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Spaces of neutrosophic λ-statistical convergence sequences and their properties

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Abstract

In this paper, we use the notion of λ-statistical convergence in order to generalize these concepts. We establish some inclusion relations between them. We define the statistical convergence and λ-statistical convergence in neutrosophic normed space. We give the λ-statistically Cauchy sequence in neutrosophic normed space and present the λ-statistically completeness in connection with a neutrosophic normed space. Some interesting examples are also displayed here in support of our definitions and results.

Keywords: t-norm, t-conorm, neutrosophic normed space, statistical convergence, λ-statistical convergence, λ-statistical Cauchy.

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1. Introduction

Fuzzy sets (FSs) put forward by Zadeh [16] has influenced deeply all the scientific fields since the publication of the paper. This concept is very important for real-life situations. Atanassov [1] initiated intuitionistic fuzzy sets (IFSs) for such cases. Neutrosophic set (NS) is a new version of the idea of the classical set which is defined by Smarandache [15]. Examples of generalizations of FS are interval-valued FS, IFS, interval-valued IFS, the sets paraconsistent, dialetheist, paradoxist, and tautological, Pythagorean fuzzy sets. One of the most important problems in fuzzy topology is to obtain an appropriate concept of fuzzy metric space. Park [13] introduced and discussed in a notion of intuitionistic fuzzy metric spaces which is based both on the idea of intuitionistic fuzzy set due to Atanassov [1] and the concept of a fuzzy metric space by George and Veeramani [10]. After defined of the intuitionistic fuzzy set (IFS), it was used in all areas where FS theory was studied. Park [13] defined IF metric space (IFMS), which is a generalization of FMSs. Park used George and Veeramani's [10] thought of applying t-norm and t-conorm to FMS meanwhile defining IFMS and studying its basic features. Esi and Hazarika [6], introduced λ-Ideal convergence in intuitionistic fuzzy 2-normed linear space. Bera and Mahapatra [2] defined the

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neutrosophic soft linear spaces (NSLSs). In [2] neutrosophic norm, Cauchy sequence in NSNLS, convexity of NSNLS, metric in NSNLS were studied.

The notion of the statistical convergence was introduced by Fast [8]. Later on it was analyzed by Fridy [9] from the sequence point of view and linked it with the summability theory. Esi and Braha [5] examined on asymptotically statistical equivalent sequences of interval numbers. Esi [4] studied asymptotically λ-invariant statistical equivalent sequences of fuzzy numbers

2. Definitions and preliminaries

Definition 2.1. In terms of the set N of positive integers, let

$$\{K \subseteq N \text{ and } K_n = j : j \leqslant n \text{ and } j \in K\}.$$

Then the natural density of K is defined by

$$\delta(K) = \lim_{n\to\infty} \frac{|K_n|}{n},$$

if the limit exists, where $|K_n|$ denotes the cardinality of the set K_n.

Definition 2.2. A sequence $x = \{x_k; k = 1, 2, 3, \ldots\}$ is statistically convergent to l if

$$(\{k : |x_k - l| \geqslant \epsilon\}) = 0$$

for every $\epsilon > 0$, we have

$$\lim_n \frac{1}{n}\{j : j \leqslant n \text{ and } |x_j - l| \geqslant \epsilon\} = 0.$$

Example 2.3. Let us consider the sequence $x = \{x_k; k = 1, 2, 3 \ldots, \}$ whose terms are

$$x_k = \begin{cases} k, & \text{if } k = n^2, n = 1, 2, 3, \ldots, \\ \frac{1}{k}, & \text{otherwise .} \end{cases}$$

The concept of λ-statistical convergence was introduced recently in [4] as follows.

Definition 2.4. Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers tending to ∞ such that $\lambda_{n+1} \leqslant \lambda_n + 1$ and $\lambda_1 = 1$. Also let $I_n = [n - \lambda_n + 1, n]$ and $K \subseteq \mathbb{N}$. Then the λ-density of K is defined by

$$\delta_\lambda(K) = \lim_{n\to\infty} |\{j : j \in I_n \text{ and } j \in K\}|.$$

It is clear that any finite subset of $\mathbb{N}$ has zero λ-density and $\delta_\lambda(K^c) = 1 - \delta_\lambda(K)$ does not hold for $0 < \alpha < 1$ in general[7], when $\lambda_n = n$, the λ-density reduces to the above-defined natural density.

Definition 2.5. A sequence $x = (x_n)$ is said to be λ-statistically convergent or S_λ-convergent to l if for every $\epsilon > 0$

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n : |x_n - l| \geqslant \epsilon\}| = 0.$$

In this case we write $S_\lambda - \lim x = l$ or $x_n \to l(S_\lambda)$, and

$$S_\lambda : \{x : \exists \;\; l \in \mathbb{R}, S_\lambda - \lim x = l\}.$$

Remark 2.6. If $A_n = n$, then S_λ is the same as S.

Remark 2.7. λ-statistical convergence is a special case of A-statistical convergence [3, 11] if the matrix $A = (a_{nk})$ is taken as

$$x_k = \begin{cases} \frac{1}{\lambda_n}, & \text{if } k \in I_n, \\ 0, & \text{if } k \notin I_n. \end{cases}$$

Triangular norms (t-norms) (TN) were initiated by Menger [12] . Triangular conorms (t-conorms) (TC) are known as dual operations of TNs. TNs and TCs are very significant for fuzzy operations (intersections and unions).

Definition 2.8 ([14])**.** A binary operation $* : [0,1] \times [0,1] \longrightarrow [0,1]$ is said to be a continuous t-norm if it satisfies the following conditions:

(a) $*$ is associative and commutative;
(b) $*$ is continuous;
(c) $a * 1 = a$ for all $a \in [0,1]$;
(d) $a * b \leqslant c * d$ whenever $a \leqslant c$ and $b \leqslant d$ for each $a, b, c, d \in [0,1]$.

Definition 2.9 ([14])**.** A binary operation $\diamond : [0,1] \times [0,1] \longrightarrow [0,1]$ is said to be a continuous t-conorm if it satisfies the following conditions:

(a) $\diamond$ is associative and commutative;
(b) $\diamond$ is continuous;
(c) $a \diamond 0 = a$ for all $a \in [0,1]$;
(d) $a \diamond b \leqslant c \diamond d$ whenever $a \leqslant c$ and $b \leqslant d$ for each $a, b, c, d \in [0,1]$.

Form above definitions, we note that if we choose $0 < e_1, e_2 < 1$ for $e_1 > e_2$, then there exist $0 < e_3, e_4 < 0, 1$ such that $e_1 * e_3 \geqslant e_2, e_1 \geqslant e_4 \diamond e_2$. Further, if we choose $e_5 \in (0,1)$, then there exist $e_6, e_7 \in (0,1)$ such that $e_6 * e_6 \geqslant e_5$ and $e_7 \diamond e_7 \leqslant e_5$.

In this paper, neutrosophic normed space (NNS) is defined and the definition statistical convergence with respect to NNS is given. The fundamental properties of NNS and statistical convergence with respect to NNS are investigated.

Definition 2.10 ([2])**.** Take F as a vector space, $N = \{< u, \mathcal{G}(u), \mathcal{B}(u), \mathcal{Y}(u) >: u \in F\}$ be a normed space(NS) such that $N : F \times \mathbb{R}^+ \rightarrow [0,1]$ show continuous TN and continuous TC, respectively. If the following conditions are hold, then the four-tuple $V = (F, N, *, \diamond)$ is called NNS, for all $u, v \in F$ and $\beta, \mu > 0$ and for each $\sigma \neq 0$,

(a) $0 \leqslant \mathcal{G}(u,\beta) \leqslant 1, 0 \leqslant \mathcal{B}(u,\beta) \leqslant 1, 0 \leqslant \mathcal{Y}(u,\beta) \leqslant 1$;
(b) $\mathcal{G}(u,\beta) + \mathcal{B}(u,\beta) + \mathcal{Y}(u,\beta) \leqslant 3, \quad \beta \in \mathbb{R}^+$;
(c) $\mathcal{G}(u,\beta) = 1$ (for $\beta > 0$) if and only if $u = 0$;
(d) $\mathcal{G}(\sigma u,\beta) = \mathcal{G}(u, \frac{\beta}{|\sigma|})$ for each $\sigma \neq 0$;
(e) $\mathcal{G}(u,\beta) * \mathcal{G}(v,\mu) \leqslant \mathcal{G}(u+v, \beta+\mu)$;
(f) $\mathcal{G}(u, .)$ is continuous non-decreasing function;
(g) $\lim_{\beta \to \infty} \mathcal{G}(u,\beta) = 1$;
(h) $\mathcal{B}(u,\beta) = 0$ (for $\beta > 0$) if and only if $u = 0$;
(i) $\mathcal{B}(\sigma u,\beta) = \mathcal{B}(u, \frac{\beta}{|\sigma|})$ for each $\sigma \neq 0$;
(j) $\mathcal{B}(u,\beta) * \mathcal{B}(v,\mu) \geqslant \mathcal{B}(u+v, \beta+\mu)$;
(k) $\mathcal{B}(u, .)$ is continuous non-increasing function;
(l) $\lim_{\beta \to \infty} \mathcal{B}(u,\beta) = 1$;
(m) $\mathcal{Y}(u,\beta) = 0$ (for $\beta > 0$) if and only if $u = 0$;
(n) $\mathcal{Y}(\sigma u,\beta) = \mathcal{Y}(u, \frac{\beta}{|\sigma|})$ for each $\sigma \neq 0$;
(o) $\mathcal{Y}(u,\beta) * \mathcal{Y}(v,\mu) \geqslant \mathcal{Y}(u+v, \beta+\mu)$;
(p) $\mathcal{Y}(u, .)$ is continuous non-increasing function;
(q) $\lim_{\beta \to \infty} \mathcal{Y}(u,\beta) = 1$;
(r) if $\beta \leqslant 0$, then $\mathcal{G}(u,\beta) = 0, \mathcal{B}(u,\beta) = 1$ and $\mathcal{Y}(u,\beta) = 1$.

Then $(N, \mathcal{G}, \mathcal{B}, \mathcal{Y})$ is called neutrosophic norm (NN).

Example 2.11. Let $(F, \|.\|)$ be a normed space. Give the operations $*$ and $\diamond$ being TN and TC, respectively, and $u * v = uv$, $u \diamond v = u + v - uv$. For $\beta > \|u\|$,

$$\mathcal{G}(u, \beta) = \frac{\beta}{\beta + \|u\|}, \mathcal{B}(u, \beta) = \frac{\|u\|}{\beta + \|u\|}, \mathcal{Y}(u, \beta) = \frac{\|u\|}{\beta}$$

for all $u, v \in F$ and $\beta > 0$. If we take $\beta \leqslant \|u\|$, then $\mathcal{G}(u, \beta) = 0, \mathcal{B}(u, \beta) = 1$ and $\mathcal{Y}(u, \beta) = 1$. Then $(F, N, *, \diamond)$ is NNS such that $N : F \times \mathbb{R}^+ \to [0, 1]$.

Definition 2.12. Let V be a neutrosophic normed space (NNS), and let the sequence (x_k) in V, $0 < \epsilon < 1$ and $\beta > 0$. Then, the sequence (x_k) is convergence to l if and only if there exists $n \in \mathbb{N}$, such that $\mathcal{G}(x_k - l, \beta) > 1 - \epsilon, \mathcal{B}(x_k - l, \beta) < \epsilon, \mathcal{Y}(x_k - l, \beta) < \epsilon$. That is $\lim_{k\to\infty} \mathcal{G}(x_k - l, \beta) = 1, \lim_{k\to\infty} \mathcal{B}(x_k - l, \beta) = 0$, $\lim_{k\to\infty} \mathcal{Y}(x_k - l, \beta) = 0$. In that case, the sequence (x_k) is called a convergent sequence in V. The convergent in NNS is denoted by $N - \lim x_k = l$.

Definition 2.13. Take a NNS V. A sequence (x_k) is called statistical convergence with respect to NN (SC-NN), if there exist $l \in F$ such that the set

$$K_\epsilon := \{k \leqslant n : \mathcal{G}(x_k - l, \beta) \leqslant 1 - \epsilon \text{ or } \mathcal{B}(x_k - l, \beta) \geqslant \epsilon, \mathcal{Y}(x_k - l, \beta) \geqslant \epsilon\}$$

has ND zero, for every $\epsilon > 0$ and $\beta > 0$ or equivalently,

$$\lim_n \frac{1}{n} |\{k \leqslant n : \mathcal{G}(x_k - l, \beta) \leqslant 1 - \epsilon \text{ or } \mathcal{B}(x_k - l, \beta) \geqslant \epsilon, \mathcal{Y}(x_k - l, \beta) \geqslant \epsilon\}| = 0.$$

Therefore, we write $S_N - \lim x_k = l$ or $x_k \to l(S_N)$. The set of SC-NN will be denoted by SN. If $l = 0$, then we will write S_0N.

3. λ-statistical convergence on NNS

In this section, we introduce the λ-statistical convergence on NNS which are needed in our subsequent discussions:

Definition 3.1. Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers tending to infinity such that $\lambda_{n+1} \leqslant \lambda_n + 1$ and $\lambda_1 = 1$. Also let $I_n = [n - \lambda_n + 1, n]$. Take a NNS V. A sequence (x_k) is called λ-statistical convergence with respect to NN (SC-NN), if there exist $l \in F$ such that the set

$${}_\lambda K_\epsilon := \{k \in I_n : \mathcal{G}(x_k - l, \beta) \leqslant 1 - \epsilon \text{ or } \mathcal{B}(x_k - l, \beta) \geqslant \epsilon, \mathcal{Y}(x_k - l, \beta) \geqslant \epsilon\}$$

has λ-density zero, for every $\epsilon > 0$ and $\beta > 0$ or equivalently,

$$\lim_n \frac{1}{\lambda_n} |\{k \in I_n : \mathcal{G}(x_k - l, \beta) \leqslant 1 - \epsilon \text{ or } \mathcal{B}(x_k - l, \beta) \geqslant \epsilon, \mathcal{Y}(x_k - l, \beta) \geqslant \epsilon\}| = 0.$$

Therefore, we write ${}_\lambda S_N - \lim x_k = l$ or $x_k \to l({}_\lambda S_N)$. The set of $\lambda - SC - NN$ will be denoted by ${}_\lambda S_N$. If $l = 0$, then we will write ${}_\lambda S_0N$.

Example 3.2. Let $(F, \|.\|)$ be a NS. For all $u, v \in [0, 1]$, define the TN $u * v = uv$ and TC $u \diamond v = \min\{u + v, 1\}$. We take

$$\mathcal{G}(u, \beta) = \frac{\beta}{\beta + \|u\|}, \mathcal{B}(u, \beta) = \frac{\|u\|}{\beta + \|u\|}, \mathcal{Y}(u, \beta) = \frac{\|u\|}{\beta}$$

for all $u \in F, \beta > 0$. Then, V is a NNS. Define,

$$x_k = \begin{cases} 1, & \text{if } k = m^2 (m \in \mathbb{N}), \\ 0, & \text{otherwise.} \end{cases}$$

Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers tending to ∞ such that $\lambda_{n+1} \leqslant \lambda_n + 1$ and $\lambda_1 = 1$. Also let $I_n = [n - \lambda_n + 1, n]$. Consider

$$_{\lambda}K_n(\epsilon, \beta) = \{k \in I_n : \mathcal{G}(u, \beta) \leqslant 1 - \epsilon \text{ or } \mathcal{B}(u, \beta) \geqslant \epsilon, \mathcal{Y}(u, \beta) \geqslant \epsilon\}$$

for every $\epsilon \in (0, 1)$ and for any $\beta > 0$. Then, we have

$$\begin{aligned} _{\lambda}K_n(\epsilon, \beta) &= \{k \in I_n : \frac{\beta}{\beta + \|u\|} \leqslant 1 - \epsilon \text{ or } \frac{\|u\|}{\beta + \|u\|} \geqslant \epsilon, \frac{\|u\|}{\beta} \geqslant \epsilon\} \\ &= \{k \in I_n : \|u\| \geqslant \frac{\beta\epsilon}{1-\epsilon} \text{ or } \|u\| \geqslant \beta\epsilon\} \\ &= \{k \in I_n : x_k = 1\} = \{k \leqslant n : k = m^2 \text{ and } m \in \mathbb{N}\} \leqslant \frac{\sqrt{n}}{n}. \end{aligned}$$

That is, when n becomes sufficiently large, quantity $\mathcal{G}(x_k, \beta)$ becomes less that $1 - \epsilon$ and similarly the quantities $\mathcal{B}(x_k, \beta)$ and $\mathcal{Y}(x_k, \beta)$ become larger than ϵ. So

$$\frac{1}{\lambda_n} |_{\lambda}K_n(\epsilon, \beta)| = 0 \text{ for } \epsilon > 0 \text{ and } \beta > 0.$$

Lemma 3.3 may be easily obtained by using the definitions and properties of λ-density.

Lemma 3.3. *Choose a NNS V. The following statements are equivalent, for every $\epsilon > 0$ and $\beta > 0$:*

1. $_{\lambda}S_N - \lim x_k = l$;
2. $\lim_n \frac{1}{\lambda_n} |\{k \in I_n : \mathcal{G}(x_k - l, \beta) \leqslant 1 - \epsilon\}| = \lim_n \frac{1}{\lambda_n} |\{k \in I_n : \mathcal{B}(x_k - l, \beta) \geqslant \epsilon\}| = \lim_n \frac{1}{\lambda_n} |\{k \in I_n : \mathcal{Y}(x_k - l, \beta) \leqslant 1 - \epsilon\}| = 0$;
3. $\lim_n \frac{1}{\lambda_n} |\{k \in I_n : \mathcal{G}(x_k - l, \beta) > 1 - \epsilon$ *and* $\mathcal{B}(x_k - l, \beta) < \epsilon, \mathcal{Y}(x_k - l, \beta) > 1 - \epsilon\}| = 1$;
4. $_{\lambda}S - \lim \mathcal{G}(x_k - l, \beta) = 1$, *and* $_{\lambda}S - \lim \mathcal{B}(x_k - l, \beta) = 0$, $_{\lambda}S - \lim \mathcal{Y}(x_k - l, \beta) = 0$.

Theorem 3.4. *Let V be a NNS. If (x_k) is λ-statistical convergent in neutrosophic normed space, then $_{\lambda}S - \lim x_k = l$ is unique.*

Proof. Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers tending to infinity such that $\lambda_{n+1} \leqslant \lambda_n + 1$ and $\lambda_1 = 1$. Also let $I_n = [n - \lambda_n + 1, n]$. Suppose that $S_\lambda - \lim x_k = l_1$ and $S_\lambda - \lim x_k = l_2$, for $l_1 \neq l_2$. Choose $\epsilon > 0$. Then, for a given $\mu > 0$ such that $(1 - \epsilon) * (1 - \epsilon) > 1 - \mu$ and $\epsilon \diamond \epsilon < \mu$. Then for any $\beta > 0$, define the following sets as:

$$K_{\mathcal{G}_1}(\epsilon, \beta) = |\{k \in I_n : \mathcal{G}(x_k - l_1, \frac{\beta}{2}) \leqslant 1 - \epsilon\}| = 0, \quad K_{\mathcal{G}_2}(\epsilon, \beta) = |\{k \in I_n : \mathcal{G}(x_k - l_2, \frac{\beta}{2}) \leqslant 1 - \epsilon\}| = 0,$$
$$K_{\mathcal{B}_1}(\epsilon, \beta) = |\{k \in I_n : \mathcal{B}(x_k - l_1, \frac{\beta}{2}) \geqslant \epsilon\}| = 0, \quad K_{\mathcal{B}_2}(\epsilon, \beta) = |\{k \in I_n : \mathcal{B}(x_k - l_2, \frac{\beta}{2}) \geqslant \epsilon\}| = 0,$$
$$K_{\mathcal{Y}_1}(\epsilon, \beta) = |\{k \in I_n : \mathcal{Y}(x_k - l_1, \frac{\beta}{2}) \geqslant \epsilon\}| = 0, \quad K_{\mathcal{Y}_2}(\epsilon, \beta) = |\{k \in I_n : \mathcal{Y}(x_k - l_2, \frac{\beta}{2}) \geqslant \epsilon\}| = 0.$$

We know that $S_\lambda - \lim x_k = l_1$. Then, using the Lemma 3.3, for all $\beta > 0$

$$\delta_\lambda(K_{\mathcal{G}_1}(\epsilon, \beta)) = \delta_\lambda(K_{\mathcal{B}_1}(\epsilon, \beta)) = \delta_\lambda(K_{\mathcal{Y}_1}(\epsilon, \beta)) = 0.$$

Further since $S_\lambda - \lim x_k = l_2$, then, using the Lemma 3.3, for all $\beta > 0$

$$\delta_\lambda(K_{\mathcal{G}_2}(\epsilon, \beta)) = \delta_\lambda(K_{\mathcal{B}_2}(\epsilon, \beta)) = \delta_\lambda(K_{\mathcal{Y}_2}(\epsilon, \beta)) = 0.$$

Let

$$K_\lambda(\epsilon, \beta) := \{K_{\mathcal{G}_1}(\epsilon, \beta) \cup K_{\mathcal{G}_2}(\epsilon, \beta)\} \cap \{K_{\mathcal{B}_1}(\epsilon, \beta) \cup K_{\mathcal{B}_2}(\epsilon, \beta)\} \cap \{K_{\mathcal{Y}_1}(\epsilon, \beta) \cup K_{\mathcal{Y}_2}(\epsilon, \beta)\}.$$

Then, observe that $\delta_\lambda(K_\lambda(\epsilon, \beta)) = 0$, which implies $\delta_\lambda(\mathbb{N}/K_\lambda(\epsilon, \beta)) = 1$. We have three possible situations, when take $k \in \mathbb{N}/K_\lambda(\epsilon, \beta)$:

(i) $k \in \mathbb{N}/\{K_{\mathcal{G}_1}(\epsilon, \beta) \cup K_{\mathcal{G}_2}(\epsilon, \beta)\}$;
(ii) $k \in \mathbb{N}/\{K_{\mathcal{B}_1}(\epsilon, \beta) \cup K_{\mathcal{B}_2}(\epsilon, \beta)\}$;
(iii) $k \in \mathbb{N}/\{K_{\mathcal{Y}_1}(\epsilon, \beta) \cup K_{\mathcal{Y}_2}(\epsilon, \beta)\}$.

Firstly, consider (i). Then, we have

$$\mathcal{G}(l_1 - l_2, \beta) \geqslant \mathcal{G}(x_k - l_1, \frac{\beta}{2}) * \mathcal{G}(x_k - l_2, \frac{\beta}{2}) > (1-\epsilon) * (1-\epsilon) > 1 - \mu. \tag{3.1}$$

Using the equation (3.1), for all $\beta > 0$, we obtain $\mathcal{G}(l_1 - l_2, \beta) = 1$, where $\mu > 0$ is arbitrary. That is, $l_1 = l_2$ is obtained.

For the situation (ii), if we take $k \in \mathbb{N}/\{K_{\mathcal{B}_1}(\epsilon, \beta) \cup K_{\mathcal{B}_2}(\epsilon, \beta)\}$, then, we can write

$$\mathcal{B}(l_1 - l_2, \beta) \leqslant \mathcal{B}(x_k - l_1, \frac{\beta}{2}) \diamond \mathcal{B}(x_k - l_2, \frac{\beta}{2}) < \epsilon \diamond \epsilon < \mu$$

for all $\beta > 0$, we obtain $\mathcal{B}(l_1 - l_2, \beta) = 0$, where $\mu > 0$ is arbitrary. That is, $l_1 = l_2$ is obtained. Again for the situation (iii), if we take $k \in \mathbb{N}/\{K_{\mathcal{Y}_1}(\epsilon, \beta) \cup K_{\mathcal{Y}_2}(\epsilon, \beta)\}$, then, we can write

$$\mathcal{Y}(l_1 - l_2, \beta) \leqslant \mathcal{Y}(x_k - l_1, \frac{\beta}{2}) \diamond \mathcal{Y}(x_k - l_2, \frac{\beta}{2}) < \epsilon \diamond \epsilon < \mu$$

for all $\beta > 0$, we obtain $\mathcal{Y}(l_1 - l_2, \beta) = 0$, where $\mu > 0$ is arbitrary. That is, $l_1 = l_2$ is obtained. This step completes the proof. □

Theorem 3.5. *If* $\lim x_k = l$ *for neutrosophic normed space (NNS)* V, *then* $S_\lambda - \lim x_k = l$.

Proof. If $\lim x_k = l$ for NNS V then, for every $\epsilon > 0$ and $\beta > 0$, there exist a number $n \in \mathbb{N}$ such that $\mathcal{G}(x_k - l, \beta) > 1 - \epsilon$, $\mathcal{B}(x_k - l, \beta) < 1 - \epsilon$ and $\mathcal{Y}(x_k - l, \beta) < 1 - \epsilon$, for all $k \geqslant n$. Therefore, the set

$$\{k \leqslant n : \mathcal{G}(x_k - l, \beta) \leqslant 1 - \epsilon, \mathcal{B}(x_k - l, \beta) \geqslant 1 - \epsilon \text{ or } \mathcal{Y}(x_k - l, \beta) \geqslant 1 - \epsilon\}$$

so that we have

$$\{k \in I_n : \mathcal{G}(x_k - l, \beta) \leqslant 1 - \epsilon, \mathcal{B}(x_k - l, \beta) \geqslant 1 - \epsilon \text{ or } \mathcal{Y}(x_k - l, \beta) \geqslant 1 - \epsilon\}$$

and has at most finitely many terms, where $I_n = [n - \lambda_n + 1, n]$ and (λ_n) be a non-decreasing sequence of positive numbers tending to infinity such that $\lambda_{n+1} \leqslant \lambda_n + 1$ and $\lambda_1 = 1$. Hence, since every finite subset of $\mathbb{N}$ has λ-density zero,

$$\begin{aligned} \lim_n \frac{1}{\lambda_n} |\{n - \lambda_n + 1 \leqslant k \leqslant n : \mathcal{G}(x_k - l, \beta) \leqslant 1 - \epsilon, \\ \mathcal{B}(x_k - l, \beta) \geqslant 1 - \epsilon \text{ or } \mathcal{Y}(x_k - l, \beta) \geqslant 1 - \epsilon\}| = 0. \end{aligned}$$

This clearly shows that $S_\lambda - \lim x_k = l$. □

Theorem 3.6. *Let* V *be an NNS.* ${}_\lambda S_N - \lim x_k = l$ *iff there exists an increasing index sequence* $J = \{j_1, j_2, \dots\} \subseteq \mathbb{N}$ *while* $\delta_\lambda(J) = 1$, *then* $N - \lim_n x_{j_n} = l$.

Proof. Suppose that ${}_\lambda S_N - \lim x_k = l$. For any $\beta > 0$ and $\mu = 1, 2, 3, \dots$

$$P_\lambda(\mu, \beta) = \left\{k \leqslant n : \mathcal{G}(x_k - l, \beta) > 1 - \frac{1}{\mu} \text{ and } \mathcal{B}(x_k - l, \beta) < \frac{1}{\mu}, \mathcal{Y}(x_k - l, \beta) < \frac{1}{\mu}\right\}$$

and

$$R_\lambda(\mu, \beta) = \left\{k \leqslant n : \mathcal{G}(x_k - l, \beta) \leqslant 1 - \frac{1}{\mu} \text{ and } \mathcal{B}(x_k - l, \beta) \geqslant \frac{1}{\mu}, \mathcal{Y}(x_k - l, \beta) \geqslant \frac{1}{\mu}\right\}.$$

Then, $\delta_\lambda(R_\lambda(\mu,\beta)) = 0$, since ${}_\lambda S_N - \lim x_k = l$. Further, for $\beta > 0$ and $\mu = 1, 2, 3 \ldots$

$$P_\lambda(\mu,\beta) \supset P_\lambda(\mu+1,\beta)$$

and so,

$$\delta_\lambda(P_\lambda(\mu,\beta)) = 1. \tag{3.2}$$

Now, we will show that for $k \in P_\lambda(\mu,\beta)$, $N - \lim x_k = l$. Assume that $N - \lim x_k \neq l$ for some $k \in P_\lambda(\mu,\beta)$. Then, there is $\rho > 0$ and a positive integer n such that $\mathcal{G}(x_k - l,\beta) \leqslant 1-\rho$ or $\mathcal{B}(x_k - l,\beta) \geqslant \rho, \mathcal{Y}(x_k - l,\beta) \geqslant \rho$ for all $k \geqslant n$. Let $\mathcal{G}(x_k - l,\beta) > 1-\rho$ or $\mathcal{B}(x_k - l,\beta) < \rho, \mathcal{Y}(x_k - l,\beta) < \rho$ for all $k \in I_n$, where $I_n = [n - \lambda_n + 1, n]$ and $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers tending to infinity such that $\lambda_{n+1} \leqslant \lambda_n + 1$ and $\lambda_1 = 1$. Hence

$$\lim_n \frac{1}{\lambda_n}\left|\{k \in I_n : \mathcal{G}(x_k - l,\beta) > 1-\rho \text{ or } \mathcal{B}(x_k - l,\beta) < \rho, \mathcal{Y}(x_k - l,\beta) < \rho\}\right| = 0.$$

Since $\rho > \frac{1}{\mu}$, we obtain $\delta_\lambda(P_\lambda(\mu,\beta)) = 0$, which contradicts (3.2). That's why, $N - \lim x_k = l$. Assume that there exists a increasing index sequence $J = \{j_1, j_2, \ldots\} \subseteq \mathbb{N}$ while $\delta_\lambda(J) = 1$, then $N - \lim_n x_{j_n} = l$, i.e., there exists a $n \in \mathbb{N}$ such that $\mathcal{G}(x_k - l,\beta) > 1-\mu, \mathcal{B}(x_k - l,\beta) < \mu, \mathcal{Y}(x_k - l,\beta) < \mu$ for every $\mu > 0$ and $\beta > 0$. In that case

$$R_\lambda(\mu,\beta) = \Big\{k \in I_n : \mathcal{G}(x_k - l,\beta) \leqslant 1-\mu \text{ and } \mathcal{B}(x_k - l,\beta) \geqslant \mu, \mathcal{Y}(x_k - l,\beta) \geqslant \mu\Big\} \subseteq n - \{j_{n+1}, j_{n+2}, \ldots\}.$$

Therefore $\delta_\lambda(R_\lambda(\mu,\beta)) \leqslant 1 - 1 = 0$, hence ${}_\lambda S_N - \lim x_k = l$. □

4. λ-statistically complete NNS

Definition 4.1. The sequence (x_k) is called λ-statistically Cauchy with respect to neutrosophic norm (NN) in neutrosophic normed space(NNS) V, if there exists $n = n(\epsilon)$, for every $\epsilon > 0$ and $\beta > 0$ such that

$${}_\lambda KC_\epsilon = \Big\{k \in I_n : \mathcal{G}(x_k - x_n,\beta) \leqslant 1-\epsilon \text{ or } \mathcal{B}(x_k - x_n,\beta) \geqslant \epsilon, \mathcal{Y}(x_k - x_n,\beta) \geqslant \epsilon\Big\}$$

has λ-density zero. That is $\delta_\lambda({}_\lambda KC_\epsilon) = 0$.

Theorem 4.2. *If a sequence x_k is λ-statistical convergent in NNS V, then it is λ-statistical Cauchy.*

Proof. Let a sequence x_k is λ-statistical convergent in NNS V. We get $(1-\epsilon) * (1-\epsilon) > 1-\mu$ and $\epsilon \diamond \epsilon < \mu$ for a given $\epsilon > 0$ and choose $\mu > 0$. Then, we have

$$\delta_\lambda\Big(A(\epsilon,\beta)\Big) = \delta_\lambda\Big(k \in I_n : \mathcal{G}(x_k - l,\frac{\beta}{2}) \leqslant 1-\epsilon \text{ or } \mathcal{B}(x_k - l,\frac{\beta}{2}) \geqslant \epsilon, \mathcal{Y}(x_k - l,\frac{\beta}{2}) \geqslant \epsilon\Big) = 0 \tag{4.1}$$

and so

$$\delta_\lambda\Big(A^c(\epsilon,\beta)\Big) = \delta_\lambda\Big(k \in I_n : \mathcal{G}(x_k - l,\frac{\beta}{2}) > 1-\epsilon \text{ or } \mathcal{B}(x_k - l,\frac{\beta}{2}) < \epsilon, \mathcal{Y}(x_k - l,\frac{\beta}{2}) < \epsilon\Big) = 1$$

for $\beta > 0$. Let $p \in A^c(\epsilon,\beta)$, then

$$\mathcal{G}(x_k - l,\frac{\beta}{2}) > 1-\epsilon \text{ and } \mathcal{B}(x_k - l,\frac{\beta}{2}) < \epsilon, \mathcal{Y}(x_k - l,\frac{\beta}{2}) < \epsilon.$$

Let

$$B(\epsilon,\beta) = \{k \in I_n : \mathcal{G}(x_k - x_n,\beta) \leqslant 1-\mu \text{ or } \mathcal{B}(x_k - x_n,\beta) \geqslant \mu, \mathcal{Y}(x_k - x_n,\beta) \geqslant \mu\}.$$

We claim that $B(\epsilon,\beta) \subset A(\epsilon,\beta)$. Let $q \in B(\epsilon,\beta)/A(\epsilon,\beta)$, then

$$\mathcal{G}(x_q - x_n, \beta) \leqslant 1-\mu \text{ and } \mathcal{G}(x_q - l, \frac{\beta}{2}) > 1-\mu,$$

in particular $\mathcal{G}(x_n - l, \frac{\beta}{2}) > 1-\epsilon$. Then

$$1-\mu \geqslant \mathcal{G}(x_q - x_n, \beta) \geqslant \mathcal{G}(x_q - l, \frac{\beta}{2}) * \mathcal{G}(x_n - l, \frac{\beta}{2}) > (1-\epsilon)*(1-\epsilon) > 1-\mu,$$

which is not possible. Moreover,

$$\mathcal{B}(x_q - x_n, \beta) \geqslant \mu \text{ and } \mathcal{B}(x_q - l, \frac{\beta}{2}) < \mu,$$

in particular $\mathcal{B}(x_n - l, \frac{\beta}{2}) < \epsilon$. Then,

$$\mu \leqslant \mathcal{B}(x_q - x_n, \beta) \leqslant \mathcal{B}(x_q - l, \frac{\beta}{2}) \diamond \mathcal{B}(x_n - l, \frac{\beta}{2}) < \epsilon \diamond \epsilon < \mu,$$

which is not possible. Similarly,

$$\mathcal{Y}(x_q - x_n, \beta) \geqslant \mu \text{ and } \mathcal{Y}(x_q - l, \frac{\beta}{2}) < \mu,$$

in particular $\mathcal{Y}(x_n - l, \frac{\beta}{2}) < \epsilon$. Then,

$$\mu \leqslant \mathcal{Y}(x_q - x_n, \beta) \leqslant \mathcal{Y}(x_q - l, \frac{\beta}{2}) \diamond \mathcal{Y}(x_n - l, \frac{\beta}{2}) < \epsilon \diamond \epsilon < \mu,$$

which is not possible. In that case $B(\epsilon,\beta) \subset A(\epsilon,\beta)$. Then, by (4.1), $\delta_\lambda\Big(A(\epsilon,\beta)\Big) = 0$, and (x_k) is λ-statistical Cauchy in NNS. □

Definition 4.3. The NNS V is called λ-statistically complete, if every λ-statistical Cauchy is λ-statistical convergent in NNS.

Theorem 4.4. *Every neutrosophic normed space* V *is λ-statistically complete.*

Proof. Let x_k be λ-statistical Cauchy but not λ-statistical convergent in NNS. Choose $\mu > 0$. We get $(1-\epsilon)*(1-\epsilon) > 1-\mu$ and $\epsilon \diamond \epsilon < \mu$, for a given $\epsilon > 0$ and $\beta > 0$. Since x_k is not λ-statistical convergent in NNS,

$$\begin{aligned}
\mathcal{G}(x_k - x_n, \beta) &\geqslant \mathcal{G}(x_k - l, \frac{\beta}{2}) * \mathcal{G}(x_n - l, \frac{\beta}{2}) > (1-\epsilon)*(1-\epsilon) > 1-\mu,\\
\mathcal{B}(x_k - x_n, \beta) &\leqslant \mathcal{B}(x_k - l, \frac{\beta}{2}) \diamond \mathcal{B}(x_n - l, \frac{\beta}{2}) < \epsilon \diamond \epsilon < \mu,\\
\mathcal{Y}(x_k - x_n, \beta) &\leqslant \mathcal{Y}(x_k - l, \frac{\beta}{2}) \diamond \mathcal{Y}(x_n - l, \frac{\beta}{2}) < \epsilon \diamond \epsilon < \mu,
\end{aligned}$$

for

$$P(\epsilon,\mu) = \{k \in I_n, \mathcal{B}_{x_k - x_n}(\epsilon) \leqslant 1-\mu\}.$$

So, $\delta_\lambda\Big(P(\epsilon,\mu)\Big) = 1$ which is a contradiction, since (x_k) was λ-statistical Cauchy in NNS. So that (x_k) must be λ-statistical convergent in NNS. Hence every NNS is λ-statistically complete.

□

5. Conclusion

In this paper, we have defined to neutrosophic normed space and λ-statistical convergence in neutrosophic normed space. The topological properties of NNSs have been established and examples are given. Further, λ-statistical convergence with respect to neutrosophic norm is introduced and some fundamental properties are examined. λ-Statistical Cauchy sequence and λ-statistically completeness for neutrosophic norm are defined.

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