J. Math. Computer Sci., 22 (2021), 399–411

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

Direct product of finite anti fuzzy normal sub-rings over non-associative rings

Nasreen Kausar[a,*], Mohammad Munir[b], Salahuddin[c], Zuhairi Baharudin[d], Badar Ul Islam[e]

[a]Department of Mathematics and Statistics, University of Agriculture, Faisalabad, Pakistan.

[b]Department of Mathematics, Government Postgraduate College, Abbottabad, Pakistan.

[c]Department of Mathematics, Jazan University, Jazan, Kingdom of Saudi Arabia.

[d]Department of Electrical and Electronics Engineering, University of Teknologi Petronas, Seri Iskandar, Malaysia.

[e]Department of Electrical Engineering, NFC IEFR FSD, Pakistan.

Abstract

In this paper, we define the concept of direct product of finite anti fuzzy normal sub-rings over non-associative and noncommutative rings LA-rings and investigate the some fundamental properties of direct product of anti fuzzy normal sub-rings.

Keywords: Direct product of fuzzy sets, anti fuzzy LA-sub-rings, anti fuzzy normal LA-sub-rings

2020 MSC: 03F55, 08A72, 20N25

1. Introduction

In 1972, a generalization of commutative semigroups was established by Kazim et al. [21]. In ternary commutative law: $abc = cba$, they introduced the braces on the left side of this law and explored a new pseudo associative law, that is: $(ab)c = (cb)a$. This law $(ab)c = (cb)a$ called the left invertive law. A groupoid S is said to be a left almost semigroup (abbreviated as LA-semigroup) if it satisfies the left invertive law: $(ab)c = (cb)a$.

In [12] (resp. [7]), a groupoid S is said to be medial (resp. paramedial) if $(ab)(cd) = (ac)(bd)$ (resp. $(ab)(cd) = (db)(ca)$). In [21], an LA-semigroup is medial, but in general an LA-semigroup needs not to be paramedial. Every LA-semigroup with left identity is paramedial by Protic et al. [28] and also satisfies $a(bc) = b(ac)$, $(ab)(cd) = (dc)(ba)$.

Kamran [14], extended the notion of LA-semigroup to the left almost group (LA-group). An LA-semigroup G is said to be a left almost group, if there exists left identity $e \in G$ such that $ea = a$ for all $a \in G$, and for every $a \in G$ there exists $b \in G$ such that $ba = e$.

*Corresponding author

Email addresses: kausar.nasreen57@gmail.com (Nasreen Kausar), dr.mohammadmunir@gmail.com (Mohammad Munir)

doi: 10.22436/jmcs.022.04.08

Received: 2020-07-12 *Revised:* 2020-08-08 *Accepted:* 2020-08-17

Shah et al. [31], discussed the left almost ring (LA-ring) of finitely nonzero functions which is a generalization of commutative semigroup ring. By a left almost ring, we mean a non-empty set R with at least two elements such that $(R,+)$ is an LA-group, $(R,\cdot)$ is an LA-semigroup, both left and right distributive laws hold. For example, from a commutative ring $(R,+,\cdot)$, we can always obtain an LA-ring $(R,\oplus,\cdot)$ by defining for all $a,b \in R$, $a \oplus b = b - a$ and $a \cdot b$ is same as in the ring. In fact an LA-ring is non-associative and non-commutative ring.

A non-empty subset A of an LA-ring R is called an LA-sub-ring of R if $a - b$ and $ab \in A$ for all $a,b \in A$. A is called a left (resp. right) ideal of R if $(A,+)$ is an LA-group and $RA \subseteq A$ (resp. $AR \subseteq A$). A is called an ideal of R if it is both a left ideal and a right ideal of R.

First time, the concept of fuzzy set introduced by Zadeh in his classical paper [34]. This concept has provided a useful mathematical tool for describing the behavior of systems that are too complex to admit precise mathematical analysis by classical methods and tools. Extensive applications of fuzzy set theory have been found in various fields such as artificial intelligence, computer science, management science, expert systems, finite state machines, Languages, robotics, coding theory and others.

Liu [24], introduced the concept of fuzzy sub-rings and fuzzy ideals of a ring. Many authors have explored the theory of fuzzy rings (for example [8–10, 22, 25, 26, 33]). Gupta et al. [10], gave the idea of intrinsic product of fuzzy subsets of a ring. Kuroki [22], characterized regular (intra-regular, both regular and intra-regular) rings in terms of fuzzy left (right, quasi, bi-) ideals.

Biswas [6], introduced the concept of anti fuzzy subgroups and studied the basic properties of groups in terms of such ideals. Hong and Jun [11], modified the Biswas idea and applied it into BCK-algebra. Akram and Dar defined anti fuzzy left h-ideals of a hemiring and discussed the basic properties of a hemiring in [4].

Sherwood [32], introduced the concept of product of fuzzy subgroups. After this, further study on this concept continued by Osman [1, 2] and Ray [29]. Zaid [3], gave the idea of normal fuzzy subgroups.

Shal et al. [30], originated the studied of intuitionistic fuzzy normal LA-sub-rings over left almost-ring. Islam et al. [17] initiated the intuitionistics fuzzy ideals with thresholds $(\alpha,\beta]$ in left almost ring. Javaid et al. [18], also studied the left almost rings by fuzzy ideals. Waqar et al. [19], studied the left almost rings by using the intuitionistic fuzzy bi-ideals. Kausar et al. [15], explored the direct product of finite intuitionistic anti fuzzy normal LA-sub-rings over LA-rings. Waqar et al. [20], investigated the direct product of finite fuzzy normal LA-sub-rings on Left Almost-rings.

Recently Munir et al. [27], discussed *on the prime fuzzy m-bi ideals in semigroups*.

In this paper, we define the concept of direct product of anti fuzzy normal LA-sub-rings. In Section 2, we investigate the some basic properties of anti fuzzy normal LA-sub-rings of an LA-ring R. Section 3, we define the direct product of fuzzy subsets μ_1,μ_2 of LA-rings R_1,R_2, respectively and investigate the some elementary properties of direct product of anti fuzzy normal LA-sub-rings of an LA-ring $R_1 \times R_2$. In Section 4, we define the direct product of fuzzy subsets $\mu_1,\mu_2,\dots,\mu_n$ of LA-rings $R_1,R_2,\dots,R_n$, respectively and examine the some fundamental properties of direct product of anti fuzzy normal LA-sub-rings of an LA-ring $R_1 \times R_2 \times \cdots \times R_n$. Specifically we show the following.

1. Let A and B are two LA-sub-rings of an LA-ring R. Then $A \cap B$ is an LA-sub-ring of R if and only if the anti characteristic function χ_Z^C of $Z = A \cap B$ is an anti fuzzy normal LA-sub-ring of R.
2. Let $X = A \times B$ and $Y = C \times D$ are two LA-sub-rings of an LA-ring $R_1 \times R_2$. Then $X \cap Y$ is an LA-sub-ring of $R_1 \times R_2$ if and only if the anti characteristic function χ_Z^C of $Z = X \cap Y$ is an anti fuzzy normal LA-sub-ring of $R_1 \times R_2$.
3. Let $A = A_1 \times A_2 \times \cdots \times A_n$ and $B = B_1 \times B_2 \times \cdots \times B_n$ are two LA-sub-rings of an LA-ring $R_1 \times R_2 \times \cdots \times R_n$. Then $A \cap B$ is an LA-sub-ring of $R_1 \times R_2 \times \cdots \times R_n$ if and only if the anti characteristic function χ_Z^C of $Z = A \cap B$ is an anti fuzzy normal LA-sub-ring of $R_1 \times R_2 \times \cdots \times R_n$.

2. Anti fuzzy normal LA-sub-rings

In this section, we investigate the some basic properties of anti fuzzy normal LA-sub-rings of an

LA-ring R.

By a fuzzy subset μ of an LA-ring R, we mean a function $\mu : R \to [0,1]$ and the complement of μ is denoted by μ', is a fuzzy subset of R defined by $\mu'(x) = 1 - \mu(x)$ for all $x \in R$.

A fuzzy subset μ of an LA-ring R is an anti fuzzy LA-sub-ring of R if $\mu(x-y) \leqslant \max\{\mu(x), \mu(y)\}$ and $\mu(xy) \leqslant \max\{\mu(x), \mu(y)\}$ for all $x, y \in R$.

An anti fuzzy LA-sub-ring of an LA-ring R is said to be an anti fuzzy normal LA-sub-ring of R if $\mu(xy) = \mu(yx)$ for all $x, y \in R$.

Let A is a non-empty subset of an LA-ring R. The anti characteristic function of A is denoted by χ_A^C and defined by

$$\chi_A^C : R \to [0,1] \mid x \to \chi_A^C(x) = \begin{cases} 0, & \text{if } x \in A, \\ 1, & \text{if } x \notin A. \end{cases}$$

Lemma 2.1. *Let A is a non-empty subset of an LA-ring R. Then A is an LA-sub-ring of R if and only if the anti characteristic function χ_A^C of A is an anti fuzzy normal LA-sub-ring of R.*

Proof. Let A is an LA-sub-ring of R and $a, b \in R$. If $a, b \in A$, then by definition of anti characteristic function $\chi_A^C(a) = 0 = \chi_A^C(b)$. Since $a - b, ab \in A$, A is an LA-sub-ring of R. This implies that

$$\chi_A^C(a-b) = 0 = 0 \vee 0 = \chi_A^C(a) \vee \chi_A^C(b) \quad \text{and} \quad \chi_A^C(ab) = 0 = 0 \vee 0 = \chi_A^C(a) \vee \chi_A^C(b).$$

Thus $\chi_A^C(a-b) \leqslant \max\{\chi_A^C(a), \chi_A^C(b)\}$ and $\chi_A^C(ab) \leqslant \max\{\chi_A^C(a), \chi_A^C(b)\}$. Since ab and $ba \in A$, so $\chi_A^C(ab) = 0 = \chi_A^C(ba)$, i.e., $\chi_A^C(ab) = \chi_A^C(ba)$. Similarly we have

$$\chi_A^C(a-b) \leqslant \max\{\chi_A^C(a), \chi_A^C(b)\}, \quad \chi_A^C(ab) \leqslant \max\{\chi_A^C(a), \chi_A^C(b)\}, \quad \chi_A^C(ab) = \chi_A^C(ba),$$

when $a, b \notin A$. Hence the anti characteristic function χ_A^C of A is an anti fuzzy normal LA-sub-ring of R.

Conversely, suppose that the anti characteristic function χ_A^C of A is an anti fuzzy normal LA-sub-ring of R. Let $a, b \in A$, then by definition $\chi_A^C(a) = 0 = \chi_A^C(b)$. By our supposition

$$\chi_A^C(a-b) \leqslant \chi_A^C(a) \vee \chi_A^C(b) = 0 \vee 0 = 0 \quad \text{and} \quad \chi_A^C(ab) \leqslant \chi_A^C(a) \vee \chi_A^C(b) = 0 \vee 0 = 0.$$

Thus $\chi_A^C(a-b) = 0 = \chi_A^C(ab)$, i.e., $a - b, ab \in A$. Hence A is an LA-sub-ring of R. □

Lemma 2.2. *If A and B are two LA-sub-rings of an LA-ring R, then their intersection $A \cap B$ is also an LA-sub-ring of R.*

Proof. Straight forward. □

Proposition 2.3. *Let A and B are two LA-sub-rings of an LA-ring R. Then $A \cap B$ is an LA-sub-ring of R if and only if the anti characteristic function χ_Z^C of $Z = A \cap B$ is an anti fuzzy normal LA-sub-ring of R.*

Proof. Let $Z = A \cap B$ is an LA-sub-ring of R and $a, b \in R$. If $a, b \in Z = A \cap B$, then by definition of anti characteristic function $\chi_Z^C(a) = 0 = \chi_Z^C(b)$. Since $a - b, ab \in A, B$, A and B are LA-sub-rings of R. This implies that

$$\chi_Z^C(a-b) = 0 = 0 \vee 0 = \chi_Z^C(a) \vee \chi_Z^C(b) \quad \text{and} \quad \chi_Z^C(ab) = 0 = 0 \vee 0 = \chi_Z^C(a) \vee \chi_Z^C(b).$$

Thus $\chi_Z^C(a-b) \leqslant \max\{\chi_Z^C(a), \chi_Z^C(b)\}$ and $\chi_Z^C(ab) \leqslant \max\{\chi_Z^C(a), \chi_Z^C(b)\}$. As ab and $ba \in Z$, so $\chi_Z^C(ab) = 0 = \chi_Z^C(ba)$, i.e., $\chi_Z^C(ab) = \chi_Z^C(ba)$. Similarly we have

$$\chi_Z^C(a-b) \leqslant \max\{\chi_Z^C(a), \chi_Z^C(b)\}, \quad \chi_Z^C(ab) \leqslant \max\{\chi_Z^C(a), \chi_Z^C(b)\}, \quad \chi_Z^C(ab) = \chi_Z^C(ba),$$

when $a, b \notin Z$. Hence the anti characteristic function χ_Z^C of Z is an anti fuzzy normal LA-sub-ring of R.

Conversely, assume that the anti characteristic function χ_Z^C of $Z = A \cap B$ is an anti fuzzy normal LA-sub-ring of R. Let $a, b \in Z = A \cap B$, this means that $\chi_Z^C(a) = 0 = \chi_Z^C(b)$. By our assumption

$$\chi_Z^C(a-b) \leqslant \chi_Z^C(a) \vee \chi_Z^C(b) = 0 \vee 0 = 0 \quad \text{and} \quad \chi_Z^C(ab) \leqslant \chi_Z^C(a) \vee \chi._Z^C(b) = 0 \vee 0 = 0.$$

Thus $\chi_Z^C(a-b) = 0 = \chi_Z^C(ab)$, i.e., $a - b$ and $ab \in Z$. Hence Z is an LA-sub-ring of R. □

Corollary 2.4. *Let $\{A_i\}_{i\in I}$ is a family of LA-sub-rings of an LA-ring R, then $A = \cap A_i$ is an LA-sub-ring of R if and only if the anti characteristic function χ_A^C of $A = \cap A_i$ is an anti fuzzy normal LA-sub-ring of R.*

Lemma 2.5. *If μ and γ are two anti fuzzy normal LA-sub-rings of an LA-ring R, then their union $\mu \cup \gamma$ is also an anti fuzzy normal LA-sub-ring of R.*

Proof. Let μ and γ are two anti fuzzy normal LA-sub-rings of an LA-ring R. We have to show that $\beta = \mu \cup \gamma$ is also an anti fuzzy normal LA-sub-ring of R. Now

$$\begin{aligned}\beta(z_1 - z_2) &= (\mu \cup \gamma)(z_1 - z_2)\\ &= \max\{\mu(z_1 - z_2), \gamma(z_1 - z_2)\}\\ &\leqslant \{\{\mu(z_1) \vee \mu(z_2)\} \vee \{\gamma(z_1) \vee \gamma(z_2)\}\}\\ &= \{\mu(z_1) \vee \{\mu(z_2) \vee \gamma(z_1)\} \vee \gamma(z_2)\}\\ &= \{\mu(z_1) \vee \{\gamma(z_1) \vee \mu(z_2)\} \vee \gamma(z_2)\}\\ &= \{\{\mu(z_1) \vee \gamma(z_1)\} \vee \{\mu(z_2) \vee \gamma(z_2)\}\}\\ &= \max\{(\mu \cup \gamma)(z_1), (\mu \cup \gamma)(z_2)\}\\ &= \max\{\beta(z_1), \beta(z_2)\}.\\ &\Rightarrow \beta(z_1 - z_2) \leqslant \max\{\beta(z_1), \beta(z_2)\}.\end{aligned}$$

Similarly, we have $\beta(z_1 \circ z_2) \leqslant \max\{\beta(z_1), \beta(z_2)\}$. Thus β is an anti fuzzy LA-sub-ring of an LA-ring R. Now

$$\begin{aligned}\beta(z_1 \circ z_2) = (\mu \cup \gamma)(z_1 \circ z_2) &= \max\{\mu(z_1 \circ z_2), \gamma(z_1 \circ z_2)\}\\ &= \max\{\mu(z_2 \circ z_1), \gamma(z_2 \circ z_1)\} = (\mu \cup \gamma)(z_2 \circ z_1) = \beta(z_2 \circ z_1).\end{aligned}$$

Hence $\beta = \mu \cup \gamma$ is an anti fuzzy normal LA-sub-ring of R. □

Corollary 2.6. *If $\{\mu_i\}_{i\in I}$ is a family of anti fuzzy normal LA-sub-rings of an LA-ring R, then $\mu = \cup \mu_i$ is also an anti fuzzy normal LA-sub-ring of R.*

3. Direct product of anti fuzzy normal LA-sub-rings

In this section, we define the direct product of fuzzy subsets μ_1, μ_2 of LA-rings R_1, R_2, respectively and investigate the some elementary properties of direct product of anti fuzzy normal LA-sub-rings of an LA-ring $R_1 \times R_2$.

Let μ_1, μ_2 is fuzzy subsets of LA-rings R_1, R_2, respectively. The direct product of fuzzy subsets μ_1, μ_2 of LA-rings R_1, R_2, is denoted by $\mu_1 \times \mu_2$ and defined by $(\mu_1 \times \mu_2)(x_1, x_2) = \max\{\mu_1(x_1), \mu_2(x_2)\}$.

A fuzzy subset $\mu_1 \times \mu_2$ of an LA-ring $R_1 \times R_2$ is said to be an anti fuzzy LA-sub-ring of $R_1 \times R_2$ if

(1) $(\mu_1 \times \mu_2)(x - y) \leqslant \max\{\mu_1(x), \mu_2(y)\}$;
(2) $(\mu_1 \times \mu_2)(xy) \leqslant \max\{\mu_1(x), \mu_2(y)\}$ for all $x = (x_1, x_2), y = (y_1, y_2) \in R_1 \times R_2$.

An anti fuzzy LA-sub-ring of an LA-ring $R_1 \times R_2$ is said to be an anti fuzzy normal LA-sub-ring of $R_1 \times R_2$ if $(\mu_1 \times \mu_2)(xy) = (\mu_1 \times \mu_2)(yx)$ for all $x = (x_1, x_2), y = (y_1, y_2) \in R_1 \times R_2$.

Let $A \times B$ is a non-empty subset of an LA-ring $R_1 \times R_2$. The anti characteristic function of $A \times B$ is denoted by $\chi_{A\times B}^C$ and defined by

$$\chi_{A\times B}^C : R_1 \times R_2 \to [0,1] \mid x = (x_1, x_2) \to \chi_{A\times B}^C(x) = \begin{cases} 0, & \text{if } x \in A \times B,\\ 1, & \text{if } x \notin A \times B.\end{cases}$$

Lemma 3.1 ([30, Lemma 4.2]). *If A and B are LA-sub-rings of LA-rings R_1 and R_2, respectively, then $A \times B$ is an LA-sub-ring of an LA-ring $R_1 \times R_2$ under the same operations defined as in $R_1 \times R_2$.*

Lemma 3.2. *Let* A *and* B *are LA-sub-rings of LA-rings* R_1 *and* R_2*, respectively. Then* $A \times B$ *is an LA-sub-ring of an LA-ring* $R_1 \times R_2$ *if and only if the anti characteristic function* χ_Z^C *of* $Z = A \times B$ *is an anti fuzzy normal LA-sub-ring of* $R_1 \times R_2$.

Proof. Let $Z = A \times B$ is an LA-sub-ring of $R_1 \times R_2$ and $a = (a_1, a_2), b = (b_1, b_2) \in R_1 \times R_2$. If $a, b \in Z = A \times B$, then by definition of anti characteristic function $\chi_Z^C(a) = 0 = \chi_Z^C(b)$. Since $a - b$ and $ab \in Z$, Z is an LA-sub-ring of an LA-ring $R_1 \times R_2$. This implies that

$$\chi_Z^C(a-b) = 0 = 0 \vee 0 = \chi_Z^C(a) \vee \chi_Z^C(b) \quad \text{and} \quad \chi_Z^C(ab) = 0 = 0 \vee 0 = \chi_Z^C(a) \vee \chi_Z^C(b).$$

Thus $\chi_Z^C(a-b) \leqslant \max\{\chi_Z^C(a), \chi_Z^C(b)\}$ and $\chi_Z^C(ab) \leqslant \max\{\chi_Z^C(a), \chi_Z^C(b)\}$. Since ab and $ba \in Z$, so $\chi_Z^C(ab) = 0 = \chi_Z^C(ba)$, i.e., $\chi_Z^C(ab) = \chi_Z^C(ba)$. Similarly we have

$$\chi_Z^C(a-b) \leqslant \max\{\chi_Z^C(a), \chi_Z^C(b)\}, \qquad \chi_Z^C(ab) \leqslant \max\{\chi_Z^C(a), \chi_Z^C(b)\}, \qquad \chi_Z^C(ab) = \chi_Z^C(ba),$$

when $a, b \notin Z$. Hence the anti characteristic function χ_Z^C of $Z = A \times B$ is an anti fuzzy normal LA-sub-ring of $R_1 \times R_2$.

Conversely, suppose that the anti characteristic function χ_Z^C of $Z = A \times B$ is an anti fuzzy normal LA-sub-ring of $R_1 \times R_2$. We have to show that $Z = A \times B$ is an LA-sub-ring of $R_1 \times R_2$. Let $a, b \in Z$, where $a = (a_1, a_2)$ and $b = (b_1, b_2)$, $a_1, b_1 \in A$, $a_2, b_2 \in B$, this means that $\chi_Z^C(a) = 0 = \chi_Z^C(b)$. By our supposition

$$\chi_Z^C(a-b) \leqslant \chi_Z^C(a) \vee \chi_Z^C(b) = 0 \vee 0 = 0 \quad \text{and} \quad \chi_Z^C(ab) \leqslant \chi_Z^C(a) \vee \chi_Z^C(b) = 0 \vee 0 = 0.$$

Thus $\chi_Z^C(a-b) = 0 = \chi_Z^C(ab)$, i.e., $a - b$ and $ab \in Z$. Hence $Z = A \times B$ is an LA-sub-ring of $R_1 \times R_2$. □

Lemma 3.3. *If* $X = A \times B$ *and* $Y = C \times D$ *are two LA-sub-rings of an LA-ring* $R_1 \times R_2$*, then their intersection* $X \cap Y$ *is also an LA-sub-ring of* $R_1 \times R_2$.

Proof. Straight forward. □

Theorem 3.4. *Let* $X = A \times B$ *and* $Y = C \times D$ *are two LA-sub-rings of an LA-ring* $R_1 \times R_2$*. Then* $X \cap Y$ *is an LA-sub-ring of* $R_1 \times R_2$ *if and only if the anti characteristic function* χ_Z^C *of* $Z = X \cap Y$ *is an anti fuzzy normal LA-sub-ring of* $R_1 \times R_2$.

Proof. Let $Z = X \cap Y$ is an LA-sub-ring of an LA-ring $R_1 \times R_2$ and $a = (a_1, a_2), b = (b_1, b_2) \in R_1 \times R_2$. If $a, b \in Z = X \cap Y$, then by definition of anti characteristic function $\chi_Z^C(a) = 0 = \chi_Z^C(b)$. Since $a - b$ and $ab \in Z$, Z is an LA-sub-ring of $R_1 \times R_2$. This implies that

$$\chi_Z^C(a-b) = 0 = 0 \vee 0 = \chi_Z^C(a) \vee \chi_Z^C(b) \quad \text{and} \quad \chi_Z^C(ab) = 0 = 0 \vee 0 = \chi_Z^C(a) \vee \chi_Z^C(b).$$

Thus $\chi_Z^C(a-b) \leqslant \max\{\chi_Z^C(a), \chi_Z^C(b)\}$ and $\chi_Z^C(ab) \leqslant \max\{\chi_Z^C(a), \chi_Z^C(b)\}$. Since ab and $ba \in Z$, then by definition $\chi_Z^C(ab) = 0 = \chi_Z^C(ba)$, i.e., $\chi_Z^C(ab) = \chi_Z^C(ba)$. Similarly we have

$$\chi_Z^C(a-b) \leqslant \max\{\chi_Z^C(a), \chi_Z^C(b)\}, \qquad \chi_Z^C(ab) \leqslant \max\{\chi_Z^C(a), \chi_Z^C(b)\}, \qquad \chi_Z^C(ab) = \chi_Z^C(ba),$$

when $a, b \notin Z$. Hence the anti characteristic function χ_Z^C of Z is an anti fuzzy normal LA-sub-ring of $R_1 \times R_2$.

Conversely, assume that the anti characteristic function χ_Z^C of $Z = X \cap Y$ is an anti fuzzy normal LA-sub-ring of an LA-ring $R_1 \times R_2$. Let $a, b \in Z = X \cap Y$, this means that $\chi_Z^C(a) = 0 = \chi_Z^C(b)$. By our assumption

$$\chi_Z^C(a-b) \leqslant \chi_Z^C(a) \vee \chi_Z^C(b) = 0 \vee 0 = 0 \quad \text{and} \quad \chi_Z^C(ab) \leqslant \chi_Z^C(a) \vee \chi_Z^C(b) = 0 \vee 0 = 0.$$

Thus $\chi_Z^C(a-b) = 0 = \chi_Z^C(ab)$, i.e., $a - b$ and $ab \in Z$. Hence Z is an LA-sub-ring of an LA-ring $R_1 \times R_2$. □

Corollary 3.5. *Let* $\{C_i\}_{i\in I} = \{A_i \times B_i\}_{i\in I}$ *is a family of LA-sub-rings of an LA-ring* $R_1 \times R_2$*, then* $C = \cap C_i$ *is an LA-sub-ring of* $R_1 \times R_2$ *if and only if the anti characteristic function* χ_C^C *of* $C = \cap C_i$ *is an anti fuzzy normal LA-sub-ring of* $R_1 \times R_2$.

Lemma 3.6. *If* μ *and* γ *are anti fuzzy normal LA-sub-rings of LA-rings* R_1 *and* R_2*, respectively, then* $\mu \times \gamma$ *is an anti fuzzy normal LA-sub-ring of an LA-ring* $R_1 \times R_2$.

Proof. Let μ and γ are anti fuzzy normal LA-sub-rings of LA-ring R_1 and R_2, respectively. We have to show that $\beta = \mu \times \gamma$ is an anti fuzzy normal LA-sub-ring of an LA-ring $R_1 \times R_2$. Let $(a, b), (c, d) \in R_1 \times R_2$. Now

$$\begin{aligned}
\beta((a,b)-(c,d)) &= (\mu\times\gamma)(a-c, b-d) \\
&= \max\{\mu(a-c), \gamma(b-d)\} \\
&= \mu(a-c) \vee \gamma(b-d) \\
&\leqslant \{\mu(a)\vee\mu(c)\} \vee \{\gamma(b)\vee\gamma(d)\} \\
&= \mu(a) \vee \{\mu(c)\vee\gamma(b)\} \vee \gamma(d) \\
&= \mu(a) \vee \{\gamma(b)\vee\mu(c)\} \vee \gamma(d) \\
&= \{\mu(a)\vee\gamma(b)\} \vee \{\mu(c)\vee\gamma(d)\} \\
&= \max\{(\mu\times\gamma)(a,b), (\mu\times\gamma)(c,d)\} \\
&= \max\{\beta(a,b), \beta(c,d)\}. \\
&\Rightarrow \beta((a,b)-(c,d)) \leqslant \max\{\beta(a,b), \beta(c,d)\}.
\end{aligned}$$

Similarly, we have $\beta((a, b) \circ (c, d)) \leqslant \max\{\beta(a, b), \beta(c, d)\}$. Thus $\mu \times \gamma$ is an anti fuzzy LA-sub-ring of $R_1 \times R_2$. Now

$$\begin{aligned}
\beta((a,b)\circ(c,d)) = (\mu\times\gamma)(ac, bd) &= \max\{\mu(ac), \gamma(bd)\} \\
&= \max\{\mu(ca), \gamma(db)\} = (\mu\times\gamma)(ca, db) = \beta((c,d)\circ(a,b)).
\end{aligned}$$

Hence $\mu \times \gamma$ is an anti fuzzy normal LA-sub-ring of $R_1 \times R_2$. □

Proposition 3.7. *If* $\mu = \mu_1 \times \mu_2$ *and* $\gamma = \gamma_1 \times \gamma_2$ *are two anti fuzzy normal LA-sub-rings of an LA-ring* $R_1 \times R$*, then their union* $\beta = \mu \cup \gamma$ *is also an anti fuzzy normal LA-sub-ring of* $R_1 \times R_2$.

Proof. Let $\mu = \mu_1 \times \mu_2$ and $\gamma = \gamma_1 \times \gamma_2$ are two anti fuzzy normal LA-sub-rings of an LA-ring $R_1 \times R_2$. We have to show that $\beta = \mu \cup \gamma$ is also an anti fuzzy normal LA-sub-ring of $R_1 \times R_2$. Now

$$\begin{aligned}
\beta((z_1,z_2)-(z_3,z_4)) &= (\mu\cup\gamma)((z_1,z_2)-(z_3,z_4)) \\
&= \max\{\mu((z_1,z_2)-(z_3,z_4)), \gamma((z_1,z_2)-(z_3,z_4))\} \\
&\leqslant \{\{\mu(z_1,z_2)\vee\mu(z_3,z_4)\} \vee \{\gamma(z_1,z_2)\vee\gamma(z_3,z_4)\}\} \\
&= \{\mu(z_1,z_2) \vee \{\mu(z_3,z_4)\vee\gamma(z_1,z_2)\} \vee \gamma(z_3,z_4)\} \\
&= \{\mu(z_1,z_2) \vee \{\gamma(z_1,z_2)\vee\mu(z_3,z_4)\} \vee \gamma(z_3,z_4)\} \\
&= \{\{\mu(z_1,z_2)\vee\gamma(z_1,z_2)\} \vee \{\mu(z_3,z_4)\vee\gamma(z_3,z_4)\}\} \\
&= \max\{(\mu\cup\gamma)(z_1,z_2), (\mu\cup\gamma)(z_3,z_4)\} \\
&= \max\{\beta(z_1,z_2), \beta(z_3,z_4)\}. \\
&\Rightarrow \beta((z_1,z_2)-(z_3,z_4)) \leqslant \max\{\beta(z_1,z_2), \beta(z_3,z_4)\}.
\end{aligned}$$

Similarly, we have $\beta((z_1, z_2) \circ (z_3, z_4)) \leqslant \max\{\beta(z_1, z_2), \beta(z_3, z_4)\}$. Thus $\beta = \mu \cup \gamma$ is an anti fuzzy LA-sub-ring of an LA-ring $R_1 \times R_2$. Now

$$\beta((z_1,z_2)\circ(z_3,z_4)) = (\mu\cup\gamma)((z_1,z_2)\circ(z_3,z_4)) = \max\{\mu((z_1,z_2)\circ(z_3,z_4)), \gamma((z_1,z_2)\circ(z_3,z_4))\}$$

$$= \max\{\mu((z_3, z_4) \circ (z_1, z_2)), \gamma((z_3, z_4) \circ (z_1, z_2))\}$$
$$= (\mu \cup \gamma)((z_3, z_4) \circ (z_1, z_2)) = \beta((z_3, z_4) \circ (z_1, z_2)).$$

Hence $\beta = \mu \cup \gamma$ is an anti fuzzy normal LA-sub-ring of an LA-ring $R_1 \times R_2$. □

Corollary 3.8. *If $\{\beta_i\}_{i\in I} = \{\mu_i \times \gamma_i\}_{i\in I}$ is a family of anti fuzzy normal LA-sub-rings of an LA-ring $R_1 \times R_2$, then $\beta = \cup\beta_i$ is also an anti fuzzy normal LA-sub-ring of $R_1 \times R_2$.*

Theorem 3.9. *If $\mu = \mu_1 \times \mu_2$ and $\gamma = \gamma_1 \times \gamma_2$ are anti fuzzy normal LA-sub-rings of LA-rings $R' = R_1 \times R_2$ and $R'' = R_3 \times R_4$, respectively, then $\beta = \mu \times \gamma$ is an anti fuzzy normal LA-sub-ring of an LA-ring $R' \times R'' = (R_1 \times R_2) \times (R_3 \times R_4)$.*

Proof. Let $\mu = \mu_1 \times \mu_2$ and $\gamma = \gamma_1 \times \gamma_2$ are anti fuzzy normal LA-sub-rings of LA-rings $R' = R_1 \times R_2$ and $R'' = R_3 \times R_4$, respectively. We have to show that $\beta = \mu \times \gamma$ is an anti fuzzy normal LA-sub-ring of an LA-ring $R' \times R''$. Now

$$\begin{aligned}
\beta(((z_1, z_2), (z_3, z_4)) - ((z_5, z_6), (z_7, z_8))) &= (\mu \times \gamma)(((z_1, z_2), (z_3, z_4)) - ((z_5, z_6), (z_7, z_8)))\\
&= (\mu \times \gamma)(((z_1, z_2) - (z_5, z_6)), ((z_3, z_4) - (z_7, z_8)))\\
&= \max\{\mu((z_1, z_2) - (z_5, z_6)), \gamma((z_3, z_4) - (z_7, z_8))\}\\
&\leqslant \max\{(\mu(z_1, z_2) \vee \mu(z_5, z_6)), (\gamma(z_3, z_4) \vee \gamma(z_7, z_8))\}\\
&= \{(\mu(z_1, z_2) \vee \mu(z_5, z_6)) \vee (\gamma(z_3, z_4) \vee \gamma(z_7, z_8))\}\\
&= \{(\mu(z_1, z_2) \vee \gamma(z_3, z_4)) \vee (\mu(z_5, z_6) \vee \gamma(z_7, z_8))\}\\
&= \max\{(\mu(z_1, z_2) \vee \gamma(z_3, z_4)), (\mu(z_5, z_6) \vee \gamma(z_7, z_8))\}\\
&= \max\{(\mu \times \gamma)((z_1, z_2), (z_3, z_4)), (\mu \times \gamma)((z_5, z_6), (z_7, z_8))\}\\
&= \max\{\beta((z_1, z_2), (z_3, z_4)), \beta((z_5, z_6), (z_7, z_8))\}.
\end{aligned}$$

Similarly, we have

$$\beta(((z_1, z_2), (z_3, z_4)) \circ ((z_5, z_6), (z_7, z_8))) \leqslant \max\{\beta((z_1, z_2), (z_3, z_4)), \beta((z_5, z_6), (z_7, z_8))\}.$$

Thus $\beta = \mu \times \gamma$ is an anti fuzzy LA-sub-ring of an LA-ring $R\prime \times R''$. Now

$$\begin{aligned}
\beta(((z_1, z_2), (z_3, z_4)) \circ ((z_5, z_6), (z_7, z_8))) &= (\mu \times \gamma)(((z_1, z_2), (z_3, z_4)) \circ ((z_5, z_6), (z_7, z_8)))\\
&= (\mu \times \gamma)(((z_1, z_2) \circ (z_5, z_6)), ((z_3, z_4) \circ (z_7, z_8)))\\
&= \max\{\mu((z_1, z_2) \circ (z_5, z_6)), \gamma((z_3, z_4) \circ (z_7, z_8))\}\\
&= \max\{\mu((z_5, z_6) \circ (z_1, z_2)), \gamma((z_7, z_8) \circ (z_3, z_4))\}\\
&= (\mu \times \gamma)(((z_5, z_6) \circ (z_1, z_2)), ((z_7, z_8) \circ (z_3, z_4)))\\
&= (\mu \times \gamma)(((z_5, z_6), (z_7, z_8)) \circ ((z_1, z_2), (z_3, z_4)))\\
&= \beta(((z_5, z_6), (z_7, z_8)) \circ ((z_1, z_2), (z_3, z_4))).
\end{aligned}$$

Hence $\beta = \mu \times \gamma$ is an anti fuzzy normal LA-sub-ring of an LA-ring $R\prime \times R''$. □

Lemma 3.10. *Let μ and γ are fuzzy subsets of LA-rings R_1 and R_2 with left identities e_1 and e_2, respectively. If $\mu \times \gamma$ is an anti fuzzy LA-sub-ring of an LA-ring $R_1 \times R_2$, then at least one of the following two statements must hold.*

1. $\mu(x) \geqslant \gamma(e_2)$, *for all* $x \in R_1$;
2. $\mu(x) \geqslant \gamma(e_1)$, *for all* $x \in R_2$.

Proof. Let $\mu \times \gamma$ is an anti fuzzy LA-sub-ring of $R_1 \times R_2$. By contra-position, suppose that none of the statements 1 and 2 holds. Then we can find a and b in R_1 and R_2, respectively such that

$$\mu(a) \leqslant \gamma(e_2) \text{ and } \mu(b) \leqslant \gamma(e_1).$$

Thus we have

$$(\mu \times \gamma)(a, b) = \max\{\mu(a), \gamma(b)\} \leqslant \max\{\mu(e_1), \gamma(e_2)\} = (\mu \times \gamma)(e_1, e_2).$$

Therefore $\mu \times \gamma$ is not an anti fuzzy LA-sub-ring of $R_1 \times R_2$. Hence either $\mu(x) \geqslant \gamma(e_2)$ for all $x \in R_1$ or $\mu(x) \geqslant \gamma(e_1)$ for all $x \in R_2$. □

Lemma 3.11. *Let μ and γ are fuzzy subsets of LA-rings R_1 and R_2 with left identities e_1 and e_2, respectively and $\mu \times \gamma$ is an anti fuzzy normal LA-sub-ring of an LA-ring $R_1 \times R_2$. Then the following conditions are true.*

1. *If $\mu(x) \geqslant \gamma(e_2)$, for all $x \in R_1$, then μ is an anti fuzzy normal LA-sub-ring of R_1.*
2. *If $\mu(x) \geqslant \gamma(e_1)$, for all $x \in R_2$, then γ is an anti fuzzy normal LA-sub-ring of R_2.*

Proof. 1. Let $\mu(x) \geqslant \gamma(e_2)$ for all $x \in R_1$, and $y \in R_1$. We have to show that μ is an anti fuzzy normal LA-sub-ring of R_1. Now

$$\begin{aligned}
\mu(x - y) &= \mu(x + (-y)) \\
&= \max\{\mu(x + (-y)), \gamma(e_2 + (-e_2))\} \\
&= (\mu \times \gamma)(x + (-y), e_2 + (-e_2)) \\
&= (\mu \times \gamma)((x, e_2) + (-y, -e_2)) \\
&= (\mu \times \gamma)((x, e_2) - (y, e_2)) \\
&\leqslant (\mu \times \gamma)(x, e_2) \vee (\mu \times \gamma)(y, e_2) \\
&= \max\{\max\{\mu(x), \gamma(e_2)\}, \max\{\mu(y), \gamma(e_2)\}\} \\
&= \mu(x) \vee \mu(y),
\end{aligned}$$

and

$$\begin{aligned}
\mu(xy) &= \max\{\mu(xy), \gamma(e_2 e_2)\} \\
&= (\mu \times \gamma)(xy, e_2 e_2) \\
&= (\mu \times \gamma)((x, e_2) \circ (y, e_2)) \\
&\leqslant (\mu \times \gamma)(x, e_2) \vee \mu \times \gamma(y, e_2) \\
&= \max\{\max\{\mu(x), \gamma(e_2)\}, \max\{\mu(y), \gamma(e_2)\}\} \\
&= \mu(x) \vee \mu(y).
\end{aligned}$$

Thus μ is an anti fuzzy LA-sub-ring of R_1. Now

$$\begin{aligned}
\mu(xy) &= \max\{\mu(xy), \gamma(e_2 e_2)\} \\
&= (\mu \times \gamma)(xy, e_2 e_2) \\
&= (\mu \times \gamma)((x, e_2) \circ (y, e_2)) \\
&= (\mu \times \gamma)((y, e_2) \circ (x, e_2)) = (\mu \times \gamma)(yx, e_2 e_2) = \max\{\mu(yx), \gamma(e_2 e_2)\} = \mu(yx).
\end{aligned}$$

Hence μ is an anti fuzzy normal LA-sub-ring of R_1. 2 is same as 1. □

4. Direct product of finite anti fuzzy normal LA-sub-rings

In this section, we define the direct product of fuzzy subsets $\mu_1, \mu_2, \dots, \mu_n$ of LA-rings $R_1, R_2, \dots, R_n$, respectively and examine the some fundamental properties of direct product of anti fuzzy normal LA-sub-rings of an LA-ring $R_1 \times R_2 \times \cdots \times R_n$.

Let $\mu_1, \mu_2, \dots, \mu_n$ are fuzzy subsets of LA-rings $R_1, R_2, \dots, R_n$, respectively. The direct of fuzzy subsets $\mu_1, \mu_2, \dots, \mu_n$, is denoted by $\mu_1 \times \mu_2 \times \cdots \times \mu_n$ and defined by $(\mu_1 \times \mu_2 \times \cdots \times \mu_n)(x_1, x_2, \dots, x_n) = \max\{\mu_1(x_1), \mu_2(x_2), \dots, \mu_n(x_n)\}$.

A fuzzy subset $\mu_1 \times \mu_2 \times \cdots \times \mu_n$ of an LA-ring $R_1 \times R_2 \times \cdots \times R_n$ is said to be an anti fuzzy LA-sub-ring of $R_1 \times R_2 \times \cdots \times R_n$ if

1. $(\mu_1 \times \mu_2 \times \cdots \times \mu_n)(x - y) \leqslant \max\{(\mu_1 \times \mu_2 \times \cdots \times \mu_n)(x), (\mu_1 \times \mu_2 \times \cdots \times \mu_n)(y)\}$;
2. $(\mu_1 \times \mu_2 \times \cdots \times \mu_n)(xy) \leqslant \max\{(\mu_1 \times \mu_2 \times \cdots \times \mu_n)(x), (\mu_1 \times \mu_2 \times \cdots \times \mu_n)(y)\}$ for all $x = (x_1, x_2, \dots, x_n)$, $y = (y_1, y_2, \dots, y_n) \in R_1 \times R_2 \times \cdots \times R_n$.

An anti fuzzy LA-sub-ring of an LA-ring $R_1 \times R_2 \times \cdots \times R_n$ is said to be an anti fuzzy normal LA-sub-ring of $R_1 \times R_2 \times \cdots \times R_n$ if $(\mu_1 \times \mu_2 \times \cdots \times \mu_n)(xy) = (\mu_1 \times \mu_2 \times \cdots \times \mu_n)(yx)$ for all $x = (x_1, x_2, \dots, x_n)$, $y = (y_1, y_2, \dots, y_n) \in R_1 \times R_2 \times \cdots \times R_n$.

Let $A_1 \times A_2 \times \cdots \times A_n$ is a non-empty subset of an LA-ring $R = R_1 \times R_2 \times \cdots \times R_n$. The anti characteristic function of $A = A_1 \times A_2 \times \cdots \times A_n$ is denoted by χ_A^C and defined by

$$\chi_A^C : R \to [0,1] \mid x = (x_1, x_2, \dots, x_n) \to \chi_A^C(x) = \begin{cases} 0, & \text{if } x \in A, \\ 1, & \text{if } x \notin A. \end{cases}$$

Lemma 4.1. *If $A_1, A_2, \dots, A_n$ are LA-sub-rings of LA-rings $R_1, R_2, \dots, R_n$, respectively, then $A_1 \times A_2 \times \cdots \times A_n$ is an LA-sub-ring of an LA-ring $R_1 \times R_2 \times \cdots \times R_n$ under the same operations defined as in* [30].

Proof. Straight forward. □

Proposition 4.2. *Let $A_1, A_2, \dots, A_n$ are LA-sub-rings of LA-rings $R_1, R_2, \dots, R_n$, respectively. Then $A_1 \times A_2 \times \cdots \times A_n$ is an LA-sub-ring of an LA-ring $R_1 \times R_2 \times \cdots \times R_n$ if and only if the anti characteristic function χ_A^C of $A = A_1 \times A_2 \times \cdots \times A_n$ is an anti fuzzy normal LA-sub-ring of $R_1 \times R_2 \times \cdots \times R_n$.*

Proof. Let $A = A_1 \times A_2 \times \cdots \times A_n$ is an LA-sub-ring of $R_1 \times R_2 \times \cdots \times R_n$ and $a = (a_1, a_2, \dots, a_n), b = (b_1, b_2, \dots, b_n) \in R_1 \times R_2 \times \cdots \times R_n$. If $a, b \in A = A_1 \times A_2 \times \cdots \times A_n$, then by definition of anti characteristic function $\chi_A^C(a) = 0 = \chi_A^C(b)$. Since $a - b$ and $ab \in A$, A is an LA-sub-ring of $R_1 \times R_2 \times \cdots \times R_n$. This implies that

$$\chi_A^C(a - b) = 0 = 0 \vee 0 = \chi_A^C(a) \vee \chi_A^C(b) \quad \text{and} \quad \chi_A^C(ab) = 0 = 0 \vee 0 = \chi_A^C(a) \vee \chi_A^C(b).$$

Thus $\chi_A^C(a - b) \leqslant \max\{\chi_A^C(a), \chi_A^C(b)\}$ and $\chi_A^C(ab) \leqslant \max\{\chi_A^C(a), \chi_A^C(b)\}$. Since ab and $ba \in A$, so $\chi_A^C(ab) = 0 = \chi_A^C(ba)$, i.e., $\chi_A^C(ab) = \chi_A^C(ba)$. Similarly, we have

$$\chi_A^C(a - b) \leqslant \max\{\chi_A^C(a), \chi_A^C(b)\}, \qquad \chi_A^C(ab) \leqslant \max\{\chi_A^C(a), \chi_A^C(b)\}, \qquad \chi_A^C(ab) = \chi_A^C(ba),$$

when $a, b \notin A$. Hence the anti characteristic function χ_A^C of $A = A_1 \times A_2 \times \cdots \times A_n$ is an anti fuzzy normal LA-sub-ring of $R_1 \times R_2 \times \cdots \times R_n$.

Conversely, assume that the anti characteristic function χ_A^C of $A = A_1 \times A_2 \times \cdots \times A_n$ is an anti fuzzy normal LA-sub-ring of $R_1 \times R_2 \times \cdots \times R_n$. We have to show that $A = A_1 \times A_2 \times \cdots \times A_n$ is an LA-sub-ring of $R_1 \times R_2 \times \cdots \times R_n$. Let $a, b \in A$, where $a = (a_1, a, \dots, a_n)$ and $b = (b_1, b_2, \dots, b_n)$, this means that $\chi_A^C(a) = 0 = \chi_A^C(b)$. By our supposition

$$\chi_A^C(a - b) \leqslant \chi_A^C(a) \vee \chi_A^C(b) = 0 \vee 0 = 0 \quad \text{and} \quad \chi_A^C(ab) \leqslant \chi_A^C(a) \vee \chi_A^C(b) = 0 \vee 0 = 0.$$

Thus $\chi_A^C(a - b) = 0 = \chi_A^C(ab)$, i.e., $a - b$ and $ab \in A$. Hence $A = A_1 \times A_2 \times \cdots \times A_n$ is an LA-sub-ring of an LA-ring $R_1 \times R_2 \times \cdots \times R_n$. □

Lemma 4.3. *If* $A = A_1 \times A_2 \times \cdots \times A_n$ *and* $B = B_1 \times B_2 \times \cdots \times B_n$ *are two LA-sub-rings of an LA-ring* $R_1 \times R_2 \times \cdots \times R_n$, *then their intersection* $A \cap B$ *is also an LA-sub-ring of* $R_1 \times R_2 \times \cdots \times R_n$.

Proof. Straight forward. □

Theorem 4.4. *Let* $A = A_1 \times A_2 \times \cdots \times A_n$ *and* $B = B_1 \times B_2 \times \cdots \times B_n$ *are two LA-sub-rings of an LA-ring* $R_1 \times R_2 \times \cdots \times R_n$. *Then* $A \cap B$ *is an LA-sub-ring of* $R_1 \times R_2 \times \cdots \times R_n$ *if and only if the anti characteristic function* χ_Z^C *of* $Z = A \cap B$ *is an anti fuzzy normal LA-sub-ring of* $R_1 \times R_2 \times \cdots \times R_n$.

Proof. Let $Z = A \cap B$ is an LA-sub-ring of $R_1 \times R_2 \times \cdots \times R_n$ and $a = (a_1, a_2, \ldots, a_n), b = (b_1, b_1, \ldots, b_n) \in R_1 \times R_2 \times \cdots \times R_n$. If $a, b \in Z = A \cap B$, then by definition of anti characteristic function $\chi_Z^C(a) = 0 = \chi_Z^C(b)$. Since $a - b$ and $ab \in Z$. This implies that

$$\chi_Z^C(a-b) = 0 = 0 \vee 0 = \chi_Z^C(a) \vee \chi_Z^C(b) \quad \text{and} \quad \chi_Z^C(ab) = 0 = 0 \vee 0 = \chi_Z^C(a) \vee \chi_Z^C(b).$$

Thus $\chi_Z^C(a-b) \leqslant \max\{\chi_Z^C(a), \chi_Z^C(b)\}$ and $\chi_Z^C(ab) \leqslant \max\{\chi_Z^C(a), \chi_Z^C(b)\}$. As ab and $ba \in Z$, by definition $\chi_Z^C(ab) = 0 = \chi_Z^C(ba)$, i.e., $\chi_Z^C(ab) = \chi_Z^C(ba)$. Similarly, we have

$$\chi_Z^C(a-b) \leqslant \max\{\chi_Z^C(a), \chi_Z^C(b)\}, \qquad \chi_Z^C(ab) \leqslant \max\{\chi_Z^C(a), \chi_Z^C(b)\}, \qquad \chi_Z^C(ab) = \chi_Z^C(ba),$$

when $a, b \notin Z$. Hence the anti characteristic function χ_Z^C of Z is an anti fuzzy normal LA-sub-ring of $R_1 \times R_2 \times \cdots \times R_n$.

Conversely, assume that the anti characteristic function χ_Z^C of $Z = A \cap B$ is an anti fuzzy normal LA-sub-ring of $R_1 \times R_2 \times \cdots \times R_n$. Let $a, b \in Z = A \cap B$, this means that $\chi_Z^C(a) = 0 = \chi_Z^C(b)$. By our supposition

$$\chi_Z^C(a-b) \leqslant \chi_Z^C(a) \vee \chi_Z^C(b) = 0 \vee 0 = 0 \quad \text{and} \quad \chi_Z^C(ab) \leqslant \chi_Z^C(a) \vee \chi_Z^C(b) = 0 \vee 0 = 0.$$

Thus $\chi_Z^C(a-b) = 0 = \chi_Z^C(ab)$, i.e., $a - b$ and $ab \in Z$. Hence Z is an LA-sub-ring of $R_1 \times R_2 \times \cdots \times R_n$. □

Corollary 4.5. *Let* $\{A_i\}_{i \in I} = \{A_{i1} \times A_{i2} \times \cdots \times A_{in}\}_{i \in I}$ *is a family of LA-sub-rings of an LA-ring* $R_1 \times R_2 \times \cdots \times R_n$, *then* $A = \cap A_i$ *is an LA-sub-ring of* $R_1 \times R_2 \times \cdots \times R_n$ *if and only if the anti characteristic function* χ_A^C *of* $A = \cap A_i$ *is an anti fuzzy normal LA-sub-ring of* $R_1 \times R_2 \times \cdots \times R_n$.

Theorem 4.6. *If* $\mu = \mu_1 \times \mu_2 \times \cdots \times \mu_n$ *and* $\gamma = \gamma_1 \times \gamma_2 \times \cdots \times \gamma_n$ *are two anti fuzzy normal LA-sub-rings of an LA-ring* $R_1 \times R_2 \times \cdots \times R_n$, *then their union* $\beta = \mu \cup \gamma$ *is also an anti fuzzy normal LA-sub-ring of* $R_1 \times R_2 \times \cdots \times R_n$.

Proof. Let $\mu = \mu_1 \times \mu_2 \times \cdots \times \mu_n$ and $\gamma = \gamma_1 \times \gamma_2 \times \cdots \times \gamma_n$ are two anti fuzzy normal LA-sub-rings of an LA-ring $R_1 \times R_2 \times \cdots \times R_n$. We have to show that $\beta = \mu \cup \gamma$ is also an anti fuzzy normal LA-sub-ring of $R_1 \times R_2 \times \cdots \times R_n$. Let $z = (z_1, z_2, \ldots, z_n)$ and $w = (w_1, w_2, \ldots, w_n) \in R_1 \times R_2 \times \cdots \times R_n$. Now

$$\begin{aligned}
\beta(z-w) = (\mu \cup \gamma)(z-w) &= \max\{\mu(z-w), \gamma(z-w)\} \\
&\leqslant \{\{\mu(z) \vee \mu(w)\} \vee \{\gamma(z) \vee \gamma(w)\}\} \\
&= \{\mu(z) \vee \{\mu(w) \vee \gamma(z)\} \vee \gamma(w)\} \\
&= \{\mu(z) \vee \{\gamma(z) \vee \mu(w)\} \vee \gamma(w)\} \\
&= \{\{\mu(z) \vee \gamma(z)\} \vee \{\mu(w) \vee \gamma(w)\}\} \\
&= \max\{(\mu \cup \gamma)(z), (\mu \cup \gamma)(w)\} \\
&= \max\{\beta(z), \beta(w)\}.
\end{aligned}$$

Thus $\beta((z_1, z_2, \ldots, z_n) - (w_1, w_2, \ldots, w_n)) \leqslant \max\{\beta(z_1, z_2, \ldots, z_n), \beta(w_1, w_2, \ldots, w_n)\}$. Similarly, we have

$$\beta((z_1, z_2, \ldots, z_n) \circ (w_1, w_2, \ldots, w_n)) \leqslant \max\{\beta(z_1, z_2, \ldots, z_n), \beta(w_1, w_2, \ldots, w_n)\}.$$

Thus $\beta = \mu \cup \gamma$ is an anti fuzzy LA-sub-ring of an LA-ring $R_1 \times R_2 \times \cdots \times R_n$. Now

$$\begin{aligned}
\beta((z_1, z_2, \ldots, z_n) \circ (w_1, w_2, \ldots, w_n)) &= (\mu \cup \gamma)(z_1w_1, z_2w_2, \ldots, z_nw_n) \\
&= \max\{\mu(z_1w_1, z_2w_2, \ldots, z_nw_n), \gamma(z_1w_1, z_2w_2, \ldots, z_nw_n)\} \\
&= \max\{\mu(w_1z_1, w_2z_2, \ldots, w_nz_n), \gamma(w_1z_1, w_2z_2, \ldots, w_nz_n)\} \\
&= (\mu \cup \gamma)(w_1z_1, w_2z_2, \ldots, w_nz_n) \\
&= \beta((w_1, w_2, \ldots, w_n) \circ (z_1, z_2, \ldots, z_n)).
\end{aligned}$$

Hence $\beta = \mu \cup \gamma$ is an anti fuzzy normal LA-sub-ring of an LA-ring $R_1 \times R_2 \times \cdots \times R_n$. □

Corollary 4.7. *If* $\{\mu_i\}_{i \in I} = \{\mu_{i1} \times \mu_{i2} \times \cdots \times \mu_{in}\}_{i \in I}$ *is a family of anti fuzzy normal LA-sub-rings of an LA-ring* $R_1 \times R_2 \times \cdots \times R_n$*, then* $\mu = \cup \mu_i$ *is also an anti fuzzy normal LA-sub-ring of* $R_1 \times R_2 \times \cdots \times R_n$.

Proposition 4.8. *Let* $\mu = \mu_1 \times \mu_2 \times \cdots \times \mu_n$ *and* $\gamma = \gamma_1 \times \gamma_2 \times \cdots \times \gamma_n$ *are fuzzy subsets of LA-rings* $R = R_1 \times R_2 \times \cdots \times R_n$ *and* $R' = R'_1 \times R'_2 \times \cdots \times R'_n$ *with left identities* $e = (e_1, e_2, \ldots, e_n)$ *and* $e' = (e_1\prime, e_2\prime, \ldots, e_n\prime)$*, respectively. If* $\mu \times \gamma$ *is an anti fuzzy LA-sub-ring of an LA-ring* $R \times R'$*, then at least one of the following two statements must hold.*

1. $\mu(x) \geqslant \gamma(e')$ *for all* $x \in R$;
2. $\mu(x) \geqslant \gamma(e)$ *for all* $x \in R'$.

Proof. Let $\mu \times \gamma$ is an anti fuzzy LA-sub-ring of $R \times R'$. By contraposition, suppose that none of the statements 1 and 2 holds. Then we can find a and b in R and R', respectively such that

$$\mu(a) \leqslant \gamma(e') \text{ and } \mu(b) \leqslant \gamma(e).$$

Thus, we have

$$(\mu \times \gamma)(a, b) = \max\{\mu(a), \gamma(b)\} \leqslant \max\{\mu(e), \gamma(e')\} = (\mu \times \gamma)(e, e\prime).$$

Therefore $\mu \times \gamma$ is not an anti fuzzy LA-sub-ring of $R \times R'$. Hence either $\mu(x) \geqslant \gamma(e')$ for all $x \in R$ or $\mu(x) \geqslant \gamma(e)$ for all $x \in R'$. □

Proposition 4.9. *Let* $\mu = \mu_1 \times \mu_2 \times \cdots \times \mu_n$ *and* $\gamma = \gamma_1 \times \gamma_2 \times \cdots \times \gamma_n$ *are fuzzy subsets of LA-rings* $R = R_1 \times R_2 \times \cdots \times R_n$ *and* $R' = R'_1 \times R'_2 \times \cdots \times R'_n$ *with left identities* $e = (e_1, e_2, \ldots, e_n)$ *and* $e' = (e_1\prime, e_2\prime, \ldots, e_n\prime)$*, respectively and* $\mu \times \gamma$ *is an anti fuzzy normal LA-sub-ring of an LA-ring* $R \times R'$*. Then the following conditions are true.*

1. *If* $\mu(x) \geqslant \gamma(e')$ *for all* $x \in R$*, then* μ *is an anti fuzzy normal LA-sub-ring of* R.
2. *If* $\mu(x) \geqslant \gamma(e)$ *for all* $x \in R'$*, then* γ *is an anti fuzzy normal LA-sub-ring of* R'.

Proof.

1. Let $\mu(x) \geqslant \gamma(e')$, for all $x \in R$, and $y \in R$. We have to show that μ is an anti fuzzy normal LA-sub-ring of R. Now

$$\begin{aligned}
\mu(x - y) &= \mu(x + (-y)) \\
&= \max\{\mu(x + (-y)), \gamma(e' + (-e'))\} \\
&= (\mu \times \gamma)(x + (-y), e' + (-e')) \\
&= (\mu \times \gamma)((x, e') + (-y, -e')) \\
&= (\mu \times \gamma)((x, e') - (y, e')) \\
&\leqslant (\mu \times \gamma)(x, e') \vee \mu \times \gamma(y, e') = \max\{\max\{\mu(x), \gamma(e')\}, \max\{\mu(y), \gamma(e')\}\} \\
&= \mu(x) \vee \mu(y),
\end{aligned}$$

and

$$\begin{aligned}\mu(xy) &= \max\{\mu(xy), \gamma(e'e')\}\\ &= (\mu\times\gamma)(xy, e'e')\\ &= (\mu\times\gamma)((x,e')\circ(y,e'))\\ &\leqslant (\mu\times\gamma)(x,e')\vee\mu\times\gamma(y,e')\\ &= \max\{\max\{\mu(x),\gamma(e')\},\max\{\mu(y),\gamma(e')\}\}\\ &= \mu(x)\vee\mu(y).\end{aligned}$$

Thus μ is an anti fuzzy LA-sub-ring of R. Now

$$\begin{aligned}\mu(xy) &= \max\{\mu(xy), \gamma(e'e')\}\\ &= (\mu\times\gamma)\left(xy, e'e'\right)\\ &= (\mu\times\gamma)\left((x,e')\circ(y,e')\right)\\ &= (\mu\times\gamma)\left((y,e')\circ(x,e')\right)\\ &= (\mu\times\gamma)(yx, e'e')\\ &= \max\{\mu(yx),\gamma(e'e')\}\\ &= \mu(yx).\end{aligned}$$

Hence μ is an anti fuzzy normal LA-sub-ring of R. 2 is same as 1. □

References

[1] M. T. Abu Osman, *On the direct product of fuzzy subgroups*, Fuzzy Sets and Systems, **12** (1984), 87–91. 1
[2] M. T. Abu Osman, *On some product of fuzzy subgroups*, Fuzzy Sets and Systems, **24** (1987), 79–86. 1
[3] S. Abou-Zaid, *On normal fuzzy subgroups*, J. Fac. Educ. Ain Shams Univ. Cairo., **13** (1988), 115–125. 1
[4] M. Akram, K. H. Dar, *On anti fuzzy left h-ideals in hemirings*, Int. Math. Forum, **2** (2007), 2295–2304. 1
[5] E. F. Alharfie, N. Muthana, *The commutativity of prime rings with homoderivations*, Int. J. Adv. Appl. Sci., **5** (2018), 79–81.
[6] R. Biswas, *Fuzzy subgroups and anti fuzzy subgroups*, Fuzzy Sets and Systems, **35** (1990), 121–124. 1
[7] J. R. Cho, J. Jezek, T. Kepka, *Paramedial groupoids*, Czechoslovak Math. J., **49** (1999), 277–290. 1
[8] K. A. Dib, N. Galhum, A. A. M. Hassan, *Fuzzy rings and fuzzy ideals*, Fuzzy Math., **4** (1996), 245–261. 1
[9] V. N. Dixit, R. Kumar, N. Ajmal, *Fuzzy ideals and fuzzy prime ideals of a ring*, Fuzzy Sets and Systems, **44** (1991), 127–138.
[10] K. C. Gupta, M. K. Kantroo, *The intrinsic product of fuzzy subsets of a ring*, Fuzzy Sets and Systems, **57** (1993), 103–110. 1
[11] S. M. Hong, Y. B. Jun, *Anti fuzzy ideals in BCK-algebra*, Kyungpook Math. J., **38** (1998), 145–150. 1
[12] J. Jezek, T. Kepka, *Medial groupoids*, Rozpravy CSAV Rada Mat. a Prir. Ved., **93** (1983), 93 pages. 1
[13] T. Kadir, *In discrepancy between the traditional Fuzzy logic and inductive*, Int. J. Adv. Appl. Sci., **1** (2014), 36–43.
[14] M. S. Kamran, *Conditions for LA-semigroups to resemble associative structures*, Ph.D. Thesis, Quaid-i-Azam University, Islamabad, (1993). 1
[15] N. Kausar, *Direct product of finite intuitionistic anti fuzzy normal sub-rings over non-associative rings*, Eur. J. Pure Appl. Math., **12** (2019), 622–648. 1
[16] N. Kausar, M. Alesemi, S. Salahuddin, M. Munir, *A study on Ordered AG-groupoid by their fuzzy interior ideals*, Int. J. Adv. Appl. Sci., **7** (2020), 75–82.
[17] N. Kausar, B. ul Islam, S. A. Ahmad, M. A. Waqar, *Intuitionistics fuzzy ideals with thresholds $(\alpha, \beta]$ in LA-rings*, Eur. J. Pure Appl. Math., **12** (2019), 906–943. 1
[18] N. Kausar, B. U. Islam, M. Y. Javaid, S. A. Ahmad, U. Ijaz, *Characterizations of non-associative rings by the properties of their fuzzy ideals*, J. Taibah. Univ. Sci., **13** (2019), 820–833. 1
[19] N. Kausar, M. A. Waqar, *Characterizations of non-associative rings by their intuitionistic fuzzy bi-ideals*, Eur. J. Pure Appl. Math., **12** (2019), 226–250. 1
[20] N. Kausar, M. A. Waqar, *product of finite fuzzy normal sub-rings over non-associative rings*, Int. J. Anal. Appl., **17** (2019), 752–770. 1
[21] M. A. Kazim, M. Naseeruddin, *On almost semigroups*, Aligarh Bull. Math., **2** (1972), 1–7. 1
[22] N. Kuroki, *Regular fuzzy duo rings*, Inf. Sci., **94** (1996), 119–139. 1

[23] A. Lafi, *DFIG control: A fuzzy approach*, Int. J. Adv. Appl. Sci., **6** (2019), 107–116.

[24] W. J. Liu, *Fuzzy invariant subgroups and ideals*, Fuzzy Sets and Systems, **8** (1982), 133–139. 1

[25] T. K. Mukherjee, M. K. Sen, *On fuzzy ideals of a ring I*, Fuzzy Sets and Systems, **21** (1987), 99–104. 1

[26] T. K. Mukherjee, M. K. Sen, *Prime fuzzy ideals in rings*, Fuzzy Sets and Systems, **32** (1989), 337–341. 1

[27] M. Munir, N. Kausar, Salahuddin, Tehreem, *On the prime fuzzy m-bi ideals in semigroups*, J. Math. Computer Sci., **21** (2020), 357–365. 1

[28] P. V. Protić, N. Stevanović, *AG-test and some general properties of Abel-Grassmann's groupoids*, Pure Math. Appl., **6** (1995), 371–383. 1

[29] A. K. Ray, *On product of fuzzy subgroups*, Fuzzy sets and systems, **105** (1999), 181–183. 1

[30] T. Shah, N. Kausar, I. Rehman, *Intuitionistic fuzzy normal sub-rings over a non-associative ring*, An. Ştiinţ. Univ. "Ovidius" Constanţa Ser. Mat., **20** (2012), 369–386. 1, 3.1, 4.1

[31] T. Shah, I. Rehman, *On LA-rings of finitely non-zero functions*, Int. J. Contemp. Math. Sci., **5** (2010), 209–222. 1

[32] H. Sherwood, *On product of fuzzy subgroups*, Fuzzy Sets and Systems, **11** (1983), 79–89. 1

[33] U. M. Swamy, K. L. N. Swamy, *Fuzzy prime ideals of rings*, J. Math. Anal. Appl., **134** (1988), 94–103. 1

[34] L. A. Zadeh, *Fuzzy sets*, Information and control, **8** (1965), 338–353. 1