zeta^{(t)}(x,t,u,u_x,u_t) + O(\lambda^2).$$

Now by using (2.4) we find $\hat{u}_t$ and $\hat{u}_x$ as follows

$$\hat{u}_t = \frac{\begin{vmatrix} D_x\hat{x} & D_x\hat{u} \\ D_t\hat{x} & D_t\hat{u} \end{vmatrix}}{\begin{vmatrix} D_x\hat{x} & D_x\hat{t} \\ D_t\hat{x} & D_t\hat{t} \end{vmatrix}} = \frac{\begin{vmatrix} 1+\lambda D_x(\epsilon)+O(\lambda^2) & u_x+\lambda D_x(\tau)+O(\lambda^2) \\ \lambda D_t(\epsilon)+O(\lambda^2) & u_t+\lambda D_t(\tau)+O(\lambda^2) \end{vmatrix}}{\begin{vmatrix} 1+\lambda D_x(\epsilon)+O(\lambda^2) & \lambda D_x(\mu)+O(\lambda^2) \\ \lambda D_t(\epsilon)+O(\lambda^2) & 1+\lambda D_t(\mu)+O(\lambda^2) \end{vmatrix}},$$

we ignore terms of (λ^2) or higher, we arrive to

$$\hat{u}_t = \frac{u_t + \lambda D_t(\zeta) + \lambda u_t D_x(\epsilon) - \lambda u_x D_t(\epsilon) + O(\lambda^2)}{1 + \lambda D_t(\mu) + \lambda D_X(\epsilon) + O(\lambda^2)} = \frac{u_t + \lambda\left[D_t(\zeta) + u_t D_x(\epsilon) - u_x D_t(\epsilon)\right] + O(\lambda^2)}{1 + \lambda\left[D_t(\mu) + D_x(\epsilon)\right] + O(\lambda^2)},$$

since

$$\hat{u}_{\hat{t}} = u_t + \lambda\zeta^{(t)}(x,t,u,u_x,u_t) + O(\lambda^2),$$

then

$$u_t + \lambda\zeta^{(t)}(x,t,u,u_x,u_t) + O(\lambda^2) = \frac{u_t + \lambda\left[D_t(\zeta) + u_t D_x(\epsilon) - u_x D_t(\epsilon)\right] + O(\lambda^2)}{1 + \lambda\left[D_t(\mu) + D_x(\epsilon)\right] + O(\lambda^2)},$$

this leads to

$$\begin{aligned} u_t + \lambda\left[D_t(\zeta) + u_t D_x(\epsilon) - u_x D_t(\epsilon)\right] + O(\lambda^2) &= \left(1 + \lambda\left[D_t(\mu) + D_x(\epsilon)\right] + O(\lambda^2)\right)(u_t + \lambda\zeta^{(t)}) \\ &= u_t + u_t\lambda D_t(\mu) + u_t\lambda D_x(\epsilon) + \lambda\zeta^{(t)} + O(\lambda^2), \end{aligned} \tag{2.7}$$

by simplifying the Eq. (2.7) more yields

$$\lambda\left[D_t(\zeta) + u_t D_x(\epsilon) - u_x D_t(\epsilon) - u_t D_t(\mu) - u_t D_x(\epsilon)\right] = \lambda\zeta^{(t)},$$

so we obtain

$$\zeta^{(t)}(x,t,u,u_x,u_t) = D_t(\zeta) - u_x D_t(\epsilon) - u_t D_t(\mu).$$

In the same way we obtain

$$\zeta^{(x)}(x,t,u,u_x,u_t) = D_x(\zeta) - u_x D_x(\epsilon) - u_t D_x(\mu),$$

then

$$\begin{aligned} \zeta^{(t)} &= \left(\frac{\partial\zeta}{\partial t} + \frac{du}{dt}\frac{\partial\zeta}{\partial u}\right) - u_x\left(\frac{\partial\epsilon}{\partial t} + \frac{du}{dt}\frac{\partial\epsilon}{\partial u}\right) - u_t\left(\frac{\partial\mu}{\partial t} + \frac{du}{dt}\frac{\partial\mu}{\partial u}\right) \\ &= (\zeta_t + u_t\zeta_u) - u_x(\epsilon_t + u_t\epsilon_u) - u_t(\mu_t + u_t\mu_u) = \zeta_t - \epsilon_t u_x + (\zeta_u - \mu_t)u_t - \epsilon_u u_x u_t - \mu_u(u_t^2), \end{aligned}$$

so

$$\zeta^{(t)} = \zeta_t - \epsilon_t u_x + (\zeta_u - \mu_t)u_t - \epsilon_u u_x u_t - \mu_u(u_t^2), \tag{2.8}$$

and

$$\zeta^{(x)} = \left(\frac{\partial\zeta}{\partial x} + \frac{du}{dx}\frac{\partial\zeta}{\partial u}\right) - u_x\left(\frac{\partial\epsilon}{\partial x} + \frac{du}{dx}\frac{\partial\epsilon}{\partial u}\right) - u_t\left(\frac{\partial\mu}{\partial x} + \frac{du}{dx}\frac{\partial\mu}{\partial u}\right)$$

$$= (\zeta_x + u_x\zeta_u) - u_x(\epsilon_x + u_x\epsilon_u) - u_t(\mu_x + u_x\mu_u)$$
$$= \zeta_x + (\zeta_u - \epsilon_x)u_x - \mu_x u_t - \epsilon_u(u_x^2) - \mu_u u_x u_t,$$

so

$$\zeta^{(x)} = \zeta_x + (\zeta_u - \epsilon_x)u_x - \mu_x u_t - \epsilon_u(u_x^2) - \mu_u u_x u_t. \tag{2.9}$$

Now by using (2.5) the transformation is extend to higher-order derivatives frequently, assume that

$$\hat{u}_h = u_h + \lambda\zeta^{(h)} + O(\lambda^2),$$

where

$$u_h \equiv \frac{\partial^{h_1+h_2}u}{\partial x^{h_1}\partial t^{h_2}}, \quad \hat{u}_h \equiv \frac{\partial^{h_1+h_2}\hat{u}}{\partial \hat{x}^{h_1}\partial \hat{t}^{h_2}},$$

for some h_1 and h_2, then

$$\hat{u}_{h\hat{x}} = u_{hx} + \lambda\zeta^{(hx)} + O(\lambda^2), \quad \hat{u}_{h\hat{t}} = u_{ht} + \lambda\zeta^{(ht)} + O(\lambda^2),$$

from Eq. (2.5)

$$\hat{u}_{h\hat{x}} = \frac{\begin{vmatrix} D_x\hat{u}_h & D_x\hat{t} \\ D_t\hat{u}_h & D_t\hat{t} \end{vmatrix}}{\begin{vmatrix} D_x\hat{x} & D_x\hat{t} \\ D_t\hat{x} & D_t\hat{t} \end{vmatrix}} = \frac{\begin{vmatrix} u_{hx} + \lambda D_x(\zeta^{(h)}) + O(\lambda^2) & \lambda D_x(\mu) + O(\lambda^2) \\ u_{ht} + \lambda D_t(\zeta^{(h)}) + O(\lambda^2) & 1 + \lambda D_t(\mu) + O(\lambda^2) \end{vmatrix}}{\begin{vmatrix} 1 + \lambda D_x(\epsilon) + O(\lambda^2) & \lambda D_x(\mu) + O(\lambda^2) \\ \lambda D_t(\epsilon) + O(\lambda^2) & 1 + \lambda D_t(\mu) + O(\lambda^2) \end{vmatrix}}$$
$$= \frac{u_{hx} + \lambda u_{hx}D_t(\mu) - \lambda u_{ht}D_x(\mu) + O(\lambda^2)}{1 + \lambda(D_t(\mu) + D_x(\epsilon)) + O(\lambda^2)},$$

since

$$\hat{u}_{h\hat{x}} = u_{hx} + \lambda\zeta^{(hx)} + O(\lambda^2),$$

then

$$u_{hx} + \lambda u_{hx}D_t(\mu) + \lambda D_x(\zeta^{(h)}) - \lambda u_{ht}D_x(\mu) + O(\lambda^2) = (1 + \lambda(D_t(\mu) + D_x(\epsilon)))\left(u_{hx} + \lambda\zeta^{(hx)} + O(\lambda^2)\right)$$
$$= u_{hx} + \lambda u_{hx}D_t(\mu) + \lambda u_{hx}D_x(\epsilon) + \lambda\zeta^{(hx)} + O(\lambda^2),$$

then

$$u_{hx} + \lambda u_{hx}D_t(\mu) + \lambda D_x(\zeta^{(h)}) - \lambda u_{ht}D_x(\mu) - u_{hx} - \lambda u_{hx}D_t(\mu) - \lambda u_{hx}D_x(\epsilon) = \lambda\zeta^{(hx)},$$

then

$$\lambda\left(D_x(\zeta^h) - u_{hx}D_x(\epsilon) - u_{ht}D_x(\mu)\right) = \lambda\zeta^{(hx)},$$

so

$$\zeta^{(hx)} = D_x(\zeta^h) - u_{hx}D_x(\epsilon) - u_{ht}D_x(\mu).$$

In the same way

$$\zeta^{(ht)} = D_t(\zeta^h) - u_{hx}D_t(\epsilon) - u_{ht}D_t(\mu).$$

We introduce the infinitesimal generator X

$$X = \epsilon\partial_x + \mu\partial_t + \zeta\partial_u.$$

The infinitesimal generator is extend to derivatives by adding all terms of the form $\zeta^{(h)}\partial_{u_h}$ up to the desired order, the first extension as

$$X^{[1]} = X + \zeta^{(x)}\partial_{u_x} + \zeta^{(t)}\partial_{u_t},$$

is a point symmetry of $f(x, t, u, u_x, u_t, \ldots) = 0$ if Lie's symmetry condition

$$X^{[1]} f \mid_{f=0}$$

and the second extension $X^{[2]}$ of the operator X is

$$X^{[2]} = X^{[1]} + \zeta^{(xx)}\partial_{u_{xx}} + \zeta^{(xt)}\partial_{u_{xt}} + \zeta^{tt}\partial_{u_{tt}},$$

is a point symmetry of $f(x, t, u, u_x, u_t, \ldots) = 0$ if Lie's symmetry condition $X^{[2]} f \mid_{f=0}$ and the extended transformations are given as

$$\zeta^{(xx)} = D_x(\zeta^{(x)}) - u_{xx}D_x(\epsilon) - u_{xt}D_x(\mu),$$

$$\zeta^{(xt)} = D_t(\zeta^{(x)}) - u_{xx}D_t(\epsilon) - u_{xt}D_t(\mu),$$

and

$$\zeta^{(xt)} = D_x(\zeta^{(t)}) - u_{xt}D_x(\epsilon) - u_{tt}D_x(\mu),$$
$$\zeta^{(tt)} = D_t(\zeta^{(t)}) - u_{xt}D_t(\epsilon) - u_{tt}D_t(\mu),$$

now we expand $\zeta^{(xx)}$),$\zeta^{(xt)}$) and $\zeta^{(tt)}$), we get

$$\begin{aligned} \zeta^{(xx)} &= \zeta_{xx} + (2\zeta_{xu} - \epsilon_{xx})u_x - \mu_{xx}u_t + (\zeta_{uu} - 2\epsilon_{xu})u_x^2 \\ &\quad - 2\mu_{xu}u_xu_t - \epsilon_{uu}u_x^3 - \mu_{uu}u_x^2u_t + (\zeta_u - 2\varepsilon_x)u_{xx} \\ &\quad - 2\mu_xu_{xt} - 3\epsilon_uu_xu_{xx} - \mu_uu_tu_{xx} - 2\mu_uu_xu_{xt}, \end{aligned} \tag{2.10}$$

$$\begin{aligned} \zeta^{(xt)} &= \zeta_{xt} + (\zeta_{tu} - \epsilon_{xt})u_x + (\zeta_{xu} - \mu_{xt})u_t - \epsilon_{tu}u_x^2 \\ &\quad + (\zeta_{uu} - \epsilon_{xu} - \mu_{tu})u_xu_t - \mu_{xu}u_t^2 - \epsilon_{uu}u_x^2u_t - \epsilon_{uu}u_xu_t^2 \\ &\quad - \epsilon_tu_{xx} - \epsilon_uu_tu_{xx} + (\zeta_u - \epsilon_x - \mu_t)u_{xt} - 2\epsilon_uu_xu_{xt} - 2\epsilon_uu_tu_{xt} - \epsilon_xu_{tt} - \epsilon_uu_xu_{tt}, \end{aligned} \tag{2.11}$$

$$\begin{aligned} \zeta^{(tt)} &= \zeta_{tt} - \epsilon_{tt}u_x + (2\zeta_{tu} - \epsilon_{tt})u_t - 2\epsilon_{tu}u_xu_t \\ &\quad + (\zeta_{uu} - 2\mu_{tu})u_t^2 - \epsilon_{uu}u_xu_t^2 - \mu_{uu}u_t^3 - 2\epsilon_tu_{xt} \\ &\quad - 2\epsilon_uu_tu_{xt} + (\zeta_u - 2\mu_t)u_{tt} - \epsilon_uu_xu_{tt} - 3\mu_uu_tu_{tt}. \end{aligned} \tag{2.12}$$

Definition 2.3 (Lie algebra, [20]). A Lie algebra L is a vector space V of operations $X = \xi_i \frac{\partial}{\partial x_i}$ over a field (real or complex) together with a binary operator $[,] : V \times V \longrightarrow V$ with the following properties:

Anti-symmetry/Anti-commutativity/skew-symmetry

$$[X_1, X_2] = -[X_2, X_1],$$

for all $X_1, X_2 \in L$.

Bilinearity

$$[X_1, \alpha X_2 + \beta X_3] = \alpha[X_1, X_2] + \beta[X_1, X_3] \quad \text{and} \quad [\alpha X_1 + \beta X_2, X_3] = \alpha[X_1, X_3] + \beta[X_2, X_3],$$

where α, β are constants, for all $X_1, X_2, X_3 \in L$.

Jacobi Identity

$$[[X_1, X_2], X_3] + [[X_2, X_3], X_1] + [[X_3, X_1], X_2] = 0,$$

for all $X_1, X_2, X_3 \in L$, where

$$X_i = \xi_i \frac{\partial}{\partial x_i}, \quad i = 1, 2, \ldots, n.$$

For example, $[X_1, X_2] = X_1X_2 - X_2X_1$ is also an element of L, so $[.,.]$ is called the Lie bracket or commutator of L.

Definition 2.4 ([9]). Let $X_i = \xi_s \frac{\partial}{\partial x_s}$ and $X_j = \eta_s \frac{\partial}{\partial x_s}$, $i, j = 1, \ldots, r$ and $s = 1, \ldots, n$ be two infinitesimal generator. The commutator $[X_i, X_j]$ of X_i and X_j is the first order operator

$$[X_i, X_j] = X_iX_j - X_jX_i = \sum_{s}^{n} \sum_{m}^{n} \left(\xi_m \frac{\partial \eta_s}{\partial x_m} - \eta_m \frac{\partial \xi_s}{\partial x_m} \right) \frac{\partial}{\partial x_s}.$$

Definition 2.5 ([20]). Lie groups G is a smooth manifold and a group such that the multiplication $\rho : G \times G \longrightarrow G$ is smooth. The inversion $\sigma : G \longrightarrow G$ is also smooth.

Definition 2.6 ([11, 20]). A finite set of infinitesimal generator $X_1, X_2, \ldots, X_r$ is said to be a basis for the Lie algebra L if $X_i \in L$ and

- $X_1, X_2, \ldots, X_r$ form a basis of the vector space;
- $[X_i, X_j] = c_{ijk}X_k$.

The coefficients c_{ijk} are called the structure constant of the Lie algebra, $i, j, k = 1, 2, \ldots, r$.

Theorem 2.7 (Second fundamental theorem of Lie, [9]). *Any two infinitesimal generators of an r-parameter Lie group, satisfy commutation relation of the form $[X_i, X_j] = c_{ijk}X_k$, where $i, j, k = 1, 2, \ldots, r$. Provided the infinitesimal generators span the Lie algebra associated with the Lie group, all real (or complex) linear combinations of the X_i will also obey commutation relations and the Jacobi identity associated with the Lie algebra. That is, the given infinitesimal generators $X_1, X_2, \ldots, X_r$ form a basis for the Lie algebra.*

Theorem 2.8 ([13]). *A function $F(x)$ is an invariant under the Lie group of transformation if and only if $XF(x) = 0$, where X is an infinitesimal generator.*

3. Lie symmetry first-order PDEs [12]

Consider the PDE

$$u_t = u_x^2, \tag{3.1}$$

we determine the symmetries of Eq. (3.1). Using Theorem 2.8

$$X^{[2]} f \,|_{f=0} = 0,$$

where

$$f = u_t - u_x^2 = 0,$$

and $X^{[2]}$ is the second-order extended of

$$X = \epsilon(x, t, u) \frac{\partial}{\partial x} + \mu(x, t, u) \frac{\partial}{\partial t} + \zeta(x, t, u) \frac{\partial}{\partial u},$$

namely,

$$X^{[2]} = \epsilon \frac{\partial}{\partial x} + \mu \frac{\partial}{\partial t} + \zeta \frac{\partial}{\partial u} + \zeta^{(x)} \frac{\partial}{\partial u_x} + \zeta^{(t)} \frac{\partial}{\partial u_t} + \zeta^{(xx)} \frac{\partial}{\partial u_{xx}} + \zeta^{(xt)} \frac{\partial}{\partial u_{xt}} + \zeta^{(tt)} \frac{\partial}{\partial u_{tt}}.$$

Hence we solve

$$\left(\epsilon\frac{\partial}{\partial x}+\mu\frac{\partial}{\partial t}+\zeta\frac{\partial}{\partial u}+\zeta^{(x)}\frac{\partial}{\partial u_x}+\zeta^{(t)}\frac{\partial}{\partial u_t}+\zeta^{(xx)}\frac{\partial}{\partial u_{xx}}+\zeta^{(xt)}\frac{\partial}{\partial u_{xt}}+\zeta^{(tt)}\frac{\partial}{\partial u_{tt}}\right)\left(u_t-u_x^2\right)|_{u_t=u_x^2}=0,$$

so we get

$$\zeta^{(t)}=2u_x\zeta^{(x)}, \tag{3.2}$$

we can expand the Eq. (3.2) by using the Eqs. (2.8) and (2.9)

$$\zeta_t-\epsilon_t u_x+(\zeta_u-\mu_t)u_t-\epsilon_u u_x u_t-\mu_u u_t^2=2u_x\left(\zeta_x+(\zeta_u-\epsilon_x)u_x-\mu_x u_t-\epsilon_u u_x^2-\mu_u u_x u_t\right),$$

now replace u_x^2 by u_t we get

$$\zeta_t-\epsilon_t\mu_x+(\zeta_u-\mu_t)u_x^2-\epsilon_u u_x u_x^2-\mu_u u_x^4=2u_x\left(\zeta_x+(\zeta_u-\epsilon_x)u_x-\mu_x u_x^2-\epsilon_u u_x^2-\mu_u u_x u_x^2\right),$$

more simplification we get

$$\zeta_t-(\epsilon_t+2\zeta_x)u_x+(-\zeta_u-\mu_t+2\epsilon_x)u_x^2+(\epsilon_u+2\mu_x)u_x^3+\mu_u u_x^4=0, \tag{3.3}$$

then for Eq. (3.3) to be satisfied requires setting the coefficient of u_x to zero we are left with the system of determining equations

$$\zeta_t=0, \tag{3.4}$$
$$\epsilon_t+2\zeta_x=0, \tag{3.5}$$
$$\zeta_u+\mu_t-2\epsilon_x=0, \tag{3.6}$$
$$\epsilon_u+2\mu_x=0, \tag{3.7}$$
$$\mu_u=0, \tag{3.8}$$

we begin by solving Eq. (3.8) by integration both side we get

$$\mu=\alpha(x,t), \tag{3.9}$$

where α is an arbitrary function.

Now integrating equation (3.7) with respect to u and substituting

$$\mu_x=\alpha_x(x,t),$$

we get

$$\epsilon=-2\alpha_x u+\beta(x,t), \tag{3.10}$$

then

$$\epsilon_x=-2\alpha_{xx}u+\beta_x(x,t). \tag{3.11}$$

From equation (3.6)

$$\zeta_u=2\epsilon_x-\mu_t, \tag{3.12}$$

substituting Eq. (3.11) into Eq. (3.12) we get

$$\zeta_u=-4\alpha_{xx}u+2\beta_x(x,t)-\alpha_t(x,t), \tag{3.13}$$

now integrating Eq. (3.13) with respect to u we get

$$\zeta=-2\alpha_{xx}u^2+(2\beta_x-\alpha_t)u+\gamma(x,t), \tag{3.14}$$

for some functions β and γ. Since

$$\epsilon_t=-2\alpha_{xt}u+\beta_t(x,t), \tag{3.15}$$

$$\zeta_x = -2\alpha_{xxx}u^2 + (2\beta_{xx} - \alpha_{xt})u + \gamma_x(x,t), \tag{3.16}$$
$$\zeta_t = -2\alpha_{xxt}u^2 + (2\beta_{xt} - \alpha_{tt})u + \gamma_t(x,t), \tag{3.17}$$

now by substituting Eqs. (3.15), (3.16), and (3.17) into Eqs. (3.4) and (3.5) we get

$$-2\alpha_{xt}u + \beta_t(x,t) - 4\alpha_{xxx}u^2 + 2(2\beta_{xx} - \alpha_{xt})u + 2\gamma_x(x,t) = 0,$$

more simplifying we get

$$-4\alpha_{xxx}u^2 + 4(\beta_{xx} - \alpha_{xt})u + \beta_t(x,t) + 2\gamma_x(x,t) = 0. \tag{3.18}$$

Substituting Eq. (3.17) in to Eq. (3.4) we get

$$-2\alpha_{xxt}u^2 + (2\beta_{xt} - \alpha_{tt})u + \gamma_t(x,t) = 0, \tag{3.19}$$

such that α,β and γ are independent of u. Now splitting Eq. (3.18) and (3.19) with respect to the power of u we get

$$\gamma_t = 0, \tag{3.20}$$
$$\beta_{xx} - \alpha_{xt} = 0, \tag{3.21}$$
$$2\beta_{xt} - \alpha_{tt} = 0, \tag{3.22}$$
$$\beta_t + 2\gamma_x = 0, \tag{3.23}$$
$$\alpha_{xxx} = 0, \tag{3.24}$$
$$\alpha_{xxt} = 0, \tag{3.25}$$

integrating Eq. (3.20) we get

$$\gamma = B_1(x). \tag{3.26}$$

From equation (3.23)

$$\beta_t(x,t) = -2\gamma_x(x,t), \tag{3.27}$$

then integrating equation (3.27) with respect to t we get

$$\beta(x,t) = -2B_1'(x)t + B_2(x). \tag{3.28}$$

From Eq. (3.22) we get

$$\alpha_{tt} = -4B_1''(x), \tag{3.29}$$

by integrating Eq. (3.29) twice with respect to t we get

$$\alpha = -2B_1''(x)t^2 + B_3(x)t + B_4(x), \tag{3.30}$$

where B_1, B_2, B_3 and B_4 are function of x. From Eq. (3.26), (3.28), and (3.30) we can find

$$\begin{aligned} \alpha_{xxx} &= -2B_1'''''(x)t^2 + B_3'''(x)t + B_4'''(x), & \alpha_{xxt} &= -4B_1''''(x)t + B_3''(x), \\ \beta_{xx} &= -2B_1'''(x)t + B_2''(x), & \alpha_{xt} &= -4B_1'''(x)t + B_3'(x), \end{aligned} \tag{3.31}$$

now substituting Eq. (3.31) in to Eq. (3.21), (3.24), and (3.25) we get

$$2B_1''t + B_2''(x) - B_3'(x) = 0, \quad -2B_1'''''(x)t^2 + B_3'''(x)t + B_4'''(x) = 0, \quad -4B_1''''(x)t + B_3''(x) = 0,$$

equating powers of t and solving the resulting ODEs we get

$$B_1''' = 0, \tag{3.32}$$
$$B_2'' - B_3' = 0, \tag{3.33}$$

$$B_3'' = 0, \tag{3.34}$$
$$B_4''' = 0, \tag{3.35}$$

we begin by solving Eq. (3.32) to obtain

$$B_1(x) = \frac{1}{2}c_1x^2 + c_2x + c_3. \tag{3.36}$$

From Eqs. (3.33), (3.34), and (3.35) we obtain

$$B_2(x) = \frac{1}{2}c_4x^2 + c_6x + c_7, \tag{3.37}$$
$$B_3(x) = c_4x + c_5, \tag{3.38}$$
$$B_4(x) = \frac{1}{2}c_8x^2 + c_9x + c_{10}. \tag{3.39}$$

Now we put $\beta_1 = \frac{c_1}{2}$, $\beta_2 = \frac{c_2}{2}$, $\beta_4 = c_4$, $\beta_5 = c_5$, $\beta_6 = c_6$, $\beta_7 = c_7$, $\beta_8 = \frac{c_8}{2}$, $\beta_9 = c_9$, and $\beta_{10} = c_{10}$, where $\beta_1, \ldots, \beta_{10}$ are arbitrary constants. Then the equations (3.36), (3.37), (3.38), and (3.39) become

$$B_1(x) = \beta_1x^2 + 2\beta_2x + \beta_3,\ \ B_2(x) = \frac{1}{2}\beta_4x^2 + \beta_6x + \beta_7,\ \ B_3(x) = \beta_4x + \beta_5,\ \ B_4(x) = \beta_8x^2 + \beta_9x + \beta_{10}.$$

So the Eqs. (3.9), (3.10), and (3.14) become

$$\begin{aligned}
\epsilon &= -4\beta_1xt - 2\beta_2t + \beta_4(\frac{1}{2}x^2 - 2tu) + \beta_6x + \beta_7 - 4\beta_8xu - 2\beta_9u,\\
\mu &= -4\beta_1t^2 + \beta_4xt + \beta_5t + \beta_8x^2 + \beta_9x + \beta_{10},\\
\zeta &= \beta_1x^2 + \beta_2x + \beta_3 + \beta_4xu - \beta_5u + 2\beta_6u - 4\beta_8u^2.
\end{aligned} \tag{3.40}$$

Now we determine the symmetry of the Eq. (3.1). Since the generator

$$X = \epsilon\frac{\partial}{\partial x} + \mu\frac{\partial}{\partial t} + \zeta\frac{\partial}{\partial u}, \tag{3.41}$$

substituting the infinitesimal (3.40) into Eq. (3.41) we get

$$\begin{aligned}
X =& -4\beta_1xt\frac{\partial}{\partial x} - 2\beta_2t\frac{\partial}{\partial x} + \beta_4(\frac{1}{2}x^2 - 2tu)\frac{\partial}{\partial x} + \beta_6x\frac{\partial}{\partial x} + \beta_7\frac{\partial}{\partial x} - 4\beta_8xu\frac{\partial}{\partial x} - 2\beta_9u\frac{\partial}{\partial x}\\
&- 4\beta_1t^2\frac{\partial}{\partial t} + \beta_4xt\frac{\partial}{\partial t} + \beta_5t\frac{\partial}{\partial t} + \beta_8x^2\frac{\partial}{\partial t} + \beta_9x\frac{\partial}{\partial t} + \beta_{10}\frac{\partial}{\partial t} + \beta_1x^2\frac{\partial}{\partial u} + \beta_3\frac{\partial}{\partial u}\\
&+ \beta_4xu\frac{\partial}{\partial u} - \beta_5u\frac{\partial}{\partial u} + 2\beta_6u\frac{\partial}{\partial u} - 4\beta_8u^2\frac{\partial}{\partial u},
\end{aligned}$$

By Theorem 2.8, the Lie algebra of Equation (3.1) is spanned by the following infinitesimal generators corresponding to each β_i

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial u}, \qquad X_2 = \frac{\partial}{\partial x}, \qquad X_3 = \frac{\partial}{\partial t},\\
X_4 &= -4xt\frac{\partial}{\partial x} - 4t^2\frac{\partial}{\partial t} + x^2\frac{\partial}{\partial u},\\
X_5 &= -2t\frac{\partial}{\partial x} + x\frac{\partial}{\partial u},\\
X_6 &= (\frac{1}{2}x^2 - 2tu)\frac{\partial}{\partial x} + xt\frac{\partial}{\partial t} + xu\frac{\partial}{\partial u},\\
X_7 &= t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u},
\end{aligned}$$

$$X_8 = x\frac{\partial}{\partial t} + 2u\frac{\partial}{\partial u},$$
$$X_9 = -4xu\frac{\partial}{\partial x} + x^2\frac{\partial}{\partial t} - 4u^2\frac{\partial}{\partial u},$$
$$X_{10} = -2u\frac{\partial}{\partial x} + x\frac{\partial}{\partial t}.$$

4. Lie symmetry of Burgers equation

Burgers' equation is a fundamental partial differential equation occurring in various areas of applied mathematics, such as fluid mechanics, nonlinear acoustics, gas dynamics, traffic flow. The equation was first introduced by Harry Bateman in 1915 [7, 21] and later studied by Johannes Martinus Burgers in 1948 [10], therefore, it is necessary to find the Lie point symmetry of that equation.

In this section we will consider the one-dimensional Burgers equation to be the nonlinear parabolic PDE

$$u_t + uu_x = u_{xx}, \tag{4.1}$$

now we determine the linearized symmetry condition for Eq. (4.1) then by using Lie's invariance condition is

$$X^{[2]} f \mid_{f=0} = 0,$$

where

$$f = u_t + uu_x = u_{xx},$$

and $X^{[2]}$ is the second-order extended of

$$X = \epsilon(x,t,u)\frac{\partial}{\partial x} + \mu(x,t,u)\frac{\partial}{\partial t} + \zeta(x,t,u)\frac{\partial}{\partial u},$$

namely

$$X^{[2]} = \epsilon\frac{\partial}{\partial x} + \mu\frac{\partial}{\partial t} + \zeta\frac{\partial}{\partial u} + \zeta^{(x)}\frac{\partial}{\partial u_x} + \zeta^{(t)}\frac{\partial}{\partial u_t} + \zeta^{(xx)}\frac{\partial}{\partial u_{xx}} + \zeta^{(xt)}\frac{\partial}{\partial u_{xt}} + \zeta^{(tt)}\frac{\partial}{\partial u_{tt}},$$

then the linearized symmetry condition for Eq. (4.1) is

$$\zeta^{(t)} + u\zeta^{(x)} + u_x\zeta = \zeta^{(xx)}. \tag{4.2}$$

Substituting Eq. (2.8), (2.9), and (2.10) into equation (4.2) we get

$$\begin{aligned}
&\zeta_t - \epsilon_t u_x + (\zeta_u - \mu_t)u_t - \zeta_u u_x u_t - \mu_u u_t^2 + u\zeta_x + u(\zeta_u - \epsilon_x)u_x - u\mu_x u_t - u\epsilon_u u_x^2 - u\mu_u u_t^2 + u_x\zeta \\
&\quad = \zeta_{xx} + (2\zeta_{xu} - \zeta_{xx})u_x - \mu_{xx}u_t \\
&\quad\quad + (\zeta_{uu} - 2\zeta_{xu})u_x^2 - 2\mu_{xu}u_x u_t - \zeta_{uu}u_x^3 - \mu_{uu}u_x^2 u_t + (\zeta_u - 2\zeta\epsilon_x)u_{xx} \\
&\quad\quad - 2\mu_x u_{xt} - 3\epsilon_u u_x u_{xx} - \mu_u u_t u_{xx} - 2\mu_u u_x u_{xt},
\end{aligned} \tag{4.3}$$

now we choose the highest-order derivative terms in (4.3) have a factor u_{xt}

$$-2\mu_x u_{xt} - 2\mu_u u_x u_{xt} = 0,$$

this leads to

$$\mu_x = \mu_u = 0.$$

Now we replace u_{xx} by the left-hand side of Eq. (4.1) and removes many terms from Eq. (4.3) containing μ_x and μ_t. So the remaining terms are

$$\begin{aligned}
&\zeta_t - \epsilon_t u_x + (\zeta_u - \mu_t)u_t - \zeta_u u_x u_t + u(\zeta_x + (\zeta_u - \epsilon_x)u_x - \epsilon_u u_x^2) + u_x\zeta \\
&\quad = \zeta_{xx} + (2\zeta_{xu} - \epsilon_{xx})u_x + (\zeta_{uu} - 2\epsilon_{xu})u_x^2 - \epsilon_{uu}u_x^3 + (\zeta_u - 2\epsilon_x - 3\epsilon_u u_x)(u_t + uu_x),
\end{aligned} \tag{4.4}$$

we choose the terms multiplied by u_t, then we get

$$(\zeta_u - \mu_t)u_t - \epsilon_u u_x u_t = (\zeta_u - 2\epsilon_x - 3\epsilon_u u_x)u_t,$$

so we determine equations:

$$\zeta_u - \mu_t = 0, \qquad \epsilon_u = 0, \qquad \zeta_u - 2\epsilon_x - 3\epsilon_u u_x = 0. \tag{4.5}$$

We solve Eq. (4.5), this leads to

$$\epsilon = \frac{1}{2}\mu_t x + \alpha(t),$$

for some arbitrary function α. From equation (4.4) we obtain

$$\zeta_{uu} - 2\epsilon_{xu} = 0, \quad \epsilon_{uu} = 0,$$

since

$$\epsilon_{xu} = 0,$$

then

$$\zeta_{uu} = 0, \tag{4.6}$$

by integrating both sides of Eq. (4.6) we obtain

$$\zeta = \beta(x,t)u + \gamma(x,t),$$

where β and γ arbitrary function. From Eq. (4.5) we obtain

$$\mu = \beta(x,t)t + \delta(x),$$

such that δ is arbitrary function. by determining α, β, γ and δ we eventually arrive at the general solution

$$\epsilon = \beta_1 + t\beta_2 + x\beta_4 + xt\beta_5, \qquad \mu = \beta_3 + 2t\beta_4 + t^2\beta_5, \qquad \zeta = \beta_2 - u\beta_4 + (x - u_t)\beta_5, \tag{4.7}$$

where β_1, β_2, β_3, β_4, and β_5 are arbitrary constants. Now we determine the symmetry of the Eq. (4.1). Since the generator

$$X = \epsilon\frac{\partial}{\partial x} + \mu\frac{\partial}{\partial t} + \zeta\frac{\partial}{\partial u}, \tag{4.8}$$

substituting the infinitesimal (4.7) into Eq. (4.8) we get

$$X = \beta_1\frac{\partial}{\partial x} + t\beta_2\frac{\partial}{\partial x} + x\beta_4\frac{\partial}{\partial x} + xt\beta_5\frac{\partial}{\partial x} + \beta_3\frac{\partial}{\partial t} + 2t\beta_4\frac{\partial}{\partial t} + t^2\beta_5\frac{\partial}{\partial t} + \beta_2\frac{\partial}{\partial u} - u\beta_4\frac{\partial}{\partial u} + (x - ut)\beta_5\frac{\partial}{\partial u},$$

then for symmetries are given by

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial t}, \quad X_3 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \quad X_4 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}, \quad X_5 = xt\frac{\partial}{\partial x} + t^2\frac{\partial}{\partial t} + (x - ut)\frac{\partial}{\partial u}.$$

5. Heat equation

The heat equation is a partial differential equation that describes how the distribution of some quantity (such as heat) evolves over time in a solid medium, as it spontaneously flows from places where it is higher towards places where it is lower. It is a special case of the diffusion equation. Using a Lie symmetry group generator and a generalized for solving second order ordinary differential equations, we determine new symmetries for two dimensional heat equations, leading to new solutions.

5.1. *One-dimensional heat equation* [1]

The one dimension heat equation is given by

$$u_{xx} - u_t = 0.$$

Now we determine the symmetries of the one-dimensional heat equation from invariance condition

$$X^{[2]} f \mid_{f=0} = 0,$$

by using the extended operator

$$X^{[2]} = \epsilon\partial_x + \mu\partial_t + \zeta\partial_u + \zeta^{(x)}\partial_{u_x} + \zeta^{(t)}\partial_{u_t} + \zeta^{(xx)}\partial_{u_{xx}} + \zeta^{(xt)}\partial_{u_{xt}} + \zeta^{tt}\partial_{u_{tt}},$$

and substituting

$$f : u_{xx} = u_t,$$

we get

$$\zeta^{(t)} - \zeta^{(xx)} = 0, \tag{5.1}$$

now substituting the extended transformations equation (2.8) and (2.10) into equation (5.1) we get

$$\begin{aligned}\zeta_t - \epsilon_t u_x + (\zeta_u - \mu_t)u_t - \epsilon_u u_x u_t - \mu_u u_t^2 - \zeta_{xx} - (2\zeta_{xu} + \epsilon_{xx})u_x + \mu_{xx}u_t - (\zeta_{uu} + 2\epsilon_{xu})u_x^2 \\ + 2\mu_{xu}u_x u_t + \epsilon_{uu}u_x^3 + \mu_{uu}u_x^2 u_t - (\zeta_u + 2\varepsilon_x)u_{xx} + 2\mu_x u_{xt} + 3\epsilon_u u_x u_{xx} + \mu_u u_t u_{xx} + 2\mu_u u_x u_{xt} = 0.\end{aligned}$$

By using the heat equation to replace u_{xx} by u_t and isolating coefficients involving each power of u_x and u_t,u_{xt}, we are left with the system of determining equations

$$\begin{aligned}
&C: \ \zeta_t - \zeta_{xx} = 0, && u_x: \ -\mu_t - 2\zeta_{xu} + \mu_{xx} = 0, && u_x^2: \ -\zeta_{uu} - 2\mu_{xu} = 0,\\
&u_x^3: \ \mu_{uu} = 0, && u_t u_x: \ 2\mu_u + 2\epsilon_{xu} = 0, && u_t: \ -\epsilon_t + \epsilon_{xx} + 2\mu_x = 0,\\
&u_t u_x^2: \ \epsilon_{uu} = 0, && u_{tx}: \ 2\epsilon_x = 0, && u_x u_{tx}: \ 2\epsilon_u = 0,\\
&\epsilon_x = 0, && \epsilon_u = 0, && \mu_u = 0, \quad \zeta_{uu} = 0.
\end{aligned}$$

Since

$$-\epsilon_t + \epsilon_{xx} + 2\mu_x = 0,$$

then

$$2\mu_x - \epsilon_t = 0, \tag{5.2}$$

and equations

$$-\mu_t - 2\zeta_{xu} + \mu_{xx} = 0, \tag{5.3}$$

$$\zeta_t - \zeta_{xx} = 0. \tag{5.4}$$

From equation (5.2) we get

$$\mu = \frac{1}{2}\epsilon_t x + \alpha(t)m \tag{5.5}$$

such that α is an arbitrary function of t. Since

$$\zeta_{uu} = 0, \tag{5.6}$$

then by integration equation (5.6) twice with respect to u we get

$$\zeta = \beta(x,t)u + \gamma(x,t), \tag{5.7}$$

where β and δ are arbitrary functions of both x, t. From equations (5.2), (5.5), and (5.7) we get

$$\mu_{xx} = 0, \qquad \mu_t = \frac{1}{2}\epsilon_{tt}x + \alpha_t(t), \qquad \zeta_{xu} = \beta_x(x,t), \tag{5.8}$$

substituting equation (5.8) into equation (5.3) we get

$$-2\beta_x(x,t) - \frac{1}{2}\epsilon_{xx}x - \alpha_t(t) = 0, \quad -2\beta_x(x,t) = \frac{1}{2}\epsilon_{xx}x + \alpha_t(t),$$

then

$$\beta_x(x,t) = \frac{-1}{4}\epsilon_{xx}x - \frac{1}{2}\alpha_t(t),$$

we integrate both sides with respect to x, we get

$$\beta(x,t) = \frac{-1}{8}\epsilon_{tt}x^2 - \frac{1}{2}\alpha_t(t)x + \delta(t), \tag{5.9}$$

where $\delta(t)$ is an arbitrary function of t. Now substituting equation (5.9) into equation (5.7) yields

$$\zeta = \left(\frac{-1}{8}\epsilon_{tt}x^2 - \frac{1}{2}\alpha_t(t)x + \delta(t)\right)u + \gamma(x,t), \tag{5.10}$$

then from equation (5.10) we get

$$\zeta_{xx} = \frac{-1}{4}\epsilon_{tt}u + \gamma_{xx}(x,t), \tag{5.11}$$

$$\zeta_t = \left(-\frac{1}{8}\epsilon_{ttt}x^2 - \frac{1}{2}\alpha_{tt}(t)x + \delta_t(t)\right)u + \gamma_t(x,t), \tag{5.12}$$

substituting the equations (5.11) and (5.12) into equation (5.4) we get

$$\left(-\frac{1}{8}\epsilon_{ttt}x^2 - \frac{1}{2}\alpha_{tt}(t)x + \delta_t(t)\right)u + \gamma_t(x,t) + \frac{1}{4}\epsilon_{tt}u - \gamma_{xx}(x,t) = 0, \tag{5.13}$$

splitting equation (5.13) with respect to the powers of x and t we get

$$\epsilon_{ttt} = 0, \tag{5.14}$$

$$\alpha_{tt} = 0, \tag{5.15}$$

$$\delta_t(t) + \frac{1}{4}\epsilon_{tt} = 0, \tag{5.16}$$

$$\gamma_t(x,t) - \gamma_{xx}(x,t) = 0,$$

integrating the equations (5.14), (5.15), and (5.16) with respect to t we get

$$\epsilon_{tt} = B_1, \tag{5.17}$$

$$\epsilon = \frac{B_1}{2}t^2 + B_2t + B_3, \tag{5.18}$$

$$\alpha_t(t) = B_4, \tag{5.19}$$

$$\alpha(t) = B_4t + B_5, \tag{5.20}$$

$$\delta(t) = -\frac{1}{4}B_1t + B_6, \tag{5.21}$$

substituting equations (5.17), (5.19), (5.21) into equation (5.10) we get

$$\zeta = \left(\frac{-1}{8}B_1x^2 - \frac{1}{2}B_4x - \frac{1}{4}B_1t + B_6\right)u + \gamma(x,t), \tag{5.22}$$

now put $\beta_1 = \frac{B_1}{2}$, $\beta_2 = \frac{B_2}{2}$,$\beta_3 = B_3$,$\beta_4 = B_4$, $\beta_5 = B_5$ and $\beta_6 = B_6$, then the equation (5.22) and (5.18) become

$$\epsilon = \beta_1 t^2 + 2\beta_2 t + \beta_3, \quad \zeta = \frac{-1}{4}\beta_1 x^2 u - \frac{\beta_4}{2} xu - \frac{1}{2}\beta_1 tu + \beta_6 u + \gamma(x,t).$$

From equation (5.18), we get

$$\epsilon_t = B_1 t + B_2, \tag{5.23}$$

substituting equations (5.23) and (5.20) into equation (5.5) we get

$$\mu = \frac{1}{2}(B_1 t + B_2)x + B_4 t + B_5 = \frac{B_1}{2}xt + \frac{B_2}{2}x + B_4 t + B_5 = \beta_1 xt + \beta_2 x + \beta_4 t + \beta_5.$$

In the end we arrive the infinitesimal

$$\epsilon = \beta_1 t^2 + 2\beta_2 t + \beta_3, \quad \mu = \beta_1 xt + \beta_2 x + \beta_4 t + \beta_5, \quad \zeta = \frac{-1}{4}\beta_1 x^2 u - \frac{\beta_4}{2} xu - \frac{1}{2}\beta_1 tu + \beta_6 u + \gamma(x,t).$$

Since the generator

$$X = \epsilon \frac{\partial}{\partial x} + \mu \frac{\partial}{\partial t} + \zeta \frac{\partial}{\partial u}, \tag{5.24}$$

substituting the infinitesimal in equation (5.24) we get

$$\begin{aligned} X =& \beta_1 t^2 \frac{\partial}{\partial x} + 2\beta_2 t \frac{\partial}{\partial x} + \beta_3 \frac{\partial}{\partial x} + \beta_1 xt \frac{\partial}{\partial t} + \beta_2 x \frac{\partial}{\partial t} + \beta_4 t \frac{\partial}{\partial t} + \beta_5 \frac{\partial}{\partial t} \\ & - \frac{1}{4}\beta_1 x^2 u \frac{\partial}{\partial u} - \frac{\beta_4}{2} xu \frac{\partial}{\partial u} - \frac{1}{2}\beta_1 tu \frac{\partial}{\partial u} + \beta_6 u \frac{\partial}{\partial u} + \gamma(x,t)\frac{\partial}{\partial u}, \end{aligned}$$

therefore the symmetries are given by

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x}, & X_2 &= \frac{\partial}{\partial t}, & X_3 &= xt\frac{\partial}{\partial t} + t^2\frac{\partial}{\partial x} - (\frac{1}{4}x^2 + \frac{1}{2}u)u\frac{\partial}{\partial u}, \\ X_4 &= x\frac{\partial}{\partial t} + 2t\frac{\partial}{\partial x}, & X_5 &= u\frac{\partial}{\partial u}, & X_6 &= t\frac{\partial}{\partial t} - \frac{1}{2}xu\frac{\partial}{\partial u}, \quad X_\infty = \gamma(x,t)\frac{\partial}{\partial u}. \end{aligned}$$

5.2. Two-dimensional heat equation [19]

We use a Lie symmetry group generator and a generalized form of Manale's formula for solving second order ordinary differential equations, we determine new symmetries for the two dimensional heat equations. The two dimension heat equation is given

$$u_t - u_{xx} - u_{yy} = 0,$$

in which u is dependent variable and t, x are independent variables. The infinitesimal generator is

$$X = \epsilon(x,y,t,u)\frac{\partial}{\partial x} + \varphi(x,y,t,u)\frac{\partial}{\partial y} + \mu(x,y,t,u)\frac{\partial}{\partial t} + \zeta(x,y,t,u)\frac{\partial}{\partial u}, \tag{5.25}$$

where $X^{[2]}$ is the second extended operator of X given by

$$\begin{aligned} X^{[2]} =& X + \zeta^{(x)}\frac{\partial}{\partial u_x} + \zeta^{(y)}\frac{\partial}{\partial u_y} + \zeta^{(t)}\frac{\partial}{\partial u_t} + \zeta^{(xx)}\frac{\partial}{\partial u_{xx}} + \zeta^{(yy)}\frac{\partial}{\partial u_{yy}} \\ & + \zeta^{(tt)}\frac{\partial}{\partial u_{tt}} + \zeta^{(xy)}\frac{\partial}{\partial u_{xy}} + \zeta^{(xt)}\frac{\partial}{\partial u_{xt}} + \zeta^{(yt)}\frac{\partial}{\partial u_{yt}}, \end{aligned}$$

with the symmetry condition

$$X^{[2]}(u_t - u_{xx} - u_{yy}) \mid_{(u_{yy} = u_t - u_{xx})} = 0, \tag{5.26}$$

then we get

$$\zeta^{(t)} - \zeta^{(xx)} - \zeta^{(yy)} \mid_{(u_{yy}=u_t-u_{xx})} = 0,$$

where

$$\zeta = fu + g, \tag{5.27}$$

$$\begin{aligned}
\zeta^{(x)} &= D_x(\zeta) - u_x D_x(\epsilon) - u_t D_x(\mu) - u_y D_x(\varphi)\\
&= D_x(fu+g) - u_x D_x(\epsilon) - u_t D_x(\mu) - u_y D_x(\varphi)\\
&= g_x + uf_x + u_x(f-\epsilon_x) - u_t\mu_x - u_y\varphi_x,\\
\zeta^{(xx)} &= D_x(\zeta^{(x)}) - u_{xx}D_x(\epsilon) - u_{xt}D_x(\mu) - u_{xy}D_x(\varphi)\\
&= D_x(g_x + uf_x + u_x(f-\epsilon_x) - u_t\mu_x - u_y\varphi_x) - u_{xx}\epsilon_x - u_{xt}\mu_x - u_{xy}\varphi_x\\
&= g_{xx} + u_x f_x + uf_{xx} + u_x f_x + u_{xx}f - u_x\epsilon_{xx} - u_{xx}\epsilon_x - u_{xt}\mu_x - u_t\mu_{xx} \qquad (5.28)\\
&\quad - u_y\varphi_{xx} - u_{xy}\varphi_x - u_{xx}\epsilon_x - u_{xt}\mu_x - u_{xy}\varphi_x\\
&= g_{xx} + uf_{xx} + u_x[2f_x - \epsilon_{xx}] - u_y\varphi_{xx} - u_t\mu_{xx} + u_{xx}[f-2\epsilon_x] - 2u_{xy}\varphi_x - 2u_{xt}\mu_x,\\
\zeta^{(t)} &= D_t(\zeta) - u_t D_t(\epsilon) - u_y D_t(\varphi) - u_t D_t(\mu)\\
&= D_t(fu+g) - u_x D_t(\epsilon) - u_y D_t(\varphi) - u_t D_t(\mu) \qquad (5.29)\\
&= u_t f + uf_t + g_t - u_x\epsilon_t - u_y\varphi_t - u_t\mu_t = g_t + uf_t + u_t[f-\mu_t] - \epsilon_t u_x - u_y\varphi_t,\\
\zeta^{(y)} &= D_y(\zeta) - u_x D_y(\epsilon) - u_y D_y(\varphi) - u_t D_y(\mu) = g_y + uf_y + u_y[f-\varphi_y] - u_x\epsilon_y - u_t\mu_y,\\
\zeta^{(yy)} &= D_y(\zeta^{y}) - u_{xy}D_y(\epsilon) - u_{yy}D_y(\varphi) - u_{yt}D_y(\mu)\\
&= g_{yy} + uf_{yy} + u_y f_y + u_y f_y + u_{yy}f - u_y\varphi_{yy} - u_{yy}\varphi_y\\
&\quad - u_x\epsilon_{yy} - u_{xy}\epsilon_y - u_t\mu_{yy} - u_{yt}\mu_y - u_{xy}\epsilon_y - u_{yy}\varphi_y - u_{yt}\mu_y \qquad (5.30)\\
&= g_{yy} + uf_{yy} + u_y[2f_y - \varphi_{yy}] - u_x\epsilon_{yy} - u_t\mu_{yy} + u_{yy}[f-2\varphi_y] - 2u_{xy}\epsilon_y - 2u_{yt}\mu_y.
\end{aligned}$$

Now substituting Eqs. (5.29), (5.28), (5.30) in Eq. (5.26) and using heat equation to replace u_{yy} by $u_t - u_{xx}$ yields

$$\begin{aligned}
&g_t + uf_t + u_t[f-\mu_t] - \epsilon_t u_x - u_y\varphi_t\\
&\quad = g_{xx} + uf_{xx} + u_x[2f_x - \epsilon_{xx}] - u_y\varphi_{xx} - u_t\mu_{xx} + u_{xx}[f-2\epsilon_x] - 2u_{xy}\varphi_x - 2u_{xt}\mu_x\\
&\qquad + g_{yy} + uf_{yy} + u_y[2f_y - \varphi_{yy}] - u_x\epsilon_{yy} - u_t\mu_{yy} + (u_t - u_{xx})[f-2\varphi_y] - 2u_{xy}\epsilon_y - 2u_{yt}\mu_y,
\end{aligned}$$

more simplifying the equation above yields

$$\begin{aligned}
&(g_t - g_{xx} - g_{yy}) + u(f_t - f_{xx} - f_{yy}) + u_t(-\mu_t + \mu_{xx} + \mu_{yy} + 2\varphi_y)\\
&\quad + u_x(-\epsilon_t + \epsilon_{xx} + \epsilon_{yy} + 2f_x) + u_y(\varphi_{yy} - \varphi_t + \varphi_{xx} - 2f_y)\\
&\quad + u_{xx}(2\epsilon_x - 2\varphi_y) + u_{xy}(2\varphi_x + 2\epsilon_y) + 2u_{xt}\mu_x + 2u_{yt}\mu_y = 0,
\end{aligned}$$

separating coefficients in Eq. (5.30) we get

$$C : g_t - g_{xx} - g_{yy} = 0,$$

$$u : f_t - f_{xx} - f_{yy} = 0, \tag{5.31}$$

$$u_t : \mu_t - \mu_{xx} - \mu_{yy} - 2\varphi_y = 0, \tag{5.32}$$

$$u_x : \epsilon_t - \epsilon_{xx} - \epsilon_{yy} + 2f_x = 0, \tag{5.33}$$

$$u_y : \varphi_t - \varphi_{xx} - \varphi_{yy} + 2f_y = 0, \tag{5.34}$$

$$u_{xx} : \epsilon_x - \varphi_y = 0, \tag{5.35}$$

$$u_{xy} : \varphi_x + \epsilon_y = 0, \tag{5.36}$$

$$u_{xt} : \mu_x = 0, \tag{5.37}$$

$$u_{yt} : \mu_y = 0,$$

integrating Eq. (5.37) with respect to x we get

$$\mu = \alpha(t),$$

since $\mu_{xx} = 0$ and $\mu_{yy} = 0$, then from Eq. (5.32) we get

$$\varphi = \frac{1}{2}\alpha_t y + \beta(x,t), \tag{5.38}$$

differentiating Eq. (5.38) with respect to x and y twice yields

$$\varphi_{xx} = \beta_{xx}(x,t), \quad \varphi_{yy} = 0.$$

Differentiating Eq. (5.38) with respect to t and y we get

$$\varphi_{tt} = \frac{1}{2}\alpha_{tt} y + \beta_t(x,t), \quad \varphi_y = \frac{1}{2}\alpha_t.$$

Substituting φ_y into Eq. (5.35) yields

$$\epsilon_x = \frac{1}{2}\alpha_t, \tag{5.39}$$

integrating Eq. (5.39) we get

$$\epsilon = \frac{1}{2}\alpha_t x + \gamma(y,t), \tag{5.40}$$

differentiating Eq. (5.40) with respect to t we get

$$\epsilon_t = \frac{1}{2}\alpha_{tt} x + \gamma_t(y,t).$$

Differentiating Eq. (5.40) with respect to x and y twice yields

$$\epsilon_{xx} = 0, \quad \epsilon_{yy} = \gamma_{yy}(y,t).$$

Differentiating Eq. (5.36) with respect to x and y, respectively we get

$$\varphi_{xx} = 0 = \beta_{xx}(x,t), \quad \epsilon_{yy} = 0 = \gamma_{yy}(y,t), \tag{5.41}$$

integrating Eq. (5.41) twice yields

$$\beta(x,t) = B_1 x + B_2, \quad \gamma(y,t) = B_3 y + B_4.$$

Substituting ϵ_t, ϵ_{xx}, ϵ_{yy} into Eq. (5.33) we obtain

$$\frac{1}{2}\alpha_{tt}(t)x + \gamma_t(y,t) + 2f_x = 0,$$

then

$$f_x = \frac{-1}{4}\alpha_{tt} x - \frac{1}{2}\gamma_t(y,t), \tag{5.42}$$

integrating Eq. (5.42) with respect to x yields

$$f = \frac{-1}{8}\alpha_{tt} x^2 - \frac{1}{2}\gamma_t(y,t)x + \delta(y,t).$$

Substituting ϵ_t, ϵ_{xx}, ϵ_{yy} into Eq. (5.34) we obtain

$$f = \frac{-1}{8}\alpha_{tt}y^2 - \frac{1}{2}\beta_t(x,t)y + \kappa(x,t). \tag{5.43}$$

From Eqs. (5.43) and (5.42) we obtain

$$f = -\frac{1}{8}\alpha_{tt}x^2 - \frac{1}{8}\alpha_{tt}y^2 - \frac{1}{2}\gamma_t(y,t) - \frac{1}{2}\beta_t(x,t)y + \kappa(x,t) + \delta(y,t). \tag{5.44}$$

Differentiating Eq. (5.43) with respect to t and twice with respect to x and y we get

$$f_t = -\frac{1}{8}\alpha_{ttt}x^2 - \frac{1}{8}\alpha_{ttt}y^2 - \frac{1}{2}\gamma_t t(y,t) - \frac{1}{2}\beta_t t(x,t)y + \kappa_t(x,t) + \delta_t(y,t), \tag{5.45}$$

$$f_{xx} = -\frac{1}{4}\alpha_{tt} + \kappa_{xx}(x,t), \quad f_{yy} = -\frac{1}{2}\alpha_{tt} + \delta_{yy}(y,t), \tag{5.46}$$

substituting Eqs. (5.45) and (5.46) into Eq. (5.31) yields

$$-\frac{1}{8}\alpha_{ttt}x^2 - \frac{1}{8}\alpha_{ttt}y^2 - \frac{1}{2}\gamma_t t(y,t) - \frac{1}{2}\beta_t t(x,t)y + \kappa_t(x,t) + \delta_t(y,t) + \frac{1}{2}\alpha_{tt} - \delta_{yy}(y,t) - \kappa_{xx}(x,t) = 0. \tag{5.47}$$

Splitting Eq. (5.47) we get

$$\alpha_{ttt}(t) = 0, \tag{5.48}$$
$$\kappa_{xx}(x,t) = 0, \tag{5.49}$$
$$\delta_{yy}(y,t) = 0, \tag{5.50}$$
$$\kappa_t(x,t) = 0,$$
$$\delta_t(y,t) = 0,$$

integrating with respect to t, x, y in Eqs. (5.48), (5.49), (5.50), respectively yields

$$\alpha(t) = \frac{1}{2}B_5t^2 + B_6t + B_7, \quad \kappa(x,t) = B_8x + B_9, \quad \delta(y,t) = B_{10}y + B_{11}.$$

Since

$$\mu = \alpha(t),$$

then

$$\mu = \frac{1}{2}B_5t^2 + B_6t + B_7.$$

Substituting $\alpha_t(t)$ and $\gamma(y,t)$ into Eq. (5.40) we obtain

$$\epsilon = \frac{1}{2}B_5xt + \frac{1}{2}B_6x + B_3y + B_4.$$

Substituting $\alpha_t(t)$ and $\beta(x,t)$ in Eq. (5.38) we get

$$\varphi = \frac{1}{2}B_5yt + \frac{1}{2}B_6y + B_1x + B_2.$$

Since

$$\alpha_{tt} = B_5, \quad \gamma_t(y,t) = 0, \quad \beta_t(x,t) = 0, \quad \kappa_t(x,t) = B_8x + B_9, \quad \delta_t(y,t) = B_{10}y + B_{11},$$

then the Eq. (5.44) becomes

$$f = -\frac{1}{8}B_5x^2 - \frac{1}{8}B_5y^2 + B_8x + B_9 + B_{10}y + B_{11}.$$

Now we put $\beta_1 = \frac{1}{2}B_5$, $\beta_2 = \frac{1}{2}B_6$, $\beta_3 = B_7$, $\beta_4 = B_3$, $\beta_5 = B_4$, $\beta_6 = B_1$, $\beta_7 = B_2$, $\beta_8 = B_8$, $\beta_9 = B_{10}$, $\beta_{10} = B_9 + B_{11}$. Then we get infinitesimals:

$$\begin{aligned} &\mu = \beta_1 t^2 + 2\beta_2 t + \beta_3, && \epsilon = \beta_1 xt + \beta_2 x + \beta_4 y + \beta_5, \\ &\varphi = \beta_1 yt + \beta_2 y + \beta_6 x + \beta_7, && f = -\frac{1}{4}\beta_1(x^2+y^2) + \beta_8 x + \beta_9 y + \beta_{10}. \end{aligned} \tag{5.51}$$

From Eq. (5.27), then ζ becomes

$$\zeta = -\frac{1}{4}u\beta_1(x^2+y^2) + \beta_8 xu + \beta_9 yu + \beta_{10}u + g.$$

Since the generator X in Eq. (5.25) is

$$X = \epsilon(x,y,t,u)\frac{\partial}{\partial x} + \varphi(x,y,t,u)\frac{\partial}{\partial y} + \mu(x,y,t,u)\frac{\partial}{\partial t} + \zeta(x,y,t,u)\frac{\partial}{\partial u},$$

substituting infinitesimals Eq. (5.51) in generator X yields

$$\begin{aligned} X = &\left(t^2\frac{\partial}{\partial t} + xt\frac{\partial}{\partial x} + ty\frac{\partial}{\partial y} - \frac{1}{4}u(x^2+y^2)\frac{\partial}{\partial u}\right)\beta_1 + \left(2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\beta_2 + \frac{\partial}{\partial t}\beta_3 \\ &+ \beta_4 y\frac{\partial}{\partial x} + \beta_5\frac{\partial}{\partial x} + x\beta_6\frac{\partial}{\partial y} + \beta_7\frac{\partial}{\partial y} + \beta_8 xu\frac{\partial}{\partial u} + \beta_9 yu\frac{\partial}{\partial u} + \beta_{10}u\frac{\partial}{\partial u} + g\frac{\partial}{\partial y}, \end{aligned}$$

therefore the symmetries are given by

$$\begin{aligned} &X_1 = \frac{\partial}{\partial t}, && X_2 = \frac{\partial}{\partial x}, && X_3 = y\frac{\partial}{\partial x}, \\ &X_4 = x\frac{\partial}{\partial y}, && X_5 = \frac{\partial}{\partial y}, && X_6 = xu\frac{\partial}{\partial u}, \\ &X_7 = yu\frac{\partial}{\partial u}, && X_8 = u\frac{\partial}{\partial u}, && X_9 = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} + ty\frac{\partial}{\partial y} - \frac{1}{4}u(x^2+y^2)\frac{\partial}{\partial u}, \\ &X_{10} = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, && X_\infty = g\frac{\partial}{\partial y}. \end{aligned}$$

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