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Essentiality and fixed point results for Eilenberg-Montgomery type maps

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Abstract

In this paper we establish topological transversality theorems so in particular general Leray-Schauder type alternatives and general Furi-Pera type results for Eilenberg-Montgomery type maps.

Keywords: Eilenberg-Montgomery maps, essential maps, Leray-Schauder alternatives, continuation theory.

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1. Introduction

Coincidence theory in a general setting for acyclic maps was initiated by O'Regan in [9]. This paper is motivated by a fixed point result of Eilenberg and Montgomery [1] where they considered maps of the form fF where f is a continuous single valued map and F is an acyclic map. In this paper we present a general continuation theory for Eilenberg-Montgomery type maps. Even though some of the results presented here could be modified from the results of O'Regan [9, 10] (Φ replaced by f^{-1} there) however we feel it is more natural to construct this theory from a well known fixed point result. In particular we base our theory on a result of Gorniewicz [5, 8] since maps of Eilenberg-Montgomery type are admissible with respect to Gorniewicz. In this paper we present general Granas type topological transversality theorems [6, 7, 10], general Leray-Schauder type alternatives [3, 7] and general Furi-Pera type results [4] for Eilenberg-Montgomery type maps.

Now we describe the maps considered in this paper. Let H be the Čech homology functor with compact carriers and coefficients in the field of rational numbers K from the category of Hausdorff topological spaces and continuous maps to the category of graded vector spaces and linear maps of degree zero. Thus $H(X) = \{H_q(X)\}$ (here X is a Hausdorff topological space) is a graded vector space, $H_q(X)$ being the q-dimensional Čech homology group with compact carriers of X. For a continuous map $f : X \to X$, $H(f)$ is the induced linear map $f_\star = \{f_{\star q}\}$, where $f_{\star q} : H_q(X) \to H_q(X)$. A space X is acyclic if X is nonempty, $H_q(X) = 0$ for every $q \geqslant 1$, and $H_0(X) \approx K$.

Let X, Y and Γ be Hausdorff topological spaces. A continuous single valued map $p : \Gamma \to X$ is called a Vietoris map (written $p : \Gamma \Rightarrow X$) if the following two conditions are satisfied:

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(i) for each $x \in X$, the set $p^{-1}(x)$ is acyclic;
(ii) p is a perfect map, i.e., p is closed and for every $x \in X$ the set $p^{-1}(x)$ is nonempty and compact.

Let $\phi : X \to Y$ be a multi-valued map (note for each $x \in X$ we assume $\phi(x)$ is a nonempty subset of Y). A pair (p, q) of single valued continuous maps of the form $X \overset{p}{\leftarrow} \Gamma \overset{q}{\rightarrow} Y$ is called a selected pair of ϕ (written $(p, q) \subset \phi$) if the following two conditions hold:

(i) p is a Vietoris map;
(ii) $q(p^{-1}(x)) \subset \phi(x)$ for any $x \in X$.

Now we define the admissible maps of Gorniewicz [5]. A upper semicontinuous map $\phi : X \to Y$ with closed values is said to be admissible (and we write $\phi \in Ad(X, Y)$) provided there exists a selected pair (p, q) of ϕ. An example of an admissible map is an acyclic map. A upper semicontinuous map $\phi : X \to K(Y)$ is said to be an acyclic map; here $K(Y)$ denotes the family of nonempty, acyclic, compact subsets of Y.

By a space we mean a Hausdorff topological space. Let Q be a class of topological spaces. A space Y is an extension space for Q (written $Y \in ES(Q)$ if for all $X \in Q$ and all $K \subseteq X$ closed in X, any continuous function $f_0 : K \to Y$ extends to a continuous function $f : X \to Y$.

Now we recall the following fixed point result in the literature [5, 8].

Theorem 1.1. *Let $X \in ES(compact)$ and $\Psi \in Ad(X, X)$ a compact map. Then there exists a $x \in X$ with $x \in \Psi(x)$.*

2. Continuation theory

Let E be a completely regular topological space and U an open subset of E. In this section we fix a continuous single valued map $f : E \to E$.

Definition 2.1. We say $F \in A(\overline{U}, E)$ if $F : \overline{U} \to K(E)$ is a upper semi-continuous compact map; here $K(E)$ denotes the family of nonempty, compact, acyclic subsets of E and $\overline{U}$ denotes the closure of U in E.

Definition 2.2. We say $F \in A_{\partial U}(\overline{U}, E)$ if $F \in A(\overline{U}, E)$ with $x \notin f(F(x))$ for $x \in \partial U$; here ∂U denotes the boundary of U in E.

Definition 2.3. Let $F, G \in A_{\partial U}(\overline{U}, E)$. We say $F \cong G$ in $A_{\partial U}(\overline{U}, E)$ if there exists a upper semicontinuous compact map $H : \overline{U} \times [0, 1] \to 2^E$ with $H_t \in A(\overline{U}, E)$ for each $t \in (0, 1)$, $x \notin f(H_t(x))$ for $x \in \partial U$ and $t \in (0, 1)$ (here $H_t(x) = H(x, t)$), $H_0 = F$ and $H_1 = G$.

Remark 2.4. Note that $\cong$ in $A_{\partial U}(\overline{U}, E)$ is an equivalence relation.

Next we present the notion of an EM-essential map.

Definition 2.5. We say $F \in A_{\partial U}(\overline{U}, E)$ is EM-essential in $A_{\partial U}(\overline{U}, E)$ if for any map $J \in A_{\partial U}(\overline{U}, E)$ with $J|_{\partial U} = F|_{\partial U}$ there exists a $x \in U$ with $x \in f(J(x))$.

Theorem 2.6. *Let E be a completely regular topological space and U an open subset of E. Let $F \in A_{\partial U}(\overline{U}, E)$ and suppose $G \in A_{\partial U}(\overline{U}, E)$ is EM-essential in $A_{\partial U}(\overline{U}, E)$. Also suppose*

$$\text{for any map } J \in A_{\partial U}(\overline{U}, E) \text{ with } J|_{\partial U} = F|_{\partial U}, \text{ we have } G \cong J \text{ in } A_{\partial U}(\overline{U}, E). \tag{2.1}$$

Then F is EM-essential in $A_{\partial U}(\overline{U}, E)$.

Proof. Let $J \in A_{\partial U}(\overline{U}, E)$ with $J|_{\partial U} = F|_{\partial U}$. From (2.1) there exists a upper semi-continuous compact map $H^J : \overline{U} \times [0, 1] \to 2^E$ with $H^J_t \in A(\overline{U}, E)$ for each $t \in (0, 1)$, $x \notin f(H^J_t(x))$ for $x \in \partial U$ and $t \in (0, 1)$ (here $H^J_t(x) = H^J(x, t)$), $H^J_0 = G$ and $H^J_1 = J$. Let

$$K = \left\{ x \in \overline{U} : x \in f(H^J_t(x)) \text{ for some } t \in [0, 1] \right\}, \quad \text{and} \quad D = \left\{ (x, t) \in \overline{U} \times [0, 1] : x \in f(H^J_t(x)) \right\}.$$

Now $D \neq \emptyset$ (since G is EM-essential in $A_{\partial U}(\overline{U}, E)$) and D is closed (note f is continuous and H^J is upper semi-continuous) and so compact (note H^J is a compact map). Let $\pi : \overline{U} \times [0,1] \to \overline{U}$ be the projection. Now $K = \pi(D)$ is closed (see Kuratowski's theorem [2]) and so in fact compact (recall projections are continuous). Also note $K \cap \partial U = \emptyset$ (since $x \notin f(H^J_t(x))$ for $x \in \partial U$ and $t \in (0,1)$) so since E is Tychonoff there exists a continuous map $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(K) = 1$. Let $R(x) = H^J(x, \mu(x)) = H^J_{\mu(x)}(x)$ for $x \in \overline{U}$. Now $R \in A_{\partial U}(\overline{U}, E)$ with $R|_{\partial U} = G|_{\partial U}$ (note if $x \in \partial U$ then $R(x) = H^J(x, 0) = G(x)$). Now since G is EM-essential in $A_{\partial U}(\overline{U}, E)$ there exists a $x \in U$ with $x \in f(R(x))$, i.e., $x \in f(H^J_{\mu(x)}(x))$. Thus $x \in K$ so $\mu(x) = 1$ and as a result $x \in f(H^J_1(x)) = f(J(x))$ and we are finished. □

We now present the topological transversality theorem for $A_{\partial U}(\overline{U}, E)$ maps. To do this we need an extra assumption (which will be discussed after the proof of our next result):

$$\text{if } F, G \in A_{\partial U}(\overline{U}, E) \text{ with } F|_{\partial U} = G|_{\partial U}, \text{ then } F \cong G \text{ in } A_{\partial U}(\overline{U}, E). \tag{2.2}$$

Theorem 2.7. *Let E be a completely regular topological space and U an open subset of E. Suppose (2.2) holds. Let F and G be two maps in $A_{\partial U}(\overline{U}, E)$ with $F \cong G$ in $A_{\partial U}(\overline{U}, E)$. Now F is EM-essential in $A_{\partial U}(\overline{U}, E)$ if and only if G is EM-essential in $A_{\partial U}(\overline{U}, E)$.*

Proof. Assume G is EM-essential in $A_{\partial U}(\overline{U}, E)$. Let $J \in A_{\partial U}(\overline{U}, E)$ with $J|_{\partial U} = F|_{\partial U}$. From (2.2) we have $J \cong F$ in $A_{\partial U}(\overline{U}, E)$ and since $F \cong G$ in $A_{\partial U}(\overline{U}, E)$ we have $G \cong J$ in $A_{\partial U}(\overline{U}, E)$, i.e., (2.1) holds. Now Theorem 2.6 guarantees that F is EM-essential in $A_{\partial U}(\overline{U}, E)$. Similarly if F is EM-essential in $A_{\partial U}(\overline{U}, E)$ then G is EM-essential in $A_{\partial U}(\overline{U}, E)$. □

Now we discuss (2.2). Let E be a topological (Hausdorff) vector space and U an open convex subset of E. Suppose

$$\text{there exists a retraction } r : \overline{U} \to \partial U, \tag{2.3}$$

(note (2.3) holds if E is an infinite dimensional Banach space). We now show that (2.2) holds. To see this let r be as in (2.3) and $F, G \in A_{\partial U}(\overline{U}, E)$ with $F|_{\partial U} = G|_{\partial U}$. Let $F^\star(x) = F(r(x))$ for $x \in \overline{U}$. Note $F^\star(x) = G(r(x))$, $x \in \overline{U}$ since $F|_{\partial U} = G|_{\partial U}$. Take

$$\Lambda(x, \lambda) = G(2\lambda r(x) + (1 - 2\lambda)x) = G \circ j(x, \lambda) \text{ for } (x, \lambda) \in \overline{U} \times \left[0, \frac{1}{2}\right];$$

here $j : \overline{U} \times [0, \frac{1}{2}] \to \overline{U}$ (note $\overline{U}$ is convex) is given by $j(x, \lambda) = 2\lambda r(x) + (1 - 2\lambda)x$. Notice $G \cong F^\star$ in $A_{\partial U}(\overline{U}, E)$; note $\Lambda : \overline{U} \times [0, \frac{1}{2}] \to 2^E$ is a upper semi-continuous compact map and for fixed $x \in \overline{U}$ and $t \in [0, \frac{1}{2}]$, note $\Lambda_t(x)$ has acyclic values and note $x \notin f(\Lambda_t(x))$ for $x \in \partial U$ and $t \in [0, \frac{1}{2}]$ since if $x \in \partial U$ and $t \in [0, \frac{1}{2}]$, then $r(x) = x$ so $\Lambda_t(x) = G(x)$ and $f(\Lambda_t(x)) = f(G(x))$. Similarly if $\Theta(x, \lambda) = F((2 - 2\lambda)r(x) + (2\lambda - 1)x)$ for $(x, \lambda) \in \overline{U} \times [\frac{1}{2}, 1]$, then $F^\star \cong F$ in $A_{\partial U}(\overline{U}, E)$. Thus (2.2) holds.

To establish Leray-Schauder type alternatives first we present an example of a EM-essential in $A_{\partial U}(\overline{U}, E)$ map.

Theorem 2.8. *Let E be a locally convex metrizable topological vector space, U an open subset of E, and $f(0) \in U$. Then the zero map is EM-essential in $A_{\partial U}(\overline{U}, E)$.*

Proof. Let $G \in A_{\partial U}(\overline{U}, E)$ with $J|_{\partial U} = \{0\}|_{\partial U}$. We must show there exists a $x \in U$ with $x \in f(G(x))$. Let

$$\Psi(x) = \begin{cases} G(x), & x \in \overline{U}, \\ \{0\}, & x \in E \backslash \overline{U}. \end{cases}$$

Now $\Psi \in A(E, E)$ (a map $\theta \in A(E, E)$ if $\theta : E \to K(E)$ is a upper semi-continuous compact map) and so $f\Psi$ is an admissible compact map. Now Theorem 1.1 (note every locally convex metrizable topological vector space is an AR) guarantees that there exists a $x \in E$ with $x \in f(\Psi(x))$. If $x \in E \backslash U$, then $x = f(0)$, a contradiction since $f(0) \in U$. Thus $x \in U$ so $x \in f(G(x))$. □

Remark 2.9. Let E be a locally convex metrizable topological vector space, U an open subset of E, $f(0) \in U$, $F \in A_{\partial U}(\overline{U}, E)$ and $x \notin f(tF(x))$ for $x \in \partial U$ and $t \in (0,1)$. Then one homotopy in $A_{\partial U}(\overline{U}, E)$ from F to 0 (i.e., so $0 \cong F$ in $A_{\partial U}(\overline{U}, E)$) is $H(x,t) = tF(x)$ for $t \in [0,1]$ and $x \in \overline{U}$. To see this note $H : \overline{U} \times [0,1] \to 2^E$ is a upper semi-continuous compact map and note for a fixed $t \in [0,1]$ and a fixed $x \in \overline{U}$, then $H_t(x)$ is acyclic valued (recall homeomorphic spaces have isomorphic homology groups) so $H_t \in A_{\partial U}(\overline{U}, E)$. Finally $H_0 = 0$ and $H_1 = F$ so $0 \cong F$ in $A_{\partial U}(\overline{U}, E)$.

Theorem 2.10. *Let E be a locally convex metrizable topological vector space, U an open subset of E, $F \in A_{\partial U}(\overline{U}, E)$, $f(0) \in U$ and $x \notin f(tF(x))$ for $x \in \partial U$ and $t \in (0,1)$. Then F is EM-essential in $A_{\partial U}(\overline{U}, E)$ (so in particular there exists a $x \in U$ with $x \in f(F(x))$).*

Proof. From Theorem 2.8 we know that the zero map is EM-essential in $A_{\partial U}(\overline{U}, E)$. We will apply Theorem 2.6 to show F is EM-essential in $A_{\partial U}(\overline{U}, E)$. Note topological vector spaces are completely regular so we need only to show (2.1) holds with $G = 0$. Consider any map $J \in A_{\partial U}(\overline{U}, E)$ with $J|_{\partial U} = F|_{\partial U}$. Now let $H^J(x,t) = tJ(x)$ and similar to Remark 2.9 note $J \cong 0$ in $A_{\partial U}(\overline{U}, E)$ (note if $x \in \partial U$ and $t \in (0,1)$, then since $J|_{\partial U} = F|_{\partial U}$ we have $f(tJ(x)) = f(tF(x))$). Thus (2.1) holds. □

Remark 2.11. Theorem 2.10 gives a strong conclusion, namely F is EM-essential in $A_{\partial U}(\overline{U}, E)$. The usual conclusion in a Leray-Schauder type alternative is that there exists a $x \in U$ with $x \in f(F(x))$. We note that this can be proved directly without any reference to essential maps. Let

$$K = \{x \in \overline{U} : x \in f(tF(x)) \text{ for some } t \in [0,1]\}.$$

Note $K \neq \emptyset$ (take $t = 0$ and $x = f(0)$) is compact and $K \cap \partial U = \emptyset$ (since $x \notin f(tF(x))$ for $x \in \partial U$ and $t \in (0,1)$) so there exists a continuous map $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(K) = 1$. Let $N : E \to 2^E$ be given by

$$N(x) = \begin{cases} \mu(x)F(x), & x \in \overline{U}, \\ \{0\}, & x \in E \backslash \overline{U}. \end{cases}$$

Now $N \in A(E, E)$ so fN is an admissible compact map. Then Theorem 1.1 guarantees that there exists a $x \in E$ with $x \in f(N(x))$. If $x \in E \backslash U$, then $x = f(0)$, a contradiction since $f(0) \in U$. Thus $x \in U$ so $x \in f(\mu(x)F(x))$ and as a result $x \in K$. Thus $\mu(x) = 1$ and so $x \in f(F(x))$.

Now we prove a Furi-Pera type result. Here E will be a locally convex metrizable topological vector space and Q a closed convex subset of E. In our next result we assume $\partial Q = Q$ (the case when $\text{int}(Q) \neq \emptyset$ is also easily handled; see Remark 2.13).

Theorem 2.12. *Let E be a locally convex metrizable topological vector space, Q a closed convex subset of E, $\partial Q = Q$, $F \in A(Q, E)$ (i.e., $F : Q \to K(E)$ a upper semi-continuous, compact map) and $f : E \to E$ a continuous single valued map with $f(0) \in Q$. In addition assume*

$$\begin{cases} \text{if } \{(x_j, \lambda_j)\}_{j=1}^{\infty} \text{ is a sequence in } \partial Q \times [0,1] \text{ convergingto } (x, \lambda) \text{ with } x \in f(\lambda F(x)) \text{ and } 0 \leqslant \lambda < 1, \\ \text{then } \{f(\lambda_j F(x_j))\} \subseteq Q \text{ for } j \text{ sufficiently large.} \end{cases} \tag{2.4}$$

Then there exists a $x \in Q$ with $x \in f(F(x))$.

Proof. From Dugundji's theorem we know there exists a retraction $r : E \to Q$. Let

$$\Omega = \{x \in E : x \in f(F(r(x))\}.$$

Note $\Omega \neq \emptyset$ from Theorem 1.1 (note fFr is a compact admissible map) and Ω is compact. We claim $\Omega \cap Q \neq \emptyset$. To show this we argue by contradiction. Suppose $\Omega \cap Q = \emptyset$. Then since Ω is compact and Q is closed there exists a $\delta > 0$ with $\text{dist}(Q, \Omega) > \delta$. Choose $m \in \{1, 2, \ldots\}$ with $1 < \delta m$ and let

$$U_i = \left\{x \in E : d(x, Q) < \frac{1}{i}\right\} \text{ for } i \in \{m, m+1, \ldots\};$$

here d is the metric associated with E. Fix $i \in \{m, m+1, \dots\}$. Since $\operatorname{dist}(Q, \Omega) > \delta$ we see that $\Omega \cap \overline{U_i} = \emptyset$. Now Remark 2.11 (note Fr has acyclic values so Fr is a compact acyclic map and $f(0) \in Q \subseteq U_i$) guarantees that there exists $\lambda_i \in (0,1)$ and $y_i \in \partial U_i$ with $y_i \in f(\lambda_i Fr(y_i))$. Since $y_i \in \partial U_i$ we have

$$\{f(\lambda_i Fr(y_i))\} \not\subseteq Q \text{ for } i \in \{m, m+1, \dots\}. \tag{2.5}$$

Let

$$D = \{x \in E : x \in f(\lambda Fr(x)) \text{ for some } \lambda \in [0,1]\}.$$

Now $D \neq \emptyset$ (see Theorem 1.1 and take $\lambda = 1$) and D is compact. This together with

$$d(y_j, Q) = \frac{1}{j} \text{ and } |j_j| \leqslant 1 \text{ for } j \in \{m, m+1, \dots\}$$

implies that we may assume without loss of generality that $\lambda_j \to \lambda^\star \in [0,1]$ and $y_j \to y^\star \in \partial Q$. In addition since f and r are continuous, F is upper semicontinuous and $y_j \in f(\lambda_j Fr(y_j))$, we have $y^\star \in f(\lambda^\star Fr(y^\star))$. Thus, since $r(y^\star) = y^\star$, we have $y^\star \in f(\lambda^\star F y^\star)$. If $\lambda^\star = 1$, then $y^\star \in f(Fy^\star)(= f(Fr(y^\star)))$, which contradicts $\Omega \cap Q = \emptyset$. Thus $0 \leqslant \lambda^\star < 1$. Now (2.4) with $x_j = r(y_j)$ (note $y_j \in \partial U_j$ and $r(y_j) \in \partial Q$) and $x = y^\star = r(y^\star)$ and $y^\star \in f(\lambda^\star F(y^\star))$ implies

$$\{f(\lambda_j F x_j)\} \subseteq Q \text{ for } j \text{ sufficiently large.}$$

This contradicts (2.5). Thus $\Omega \cap Q \neq \emptyset$ so there exists a $x \in Q$ with $x \in f(Fr(x)) = f(Fx)$. □

Remark 2.13. In Theorem 2.12 we assumed $\partial Q = Q$. However this is easily removed since if $\operatorname{int}(Q) \neq \emptyset$ (assume without loss of generality that $0 \in \operatorname{int}(Q)$), then one can take the retraction $r : E \to Q$ as

$$r(x) = \frac{x}{\max\{1, \mu(x)\}} \text{ for } x \in E,$$

where μ is the Minkowski functional on Q (i.e., $\mu(x) = \inf\{\alpha > 0 : x \in \alpha Q\}$. Note $r(z) \in \partial Q$ if $z \in E \backslash Q$. The argument in Theorem 2.12 now remains the same.

However when $0 \in \operatorname{int}(Q)$, condition (2.4) can be linked to $x \notin f(\lambda F(x))$ for $x \in \partial U$ and $\lambda \in (0,1)$ in Theorem 2.10 (Remark 2.11); here $U = \operatorname{int}(Q)$. For simplicity take $f = i$ (identity), $U = \operatorname{int} Q$ and let $F : \overline{U} \to E$ be a continuous single valued compact map with $x \neq Fx$ for $x \in \partial U$. Now we <u>claim</u> if (2.4) holds (i.e., if $\{(x_j, \lambda_j)\}_{j=1}^{\infty}$ is a sequence in $\partial Q \times [0,1]$ converging to (x, λ) with $x = \lambda F x$ and $0 \leqslant \lambda < 1$, then $\lambda_j F(x_j) \in Q$ for j sufficiently large), then if $x \in \partial U$ and $\lambda \in (0,1)$, then $x \neq \lambda F x$. Suppose the claim is false. Then there exists a $x \in \partial U$ and $\lambda \in (0,1)$ with $x = \lambda F x$. Let $\{(x_j, \lambda_j)\}_{j=1}^{\infty}$ be a sequence in $\partial Q \times [0,1]$ with $x_j = x$, $\lambda_j \to \lambda$ and $\lambda_j > \lambda$. Now (2.4) implies that $\lambda_j F x \in Q$ for j sufficiently large. However

$$\mu(\lambda_j F x) = \mu\left(\frac{\lambda_j}{\lambda} \lambda F x\right) = \mu\left(\frac{\lambda_j}{\lambda} x\right) = \frac{\lambda_j}{\lambda} \mu(x) = \frac{\lambda_j}{\lambda} > 1$$

so $\lambda_j F x \notin Q$, a contradiction. Thus $x \neq \lambda F x$ for $x \in \partial U$ and $\lambda \in (0,1)$. Theorem 2.10 guarantees that F has a fixed point in U.

Remark 2.14. Note one can choose d to be a translational invariant metric associated with E so each U_i ($i \in \{1, 2, \dots\}$) in the proof of Theorem 2.12 could be convex.

To take (2.2) into account in the topological transversality theorem one could replace the definition of EM-essential in $A_{\partial U}(\overline{U}, E)$ with the following definition.

Definition 2.15. We say $F \in A_{\partial U}(\overline{U}, E)$ is EM-essential in $A_{\partial U}(\overline{U}, E)$ if for any map $J \in A_{\partial U}(\overline{U}, E)$ with $J|_{\partial U} = F|_{\partial U}$ and $J \cong F$ in $A_{\partial U}(\overline{U}, E)$ there exists a $x \in U$ with $x \in f(J(x))$.

Theorem 2.16. *Let E be a completely regular topological space and U an open subset of E. Let $F \in A_{\partial U}(\overline{U}, E)$ and suppose $G \in A_{\partial U}(\overline{U}, E)$ is EM-essential in $A_{\partial U}(\overline{U}, E)$ (Definition 2.15). Also suppose*

$$\text{for any map } J \in A_{\partial U}(\overline{U}, E) \text{ with } J|_{\partial U} = F|_{\partial U}, \text{and } J \cong F \text{ in } A_{\partial U}(\overline{U}, E), \text{ we have } G \cong J, \text{in } A_{\partial U}(\overline{U}, E). \quad (2.6)$$

Then F is EM-essential in $A_{\partial U}(\overline{U}, E)$ (Definition 2.15).

Proof. Let $J \in A_{\partial U}(\overline{U}, E)$ with $J|_{\partial U} = F|_{\partial U}$ and $J \cong F$ in $A_{\partial U}(\overline{U}, E)$. From (2.1) there exists a upper semicontinuous compact map $H^J : \overline{U} \times [0,1] \to 2^E$ with $H^J_t \in A(\overline{U}, E)$ for each $t \in (0,1)$, $x \notin f(H^J_t(x))$ for $x \in \partial U$ and $t \in (0,1)$ (here $H^J_t(x) = H^J(x,t)$), $H^J_0 = G$ and $H^J_1 = J$. Let

$$K = \left\{x \in \overline{U} : x \in f(H^J_t(x)) \text{ for some } t \in [0,1]\right\}.$$

Now $K \neq \emptyset$ is compact, $K \cap \partial U = \emptyset$ so there exists a continuous map $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(K) = 1$. Let $R(x) = H^J_{\mu(x)}(x)$ for $x \in \overline{U}$. Now $R \in A_{\partial U}(\overline{U}, E)$ with $R|_{\partial U} = G|_{\partial U}$. We now claim $G \cong R$ in $A_{\partial U}(\overline{U}, E)$. If the claim is true, then since G is EM-essential in $A_{\partial U}(\overline{U}, E)$ (Definition 2.15) there exists a $x \in U$ with $x \in f(R(x))$, i.e., $x \in f(H^J_{\mu(x)}(x))$. Thus $x \in K$ so $\mu(x) = 1$ and as a result $x \in f(H^J_1(x)) = f(J(x))$ and we are finished. It remains to prove the claim. To see this let $Q : \overline{U} \times [0,1] \to 2^E$ be given by $Q(x,t) = H^J(x, t\mu(x)) = H^J_{t\mu(x)}(x)$ and note $Q : \overline{U} \times [0,1] \to 2^E$ is a upper semicontinuous compact map with $Q_s = H^J_{s\mu(.)} \in A(\overline{U}, E)$ for each $s \in [0,1]$ and $x \notin f(Q_t(x))$ for $x \in \partial U$ and $t \in (0,1)$ (note if $x \in \partial U$ and $t \in (0,1)$, then $Q_t(x) = H^J_{t\mu(x)}(x) = H^J_t(x)$ since $x \in K$). □

Theorem 2.17. *Let E be a completely regular topological space and U an open subset of E. Let F and G be two maps in $A_{\partial U}(\overline{U}, E)$ with $F \cong G$ in $A_{\partial U}(\overline{U}, E)$. Now F is EM-essential in $A_{\partial U}(\overline{U}, E)$ (Definition 2.15) if and only if G is EM-essential in $A_{\partial U}(\overline{U}, E)$ (Definition 2.15).*

Proof. Assume G is EM-essential in $A_{\partial U}(\overline{U}, E)$ (Definition 2.15). Let $J \in A_{\partial U}(\overline{U}, E)$ with $J|_{\partial U} = F|_{\partial U}$ and $J \cong F$ in $A_{\partial U}(\overline{U}, E)$. Now since $F \cong G$ in $A_{\partial U}(\overline{U}, E)$ we have $G \cong J$ in $A_{\partial U}(\overline{U}, E)$, i.e., (2.6) holds. Now Theorem 2.16 guarantees that F is EM-essential in $A_{\partial U}(\overline{U}, E)$ (Definition 2.15). Similarly if F is EM-essential in $A_{\partial U}(\overline{U}, E)$ (Definition 2.15), then G is EM-essential in $A_{\partial U}(\overline{U}, E)$ (Definition 2.15). □

Now we present a general Leray-Schauder type result.

Theorem 2.18. *Let E be a completely regular topological space, U an open subset of E, $u_0 \in E$ with $f(u_0) \in U$, and $F \in A_{\partial U}(\overline{U}, E)$. Suppose the following:*

$$\text{for any } \Phi \in A(E,E) \text{ with } \Phi \cong \{u_0\} \text{ in } A(E,E), \text{there exists a } z \in E \text{ with } z \in f(\Phi(z)). \quad (2.7)$$

Finally suppose $F \cong \{u_0\}$ in $A_{\partial U}(\overline{U}, E)$. Then F is EM-essential in $A_{\partial U}(\overline{U}, E)$ (Definition 2.15).

Remark 2.19. We say $\Phi \cong \{u_0\}$ in $A(E,E)$ if there exists a upper semicontinuous, compact map $R : E \times [0,1] \to 2^E$ with $R_t \in A(E,E)$ for each $t \in [0,1]$, $R_0 = \Phi$ and $R_1 = \{u_0\}$.

Proof. Let $J(x) = \{u_0\}$ for $x \in E$. We show J is EM-essential in $A_{\partial U}(\overline{U}, E)$ (Definition 2.15) so, then F is EM-essential in $A_{\partial U}(\overline{U}, E)$ (Definition 2.15) from Theorem 2.17. Let $G \in A_{\partial U}(\overline{U}, E)$ with $G|_{\partial U} = J|_{\partial U}$ and $G \cong J$ in $A_{\partial U}(\overline{U}, E)$. We must show there exists a $x \in U$ with $x \in f(G(x))$. Since $G \cong J$ in $A_{\partial U}(\overline{U}, E)$ there exists a upper semicontinuous, compact map $\Psi : \overline{U} \times [0,1] \to 2^E$ with $\Psi_t \in A_{\partial U}(\overline{U}, E)$ for each $t \in [0,1]$, $\Psi_0 = J$ and $\Psi_1 = G$. Let

$$\Omega = \left\{x \in \overline{U} : x \in f(\Psi_t(x)) \text{ for some } t \in [0,1]\right\}.$$

Note $\Omega \neq \emptyset$ (take $t = 0$ and $x = f(u_0)$) is compact and $\Omega \cap (E \backslash U) = \emptyset$ (note $\Psi_t \in A_{\partial U}(\overline{U}, E)$ for $t \in [0,1]$ so $x \notin f(\Psi_t(x))$ for $x \in \partial U$ and $t \in (0,1)$). Now there exists a continuous map $\sigma : E \to [0,1]$ with $\sigma(\Omega) = 1$ and $\sigma(E \backslash U) = 0$. Define $\Theta : E \times [0,1] \to 2^E$ by

$$\Theta(x,t) = \begin{cases} \Psi(x, t\sigma(x)), & x \in \overline{U}, \\ \{u_0\}, & x \in E \backslash U. \end{cases}$$

Note $\Theta : E \times [0,1] \to 2^E$ is an upper semicontinuous, compact map with $\Theta_t \in A(E,E)$ for each $t \in [0,1]$, so as a result $\Theta_1 \cong \Theta_0 = J$ in $A(E,E)$. Now (2.7) guarantees that there exists a $x \in E$ with $x \in f(\Theta_1(x))$. If $x \in E \backslash \overline{U}$, then $x = f(u_0)$, a contradiction since $f(u_0) \in U$. Consequently $x \in U$ so $x \in f(\Psi(x, \sigma(x))) = f(\Psi_{\sigma(x)}(x))$ and as a result $x \in \Omega$ which implies $\sigma(x) = 1$ and so $x \in f(\Psi_1(x)) = f(G(x))$. □

Theorem 2.20. *Let E be a (metrizable) ANR, U an open subset of E, $u_0 \in E$ with $f(u_0) \in U$, $F \in A_{\partial U}(\overline{U}, E)$ and suppose $F \cong \{u_0\}$ in $A_{\partial U}(\overline{U}, E)$. Then F is EM-essential in $A_{\partial U}(\overline{U}, E)$ (Definition 2.15).*

Proof. The result follows from Theorem 2.18 once we show (2.7). Let $\Phi \in A(E,E)$ with $\Phi \cong \{u_0\}$ in $A(E,E)$. Then there exists a upper semicontinuous, compact map $R : E \times [0,1] \to 2^E$ with $R_t \in A(E,E)$ for each $t \in [0,1]$,$R_1 = \Phi$ and $R_0 = \{u_0\}$. Note E can be regarded as a closed subset of a normed space X (see the Arens-Eells theorem). Since $E \in ANR$ there is an open neighborhood V of E in X and a retraction (continuous) $r : \overline{V} \to E$. Let $\lambda : X \to [0,1]$ be a continuous function with $\lambda(X \backslash V) = 0$ and $\lambda(E) = 1$ and let

$$Q(x) = \begin{cases} R(r(x), \lambda(x)), & x \in \overline{V}, \\ \{u_0\}, & x \in X \backslash V. \end{cases}$$

(note if $x \in \partial V$, then $Q(x) = R(r(x), 0) = R_0(r(x)) = \{u_0\}$). Also note $Q \to 2^X$ is a upper semi-continuous, compact map and for fixed $x \in X$ note $Q(x)$ is acyclic valued, so $Q \in A(X,X)$. Thus fQ is an admissible compact map so Theorem 1.1 guarantees that there exists a $x_0 \in X$ with $x_0 \in f(Q(x_0))$. If $x_0 \in X \backslash V$, then $x_0 = f(u_0)$, a contradiction since $f(u_0) \in U \subseteq E \subseteq V$. If $x_0 \in \overline{V} \backslash E$, then since $Q : X \to 2^E$ (note $R : E \times [0,1] \to 2^E$) and since $x_0 \in f(Q(x_0))$ one has $x_0 \in E$, a contradiction. Thus $x_0 \in E$ and so $r(x_0) = x_0$, $\lambda(x_0) = 1$ and as a result $x_0 \in f(R(x_0, 1)) = f(\Phi(x_0))$, i.e., (2.7) holds. □

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