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Degenerate polyexponential-Genocchi numbers and polynomials

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Abstract

Recently, Kim et al. in [T. Kim, D. S. Kim, H. Y. Kim, L.-C. Jang, Informatica, **3** (2020), 8 pages] studied the degenerate poly-Bernoulli numbers and polynomials which are defined by using the polylogarithm function. In this paper, we study the degenerate polyexponential-Genocchi polynomials and numbers arising from polyexponential function and derive their explicit expressions and some identity involving them. In the final section, we introduce degenerate unipoly-Genocchi polynomials attached to an arithmetic function, by using polylogarithm function and investigate some identities for those polynomials.

Keywords: Polylogarithm function, degenerate poly-Bernoulli polynomials, degenerate poly-Genocchi polynomials, unipoly function.

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1. Introduction

In 1905, Hardy considered the polyexponential function [9, 10] given by

$$e(x, a|s) = \sum_{n=0}^{\infty} \frac{x^n}{(n+a)^s n!}, \quad (\Re(a) > 0).$$

For $k \in \mathbb{Z}$, Kim and Kim [24] defined the modified polyexponential function, as an inverse to the polylogarithm function by

$$\mathrm{Ei}_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k (n-1)!}. \tag{1.1}$$

It is worthy to note that $e(x, 1|k) = \frac{1}{x}\mathrm{Ei}_k(x)$ and $\mathrm{Ei}_1(x) = e^x - 1$.

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As is well known, the classical Euler and Genocchi polynomials are defined by (see [2–4, 7, 11, 15, 17, 19])

$$\frac{2}{e^z+1}e^{uz}=\sum_{j=0}^{\infty}E_j(u)\frac{z^j}{j!},\ \frac{2z}{e^z+1}e^{uz}=\sum_{j=0}^{\infty}G_j(u)\frac{z^j}{j!}. \tag{1.2}$$

In the case when $u=0$, $E_j=E_j(0)$ and $G_j=G_j(0)$ are respectively, called the Euler numbers and Genocchi numbers.

From (1.2), we see that

$$G_0(0)=0, E_j(u)=\frac{G_{j+1}(u)}{j+1},\quad (j\geqslant 0),\quad \text{(see [27])}.$$

For $k\in\mathbb{Z}$, Kim-Kim considered the type 2 poly-Bernoulli polynomials are defined by means of the following generating function

$$\frac{\mathrm{Ei}_k\left(\log(1+t)\right)}{e^t-1}e^{xt}=\sum_{n=0}^{\infty}B_n^{(k)}(x)\frac{t^n}{n!},\quad \text{(see [14])}.$$

In the case when $x=0$, $B_n^{(k)}=B_n^{(k)}(0)$ are called the type 2 poly-Bernoulli numbers.

For $k\in\mathbb{Z}$, the polylogaritm function is defined by

$$\mathrm{Li}_k(x)=\sum_{n=1}^{\infty}\frac{x^n}{n^k},\quad (|\,x\,|<1),\quad \text{(see, [12, 13, 16, 19, 30])}.$$

Note that

$$\mathrm{Li}_1(x)=\sum_{n=1}^{\infty}\frac{x^n}{n}=-\log(1-x).$$

For any nonzero $\nu\in\mathbb{R}$ (or $\mathbb{C}$), the degenerate exponential function is defined by

$$e_\nu^\xi(z)=(1+\nu z)^{\frac{\xi}{\nu}},\quad e_\nu(z)=(1+\nu z)^{\frac{1}{\nu}},\quad \text{(see [20–25, 27–29])}.$$

By Taylor expansion, we see

$$e_\nu^\xi(z)=\sum_{q=0}^{\infty}(\xi)_{q,\nu}\frac{z^q}{q!},\quad \text{(see [20, 24, 29, 31])},$$

where $(\xi)_{0,\nu}=1$, $(\xi)_{q,\nu}=\xi(\xi-\nu)\cdots(\xi-(q-1)\nu)$, $(q\geqslant 1)$.

Obviously

$$\lim_{\nu\to 0}e_\nu^\xi(z)=\sum_{q=0}^{\infty}\xi^q\frac{z^q}{q!}=e^{\xi z}.$$

In [5, 6], Carlitz considered the degenerate Bernoulli and degenerate Euler polynomials defined by

$$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}\beta_{n,\lambda}(x)\frac{t^n}{n!},$$

and

$$\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}E_{n,\lambda}(x)\frac{t^n}{n!}. \tag{1.3}$$

Putting $x = 0, \beta_{n,\lambda} = \beta_{n,\lambda}(0)$ and $E_{n,\lambda} = E_{n,\lambda}(0)$ are called the degenerate Bernoulli and degenerate Euler numbers.

In [26], Kim et al. considered the the degenerate Genocchi polynomials given by

$$\frac{2z}{e_\lambda(z)+1}e_\lambda^u(z) = \sum_{j=0}^{\infty} G_{j,\lambda}(u)\frac{z^j}{j!}. \tag{1.4}$$

In the case when $u = 0$, $G_{j,\lambda} = G_{j,\lambda}(0)$ are called the degenerate Genocchi numbers.

Very recently, Kim et al. [27] introduced the degenerate poly-Bernoulli polynomials defined by

$$\frac{\mathrm{Ei}_k(\log(1+t))}{e_\lambda(t)-1}e_\lambda^x(t) = \sum_{n=0}^{\infty} \beta_{n,\lambda}^{(k)}(x)\frac{t^n}{n!}.$$

Here, $\beta_{n,\lambda}^{(k)} = \beta_{n,\lambda}^{(k)}(0)$ are called the degenerate poly-Bernoulli numbers.

The Daehee polynomials [28] are defined by

$$\frac{\log(1+t)}{t}(1+t)^x = \sum_{n=0}^{\infty} D_n(x)\frac{t^n}{n!}, \quad \text{(see [19, 32])}.$$

When $x = 0$, $D_n = D_n(0)$ are called the Daehee numbers.

The degenerate Stirling numbers of the first kind [23] are defined by

$$\frac{1}{k!}(\log_\lambda(1+z))^k = \sum_{j=k}^{\infty} S_{1,\lambda}(j,k)\frac{z^j}{j!}, \quad (k \geqslant 0).$$

It is notice that

$$\lim_{\lambda\to 0} S_{1,\lambda}(j,k) = S_1(j,k),$$

are calling the Stirling numbers of the first kind given by

$$\frac{1}{k!}(\log(1+z))^k = \sum_{j=k}^{\infty} S_1(j,k)\frac{z^j}{j!}, \quad (k \geqslant 0), \quad \text{(see [24, 25])}.$$

The degenerate Stirling numbers of the second kind [21] are given by

$$\frac{1}{k!}(e_\lambda(z)-1)^k = \sum_{j=k}^{\infty} S_{2,\lambda}(j,k)\frac{z^j}{j!}, \quad (k \geqslant 0), \quad \text{(see [1, 8])}.$$

Note here that

$$\lim_{\lambda\to 0} S_{2,\lambda}(j,k) = S_2(j,k),$$

standing for the Stirling numbers of the second kind given by means of the following generating function:

$$\frac{1}{k!}(e^z-1)^k = \sum_{j=k}^{\infty} S_2(j,k)\frac{z^j}{j!}, \quad (k \geqslant 0), \quad \text{(see, [14, 15, 18, 20–32])}.$$

2. Degenerate polyexponential-Genocchi numbers and polynomials

In this section, we define degenerate Genocchi numbers and polynomials by using the degenerate polyexponential function which are called the degenerate polyexponential-Genocchi polynomials as follows,

$$\frac{2\mathrm{Ei}_k(\log(1+t))}{e_\lambda(t)+1}e_\lambda^x(t)=\sum_{n=0}^{\infty}G_{n,\lambda}^{(k)}(x)\frac{t^n}{n!},\quad (k\in\mathbb{Z}). \tag{2.1}$$

For $k=1$, $G_{n,\lambda}^{(1)}(x)=G_{n,\lambda}(x)$, $(n\geqslant 0)$. Here, $G_{n,\lambda}^{(k)}=G_{n,\lambda}^{(k)}(0)$ are called the degenerate polyexponential-Genocchi numbers.

From (1.2) and (2.1), we note that

$$\lim_{\lambda\to 0}G_{n,\lambda}^{(1)}(x)=G_n(x),\quad (n\geqslant 0).$$

Theorem 2.1. *For $n\geqslant 0$ and $k\in\mathbb{Z}$, we have*

$$G_{n,\lambda}^{(k)}(x)=\sum_{m=0}^{n}\sum_{l=1}^{m+1}\binom{n}{m}\frac{S_1(m+1,l)}{l^{k-1}(m+1)}G_{n-m,\lambda}(x). \tag{2.2}$$

Proof. Using equations (1.1) and (2.1), we see that

$$\begin{aligned}\mathrm{Ei}_k\left(\log(1+t)\right)&=\sum_{r=1}^{\infty}\frac{(\log(1+t))^r}{(r-1)!r^k}\\&=\sum_{r=1}^{\infty}\frac{(\log(1+t))^r r!}{(r-1)!r^k r!}=\sum_{r=1}^{\infty}\frac{1}{r^{k-1}}\sum_{q=r}^{\infty}S_1(q,r)\frac{t^q}{q!}=\sum_{q=0}^{\infty}\sum_{r=1}^{q+1}\frac{S_1(q+1,r)}{r^{k-1}(q+1)}\frac{t^{q+1}}{q!}.\end{aligned} \tag{2.3}$$

By using equations (1.4) and (2.3), equation (2.1) is

$$\frac{2t}{e_\lambda(t)+1}e_\lambda^x(t)\frac{1}{t}\mathrm{Ei}_k(\log(1+t))=\sum_{k=0}^{\infty}G_{k,\lambda}(x)\frac{t^k}{k!}\sum_{q=0}^{\infty}\sum_{r=1}^{q+1}\frac{S_1(q+1,r)}{r^{k-1}(q+1)}\frac{t^{q+1}}{q!},$$

$$\text{L.H.S}=\sum_{n=0}^{\infty}\left(\sum_{q=0}^{n}\sum_{r=1}^{q+1}\binom{n}{q}\frac{S_1(q+1,r)}{r^{k-1}(q+1)}G_{n-q,\lambda}(x)\right)\frac{t^n}{n!}.$$

Comparing the coefficients of t^n on both sides, we get the result (2.2). □

Theorem 2.2. *For $n\geqslant 0$ and $k\in\mathbb{Z}$, we have*

$$G_{n,\lambda}^{(k)}(x)=\sum_{q=1}^{n}\binom{n}{q}\sum_{r=1}^{q}\frac{S_1(q,r)}{r^{k-1}}E_{n-q,\lambda}(x).$$

Proof. It is proved by using (1.3) and (2.1) that

$$\begin{aligned}\frac{2\mathrm{Ei}_k(\log(1+t))}{e_\lambda(t)+1}e_\lambda^x(t)&=\frac{2}{e_\lambda(t)+1}e_\lambda^x(t)\mathrm{Ei}_k(\log(1+t))\\&=\frac{2}{e_\lambda(t)+1}e_\lambda^x(t)\sum_{r=1}^{\infty}\frac{(\log(1+t))^r r!}{(r-1)!r^k r!}\\&=\frac{2}{e_\lambda(t)+1}e_\lambda^x(t)\sum_{r=1}^{\infty}\frac{1}{r^{k-1}}\sum_{q=r}^{\infty}S_1(q,r)\frac{t^q}{q!}\end{aligned}$$

$$= \sum_{n=0}^{\infty} E_{n,\lambda}(x)\frac{t^n}{n!}\sum_{q=1}^{\infty}\sum_{r=1}^{q}\frac{S_1(q,r)}{r^{k-1}}\frac{t^q}{q!},$$

$$\text{L.H.S} = \sum_{n=1}^{\infty}\left(\sum_{q=1}^{n}\binom{n}{q}\sum_{r=1}^{q}\frac{S_1(q,r)}{r^{k-1}}E_{n-q,\lambda}(x)\right)\frac{t^n}{n!}.$$

Therefore, by (2.1) and above equation, we complete the proof. □

For the next theorem, we need the following well-known from ([7]) that

$$\left(\frac{t}{\log(1+t)}\right)^r (1+t)^{x-1} = \sum_{n=0}^{\infty} B_n^{(n-r+1)}(x)\frac{t^n}{n!}, \quad (r \in \mathbb{C}), \tag{2.4}$$

where $B_n^{(r)}(x)$ are called the higher-order Bernoulli polynomials which are given by the generating function

$$\left(\frac{t}{e^t-1}\right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x)\frac{t^n}{n!}.$$

Theorem 2.3. *For $n \geqslant 0, k \in \mathbb{Z}$, we have*

$$G_{n,\lambda}^{(k)} = \sum_{m=0}^{n}\binom{n}{m}\sum_{m_1+\cdots+m_{k-1}=m} G_{n-m,\lambda}(x) \times \binom{m}{m_1,\ldots,m_{k-1}}\frac{B_{m_1}^{(m_1)}(0)}{m_1+1}\frac{B_{m_2}^{(m_2)}(0)}{m_1+m_2+1}\cdots\frac{B_{m_{k-1}}^{(m_{k-1})}(0)}{m_1+m_2+\cdots+m_{k-1}+1}.$$

Proof. To prove this section, we first consider the following expression

$$\begin{aligned}\frac{d}{dx}\mathrm{Ei}_k(\log(1+x)) &= \frac{d}{dx}\sum_{n=1}^{\infty}\frac{(\log(1+x))^n}{(n+1)!n^k}\\ &= \frac{1}{(1+x)\log(1+x)}\sum_{n=1}^{\infty}\frac{(\log(1+x))^n}{(n+1)!n^{k-1}} = \frac{1}{(1+x)\log(1+x)}\mathrm{Ei}_{k-1}(\log(1+x)).\end{aligned} \tag{2.5}$$

From (2.5), $k \geqslant 2$, we have

$$\begin{aligned}\mathrm{Ei}_k(\log(1+x)) &= \int_0^x \frac{1}{(1+t)\log(1+t)}\mathrm{Ei}_{k-1}(\log(1+t))dt\\ &= \underbrace{\int_0^x \frac{1}{(1+t)\log(1+t)}\int_0^t\cdots\frac{1}{(1+t)\log(1+t)}}_{k-2\text{-times}}\int_0^t\frac{1}{(1+t)\log(1+t)}\\ &\quad\times \mathrm{Ei}_1(\log(1+x))dt\cdots dt\\ &= \underbrace{\int_0^x \frac{1}{(1+t)\log(1+t)}\int_0^t\cdots\frac{1}{(1+t)\log(1+t)}}_{k-2\text{-times}}\int_0^t\frac{t}{(1+t)\log(1+t)}dt\cdots dt.\end{aligned} \tag{2.6}$$

By (2.1), (2.6), and (2.4), we get

$$\sum_{n=0}^{\infty} G_{n,\lambda}^{(k)}\frac{x^n}{n!} = \frac{2\mathrm{Ei}_k(\log(1+x))}{e_\lambda(x)+1} = \frac{2}{e_\lambda(x)+1}$$

$$\times \underbrace{\int_0^x \frac{1}{(1+t)\log(1+t)} \int_0^t \cdots \frac{1}{(1+t)\log(1+t)} \int_0^t}_{k-2\text{-times}} \frac{t}{(1+t)\log(1+t)} dt \cdots dt$$

$$= \frac{2x}{e_\lambda(x)+1}$$

$$\times \sum_{m=0}^{\infty} \sum_{m_1+\cdots+m_{k-1}=m} \binom{m}{m_1,\ldots,m_{k-1}} \frac{B_{m_1}^{(m_1)}(0)}{m_1+1} \frac{B_{m_2}^{(m_2)}(0)}{m_1+m_2+1} \cdots \frac{B_{m_{k-1}}^{(m_{k-1})}(0)}{m_1+m_2+\cdots+m_{k-1}+1} \frac{x^m}{m!}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \binom{n}{m} \sum_{m_1+\cdots+m_{k-1}=m} \binom{m}{m_1,\ldots,m_{k-1}} G_{n-m,\lambda}(x)$$

$$\times \frac{B_{m_1}^{(m_1)}(0)}{m_1+1} \frac{B_{m_2}^{(m_2)}(0)}{m_1+m_2+1} \cdots \frac{B_{m_{k-1}}^{(m_{k-1})}(0)}{m_1+m_2+\cdots+m_{k-1}+1} \frac{x^n}{n!}.$$

Thus, we complete the proof. □

On setting $k = 2$, Theorem 2.3 gives the following result.

Corollary 2.4. *For $n \geqslant 0$, we have*

$$G_{n,\lambda}^{(2)} = \sum_{l=0}^{n} \binom{n}{l} \frac{B_l^{(l)}(0)}{l+1} G_{n-l,\lambda}.$$

Theorem 2.5. *Let $k \geqslant 1$ and $m \in \mathbb{N} \bigcup \{0\}$, $s \in \mathbb{C}$, we have*

$$\chi_{k,\nu}(-m) = (-1)^m G_{m,\nu}^{(k)}.$$

Proof. Let $k \geqslant 1$ be an integer. For $s \in \mathbb{C}$, we define the function $\chi_k(s)$ as

$$\chi_{k,\nu}(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{z^{s-1}}{e_\nu(z)+1} \mathrm{Ei}_k\left(\log(1+z)\right) dz. \tag{2.7}$$

From (2.7), we note that

$$\begin{aligned} \chi_{k,\nu}(s) &= \frac{1}{\Gamma(s)} \int_0^\infty \frac{z^{s-1}}{e_\nu(z)+1} \mathrm{Ei}_k\left(\log(1+z)\right) dz \\ &= \frac{1}{\Gamma(s)} \int_0^1 \frac{z^{s-1}}{e_\nu(z)+1} \mathrm{Ei}_k\left(\log(1+z)\right) dz + \frac{1}{\Gamma(s)} \int_1^\infty \frac{z^{s-1}}{e_\nu(z)+1} \mathrm{Ei}_k\left(\log(1+z)\right) dz. \end{aligned} \tag{2.8}$$

The second integral converges absolutely for any $s \in \mathbb{C}$ and hence, the second term on the right hand side vanishes at non-positive integers. That is,

$$\lim_{s \to -m} \left| \frac{1}{\Gamma(s)} \int_1^\infty \frac{z^{s-1}}{e_\nu(z)+1} \mathrm{Ei}_k\left(\log(1+z)\right) dz \right| \leqslant \frac{1}{\Gamma(-m)} M = 0. \tag{2.9}$$

On the other hand, for $\Re(s) > 0$, the first integral in (2.9) can be written as

$$\frac{1}{\Gamma(s)} \sum_{r=0}^{\infty} \frac{G_{r,\lambda}^{(k)}}{r!} \frac{1}{s+r},$$

which defines an entire function of s. Thus, we may include that $\chi_k(s)$ can be continued to an entire function of s.

Further, from (2.8) and (2.9), we obtain

$$\begin{aligned}\chi_k(-m) &= \lim_{s\to -m}\frac{1}{\Gamma(s)}\int_0^1 \frac{z^{s-1}}{e_\nu(z)+1}\mathrm{Ei}_k\left(\log(1+z)\right)dz\\ &= \lim_{s\to -m}\frac{1}{\Gamma(s)}\int_0^1 z^{s-1}\sum_{r=0}^{\infty}\frac{G_r^{(k)}z^r}{r!}dz = \lim_{s\to -m}\frac{1}{\Gamma(s)}\sum_{r=0}^{\infty}\frac{G_r^{(k)}}{s+r}\frac{1}{r!}\\ &= \cdots+0+\cdots+0+\lim_{s\to -m}\frac{1}{\Gamma(s)}\frac{1}{s+m}\frac{G_m^{(k)}}{m!}+0+0+\cdots\\ &= \lim_{s\to -m}\frac{\left(\frac{\Gamma(1-s)\sin\pi s}{\pi}\right)}{s+m}\frac{G_m^{(k)}}{m!} = \Gamma(1+m)\cos(\pi m)\frac{G_m^{(k)}}{m!} = (-1)^m G_m^{(k)}.\end{aligned}$$

Thus, we complete the proof of this theorem. □

Theorem 2.6. *For $n \geqslant 0$ and $k \in \mathbb{Z}$, we have*

$$G_{m,\lambda}^{(k)}S_2(n,m) = \sum_{i=0}^{n}\sum_{r=0}^{i}\sum_{j=0}^{r}\binom{n}{i}\binom{i}{r}G_{j,\lambda}S_2(r,j)B_{i-r}\frac{1}{(n-i+1)^k}.$$

Proof. By replacing t by e^t-1 in (2.1), we get

$$\begin{aligned}\frac{2}{e_\lambda(e^t-1)+1}\mathrm{Ei}_k(t) &= \sum_{m=0}^{\infty}G_{m,\lambda}^{(k)}\frac{1}{m!}(e^t-1)^m\\ &= \sum_{m=0}^{\infty}G_{m,\lambda}^{(k)}\sum_{n=m}^{\infty}S_2(n,m)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}G_{m,\lambda}^{(k)}S_2(n,m)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.10}$$

On the other hand, we see that

$$\begin{aligned}\frac{2}{e_\lambda(e^t-1)+1}\mathrm{Ei}_k(t) &= \frac{2(e^t-1)}{e_\lambda(e^t-1)+1}\frac{1}{e^t-1}\sum_{l=1}^{\infty}\frac{t^l}{(l-1)!l^k}\\ &= \frac{2(e^t-1)}{e_\lambda(e^t-1)+1}\frac{t}{e^t-1}\sum_{l=0}^{\infty}\frac{t^l}{(l+1)^k l!}\\ &= \sum_{j=0}^{\infty}G_{j,\lambda}\frac{1}{j!}(e^t-1)^j\sum_{i=1}^{\infty}B_i\frac{t^i}{i!}\sum_{l=0}^{\infty}\frac{t^l}{(l+1)^k l!}\\ &= \sum_{r=0}^{\infty}\sum_{j=0}^{r}G_{j,\lambda}S_2(r,j)\frac{t^r}{r!}\sum_{i=0}^{\infty}B_i\frac{t^i}{i!}\sum_{l=0}^{\infty}\frac{t^l}{(l+1)^k l!}\\ &= \sum_{m=0}^{\infty}\left(\sum_{r=0}^{i}\sum_{j=0}^{r}\binom{i}{r}G_{j,\lambda}S_2(r,j)B_{i-r}\right)\frac{t^i}{i!}\sum_{n=0}^{\infty}\frac{t^n}{(n+1)^k l!},\\ \text{L.H.S} &= \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\sum_{r=0}^{i}\sum_{j=0}^{r}\binom{n}{i}\binom{i}{r}G_{j,\lambda}S_2(r,j)B_{i-r}\frac{1}{(n-i+1)^k}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.11}$$

When equating (2.10) and (2.11) gives desired proof. □

3. Degenerate unipoly-Genocchi polynomials and numbers

In this section, we define the degenerate unipoly-Genocchi polynomials by using the unipoly function and derive some multifarious properties.

Let p be any arithmetic function which is a real or complex valued function defined on the set of positive integers $\mathbb{N}$. Kim and Kim [24] defined the unipoly function attached to polynomials $p(x)$ by

$$u_k(x|p)=\sum_{n=1}^{\infty}\frac{p(n)}{n^k}x^n,\quad (k\in\mathbb{Z}).$$

Moreover,

$$u_k(x|1)=\sum_{n=1}^{\infty}\frac{x^n}{n^k}=\mathrm{Li}_k(x),\quad \text{(see [12])},$$

is the ordinary polylogaritm function.

Now, we define the degenerate unipoly-Genocchi polynomials attached to polynomials $p(x)$ by

$$\frac{2}{e_\lambda(t)+1}u_k\left(\log(1+t)|p\right)e_\lambda^x(t)=\sum_{n=0}^{\infty}G_{n,\lambda,p}^{(k)}(x)\frac{t^n}{n!}. \tag{3.1}$$

In case when $x=0$, $G_{n,\lambda,p}^{(k)}=G_{n,\lambda,p}^{(k)}(0)$ are called the degenerate unipoly-Genocchi numbers attached to p. If we take $p(n)=\frac{1}{\Gamma(n)}$, then we have

$$\sum_{n=0}^{\infty}G_{n,\frac{1}{\Gamma}}^{(k)}(x)\frac{t^n}{n!}=\frac{2}{e_\lambda(t)+1}e_\lambda^x(t)u_k\left(\log(1+t)|\frac{1}{\Gamma}\right)=\frac{2}{e_\lambda(t)+1}e_\lambda^x(t)\sum_{m=1}^{\infty}\frac{(\log(1+t))^m}{m^k(m-1)!}.$$

In particular, for $k=1$, we obtain

$$\sum_{n=0}^{\infty}G_{n,\lambda,\frac{1}{\Gamma}}^{(1)}(x)\frac{t^n}{n!}=\frac{2}{e_\lambda(t)+1}e_\lambda^x(t)\sum_{m=1}^{\infty}\frac{(\log(1+t))^m}{m!}=\frac{2t}{e_\lambda(t)+1}e_\lambda^x(t). \tag{3.2}$$

Therefore by (3.1) and (3.2), we have

$$G_{n,\lambda,\frac{1}{\Gamma}}^{(1)}(x)=G_{n,\lambda}(x),\quad (n\geqslant 0).$$

Theorem 3.1. *For $n\geqslant 0$ and $k\in\mathbb{Z}$, we have*

$$G_{n,\lambda,p}^{(k)}(x)=\sum_{l=0}^{n}\sum_{m=0}^{l}\binom{n}{l}\frac{p(m+1)(m+1)!}{(m+1)^k}\frac{S_1(l+1,m+1)}{l+1}G_{n-l,\lambda}(x).$$

Moreover,

$$G_{n,\lambda,\frac{1}{\Gamma}}^{(k)}(x)=\sum_{l=0}^{n}\sum_{m=0}^{l}\binom{n}{l}\frac{1}{(m+1)^{k-1}}\frac{S_1(l+1,m+1)}{l+1}G_{n-l,\lambda}(x).$$

Proof. From (3.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty}G_{n,\lambda,p}^{(k)}(x)\frac{t^n}{n!}&=\frac{2}{e_\lambda(t)+1}e_\lambda^x(t)u_k\left(\log(1+t)|p\right)\\&=\frac{2}{e_\lambda(t)+1}e_\lambda^x(t)\sum_{m=1}^{\infty}\frac{p(m)}{m^k}(\log(1+t))^m\end{aligned}$$

$$
\begin{aligned}
&= \frac{2}{e_\lambda(t)+1} e_\lambda^x(t) \sum_{m=0}^{\infty} \frac{p(m+1)}{(m+1)^k} (\log(1+t))^{m+1} \\
&= \frac{2}{e_\lambda(t)+1} e_\lambda^x(t) \sum_{m=0}^{\infty} \frac{p(m+1)(m+1)!}{(m+1)^k} \sum_{l=m+1}^{\infty} S_1(l,m+1) \frac{t^l}{l!} \\
&= \frac{2t}{e_\lambda(t)+1} e_\lambda^x(t) \sum_{m=0}^{\infty} \frac{p(m+1)(m+1)!}{(m+1)^k} \sum_{l=m}^{\infty} \frac{S_1(l+1,m+1)}{l+1} \frac{t^l}{l!} \\
&= \sum_{n=0}^{\infty} G_{n,\lambda}(x) \frac{t^n}{n!} \sum_{l=0}^{\infty} \left(\sum_{m=0}^{l} \frac{p(m+1)(m+1)!}{(m+1)^k} \frac{S_1(l+1,m+1)}{l+1} \right) \frac{t^l}{l!},
\end{aligned}
$$

$$
\text{L.H.S} = \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \sum_{m=0}^{l} \binom{n}{l} \frac{p(m+1)(m+1)!}{(m+1)^k} \frac{S_1(l+1,m+1)}{l+1} G_{n-l,\lambda}(x) \right) \frac{t^n}{n!}.
$$

Therefore, by comparing the coefficients of the same powers in t of above equation and (2.1), we obtain the desired result. □

Theorem 3.2. *For $n \geqslant 0$ and $k \in \mathbb{Z}$, we have*

$$
G_{n,\lambda,p}^{(k)}(x) = \sum_{m=0}^{n} \binom{n}{m} G_{m,\lambda,p}^{(k)}(x)_{n-m,\lambda}.
$$

Proof. Recalling from (3.1) that

$$
\begin{aligned}
\sum_{n=0}^{\infty} G_{n,\lambda,p}^{(k)}(x) \frac{t^n}{n!} &= \frac{2}{e_\lambda(t)+1} u_k(\log(1+t)|p) e_\lambda^x(t) \\
&= \frac{2u_k\left(\log(1+t)|p\right)}{e_\lambda(t)+1} \sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!} \\
&= \sum_{m=0}^{\infty} G_{m,\lambda,p}^{(k)} \frac{t^m}{m!} \sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!}, \\
\text{L.H.S} &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{m} G_{m,\lambda,p}^{(k)}(x)_{n-m,\lambda} \right) \frac{t^n}{n!}.
\end{aligned}
$$

By comparing the coefficients of $\frac{t^n}{n!}$, we complete the proof. □

Theorem 3.3. *For $n \geqslant 0$ and $k \in \mathbb{Z}$, we have*

$$
G_{n,\lambda,p}^{(k)} = \sum_{l=0}^{n} \sum_{m=0}^{l} \sum_{j=0}^{n-l} \binom{n-l}{j} \binom{n}{l} D_{n-j-l} G_{j,\lambda} \frac{p(m+1)m!}{(m+1)^k} S_1(l,m).
$$

Proof. It is proved by using (3.1) that

$$
\begin{aligned}
\sum_{n=0}^{\infty} G_{n,\lambda,p}^{(k)} \frac{t^n}{n!} &= \frac{2}{e_\lambda(t)+1} u_k\left(\log(1+t)|p\right) \\
&= \frac{2}{e_\lambda(t)+1} \sum_{m=1}^{\infty} \frac{p(m)}{m^k} (\log(1+t))^m \\
&= \frac{2t}{e_\lambda(t)+1} \frac{\log(1+t)}{t} \sum_{m=0}^{\infty} \frac{p(m+1)m!}{(m+1)^k} \frac{(\log(1+t))^m}{m!}
\end{aligned}
$$

$$= \sum_{n=0}^{\infty} D_n \frac{t^n}{n!} \sum_{j=0}^{\infty} G_{j,\lambda} \frac{t^j}{j!} \sum_{m=0}^{\infty} \frac{p(m+1)m!}{(m+1)^k} \sum_{l=m}^{\infty} S_{1,\lambda}(l,m) \frac{t^l}{l!}$$

$$= \sum_{n=0}^{\infty} D_n \frac{t^n}{n!} \sum_{j=0}^{\infty} G_{j,\lambda} \frac{t^j}{j!} \sum_{l=0}^{\infty} \sum_{m=0}^{l} \frac{p(m+1)m!}{(m+1)^k} S_1(l,m) \frac{t^l}{l!}$$

$$= \sum_{n=0}^{\infty} \sum_{j=0}^{n} \binom{n}{j} D_{n-j} G_{j,\lambda} \frac{t^n}{n!} \sum_{l=0}^{\infty} \sum_{m=0}^{l} \frac{p(m+1)m!}{(m+1)^k} S_1(l,m) \frac{t^l}{l!}$$

$$\text{L.H.S} = \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \sum_{m=0}^{l} \sum_{j=0}^{n-l} \binom{n-l}{j} \binom{n}{l} D_{n-j-l} G_{j,\lambda} \frac{p(m+1)m!}{(m+1)^k} S_1(l,m) \right) \frac{t^n}{n!}.$$

Thus, we complete the proof of this theorem. □

4. Conclusion

In this paper, we studied the degenerate polyexponential-Genocchi numbers and polynomials and derived explicit expressions and some identity involving them. In more detail, we obtained an expression of the degenerate polyexponential-Genocchi polynomials in terms of the degenerate Bernoulli polynomials and Stirling numbers of the first kind. We also deduced an expression of the degenerate polyexponential-Genocchi numbers in terms of the degenerate Bernoulli numbers and values of higher-order Bernoulli polynomials at zero. Also, we derived an identity involving the degenerate polyexponential-Bernoulli numbers, degenerate Stirling numbers of the second kind, and the degenerate Genocchi numbers. In the last section, we defined degenerate unipoly-Genocchi polynomials attached to arithmetic function by using the modified polyexponential function and obtained the identity degenerate unipoly-Genocchi numbers and polynomials in terms of Stirlings numbers of the first kind and Daehee numbers and Bernoulli numbers.

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