J. Math. Computer Sci., 23 (2021), 26–35

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

Some new results for Horn's hypergeometric functions Γ_1 and Γ_2

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Abstract

The object of the present work is to deduce several important developments in various recursion relations, relevant differential recursion formulas, infinite summation formulas, integral representations, and integral operators for Horn's hypergeometric functions Γ_1 and Γ_2.

Keywords: Horn's functions, recursion formulas, infinite summation formulas, integral operators.

2020 MSC: 33C20, 11J72.

1. Introduction, notations, and preliminaries

The problem of recursion formulas of hypergeometric type functions with respect to their parameters was considered in numerous papers, see for example [10, 20, 24]. Recently, Opps et al. [11, 12] obtained recursion formulas for Appell's function F_2. Brychkov [2] and Brychkov et al. [3–6] gave the recursion formulas for Appell's hypergeometric functions F_1, F_2, F_3 and F_4. Brychkov and Savischenko [7] obtained some formulas for Horn functions $H_1(a, b, c; d; w, z)$ and $H_1^{(c)}(a, b; d; w, z)$. Mullen [10], Sharma [20] and Wang [24] gave various recursion formulas for Appell's functions. Sahin [17] and Sahai and Verma [13–16] gave some recursion formulas for three variables hypergeometric functions and k-Lauricella's hypergeometric functions. Sahin and Agha [18], studied the recursion relations of G_1 and G_2 Horn's hypergeometric functions. Srivastava et al. [23] introduced incomplete Hurwitz-Lerch zeta functions of two variables.

Recall that the following abbreviated notations, the Pochhammer symbol $(a)_n$ is defined in [21, 22] by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} a(a+1)(a+2)\cdots(a+n-1), & n \in \mathbb{N}; a \in \mathbb{C}-\{0\}; \\ 1, & n = 0; a \in \mathbb{C}-\{0\}, \end{cases} \tag{1.1}$$

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doi: 10.22436/jmcs.023.01.03

Received: 2020-06-16 *Revised:* 2020-07-20 *Accepted:* 2020-08-17

where $\Gamma(a)$ is a gamma function, the symbols $\mathbb{N}$ and $\mathbb{C}$ denote the sets of natural numbers and complex numbers, respectively, and

$$(a)_{-n} = \frac{(-1)^n}{(1-a)_n}, (a \neq 0, \pm 1, \pm 2, \pm 3, \dots, \forall a > n; n \in \mathbb{N}). \tag{1.2}$$

Its well known form are also used in [8] as

$$(a)_{n+1} = a(a+1)_n = (a+1)(a)_n, \quad (a+1)_n = \left(1 + \frac{n}{a}\right)(a)_n; a \neq 0.$$

For $n, m \in \mathbb{N}$, we have

$$(a)_{n+m} = (a)_n(a+n)_m = (a)_m(a+m)_n, \quad (b)_{n-m} = \frac{(-1)^m (b)_n}{(1-b-n)_m}, \quad 0 \leqslant m \leqslant n. \tag{1.3}$$

The Horn's hypergeometric functions Γ_1 and Γ_2 were defined by the series [8, 9]

$$\Gamma_1(a, b, c; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_m (b)_{n-m} (c)_{m-n}}{m!n!} x^m y^n; |x| < 1, |y| < \infty \tag{1.4}$$

and

$$\Gamma_2(b, c; x, y) = \sum_{m,n=0}^{\infty} \frac{(b)_{n-m} (c)_{m-n}}{m!n!} x^m y^n; |x| < \infty, |y| < \infty. \tag{1.5}$$

Throughout this work, supposing that none of the numerator parameters a, b, c are zero or a negative integer, and with the usual restrictions (1.1), (1.2), and (1.3), the Horn's hypergeometric functions Γ_1 and Γ_2 were defined by the series (1.4) and (1.5) taking into consideration the parameter a satisfies the condition in (1.1) and the parameters b and c satisfy the conditions in (1.2) and (1.3).

To simplify the notations, we write Γ_1 for the series $\Gamma_1(a, b, c; x, y)$, $\Gamma_1(a \pm n)$ for the series $\Gamma_1(a \pm n, b, c; x, y)$,... and $\Gamma_2(c \pm n)$ stands for the series $\Gamma_2(b, c \pm n; x, y)$ etc.

Abul-Ez and Sayyed [1] and Sayyed [19] introduced an integral operator $\hat{I}$ as follows

$$\hat{I} = \frac{1}{x} \int_0^x dx + \frac{1}{y} \int_0^y dy.$$

For $|t| < 1$ and α satisfing the condition in (1.1), we have the binomial theorem (see [22])

$$(1-t)^{-\alpha} = \sum_{r=0}^{\infty} \frac{(\alpha)_r}{r!} t^r. \tag{1.6}$$

Motivated essentially by the demonstrated potential for applications of the Horn's functions Γ_1 and Γ_2 in many diverse areas of physical, mathematical, statistical sciences and engineering, this paper is organized as follows. In Section 2, we establish a number of new recursion formulas, various differential recursion formulas and generating relations for Horn's functions Γ_1. In Section 3, we present new an investigation of several classes of recursion formulas, differential recursion formulas, infinite summation formulas and integral representations for Horn's functions Γ_2. In Section 4, we introduce and investigate of integral operators for Horn's functions Γ_1 and Γ_2. Finally, some concluding remarks and observations of the results are obtained in Section 5.

2. Recursion formulas and differential recursion formulas of Γ_1

In this section, we give some investigation of several classes of recursion formulas, generating relations and relevant differential formulas for Horn's function Γ_1.

Theorem 2.1. *For $n \in \mathbb{N}$ and $b \neq 1$, recursion formulas for Horn's function Γ_1 are as follows*

$$\Gamma_1(a+n) = \Gamma_1 + \frac{cx}{b-1}\sum_{k=1}^{n}\Gamma_1(a+k, b-1, c+1; x, y) \tag{2.1}$$

and

$$\Gamma_1(a-n) = \Gamma_1 - \frac{cx}{b-1}\sum_{k=0}^{n-1}\Gamma_1(a-k, b-1, c+1; x, y). \tag{2.2}$$

Proof. From the definition of the Γ_1 (1.4) and transformation

$$(a+1)_m = \left(1+\frac{m}{a}\right)(a)_m,\ a \neq 0,$$

we get

$$\Gamma_1(a+1) = \Gamma_1 + \frac{cx}{b-1}\Gamma_1(a+1, b-1, c+1; x, y), \quad b \neq 1. \tag{2.3}$$

If we compute the Horn's function Γ_1 with the parameter $a+n$ by relation (2.3) for n times, we obtain the formula (2.1). From the contiguous relation (2.3) and replacing a by $a-1$, we get

$$\Gamma_1(a-1) = \Gamma_1 - \frac{cx}{b-1}\Gamma_1(a, b-1, c+1; x, y), b \neq 1.$$

If we apply this relation to the Γ_1 with the parameter $a-n$ for n times, we obtain (2.2). □

Theorem 2.2. *For $b \neq 1, 2, 3, \ldots$, the Horn's function Γ_1 satisfies the identity:*

$$\Gamma_1(a+n) = \sum_{k=0}^{n}\binom{n}{k}\frac{(-x)^k(c)_k}{(1-b)_k}\Gamma_1(a+k, b-k, c+k; x, y)$$

and

$$\Gamma_1(a-n) = \sum_{k=0}^{n}\binom{n}{k}\frac{x^k(c)_k}{(1-b)_k}\Gamma_1(a, b-k, c+k; x, y).$$

Proof. The proofs are omitted (mathematical induction method). □

Theorem 2.3. *For $n \in \mathbb{N}$ and $c \neq 1$, the recursion relations hold true for Horn's function Γ_1:*

$$\begin{aligned}\Gamma_1(b+n) =& \Gamma_1 + \frac{y}{c-1}\sum_{k=1}^{n}\Gamma_1(a, b+k, c-1; x, y)\\ &- acx\sum_{k=1}^{n}\frac{\Gamma_1(a+1, b+k-2, c+1; x, y)}{(b+k-1)(b+k-2)}, b \neq 1-k, b \neq 2-k\ \forall k \in \mathbb{N}\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}\Gamma_1(b-n) =& \Gamma_1 - \frac{y}{c-1}\sum_{k=1}^{n}\Gamma_1(a+1, b-k+1, c-1; x, y)\\ &+ acx\sum_{k=1}^{n}\frac{\Gamma_1(a+1, b-k-1, c+1; x, y)}{(b-k)(b-k-1)}, b \neq k, b \neq 1+k\ \forall k \in \mathbb{N}.\end{aligned} \tag{2.5}$$

Proof. From (1.4) and the relation

$$(b+1)_{n-m} = \left(1+\frac{n}{b}-\frac{m}{b}\right)(b)_{n-m}, b \neq 0,$$

we obtain the contiguous function

$$\begin{aligned}\Gamma_1(b+1) =&\Gamma_1 + \frac{y}{c-1}\Gamma_1(a, b+1, c-1; x, y)\\ &-\frac{acx}{b(b-1)}\Gamma_1(a+1, b-1, c+1; x, y); b \neq 0, b \neq 1, c \neq 1.\end{aligned} \tag{2.6}$$

By iterating this method on the above contiguous relation for Γ_1 with $b+n$ for n times, we get (2.4).

Replacing b by $b-1$ in contiguous relation (2.6), we obtain

$$\begin{aligned}\Gamma_1(b-1) =&\Gamma_1 - \frac{ay}{c-1}\Gamma_1(a+1, b, c-1; x, y)\\ &+\frac{acx}{(b-1)(b-2)}\Gamma_1(a+1, b-2, c+1; x, y), c, b \neq 1, b \neq 2.\end{aligned} \tag{2.7}$$

If we compute the Horn's function Γ_1 with the parameter $b-n$ by the contiguous relation (2.7) for n times, we obtain the recursion formula (2.5). □

Theorem 2.4. *For $n \in \mathbb{N}$ and $b \neq 1$, the Horn's function Γ_1 satisfies the relations:*

$$\begin{aligned}\Gamma_1(c+n) =&\Gamma_1 + \frac{ax}{b-1}\sum_{k=1}^{n}\Gamma_1(a+1, b-1, c+k; x, y)\\ &-by\sum_{k=1}^{n}\frac{\Gamma_1(a, b+1, c+k-2; x, y)}{(c+k-1)(c+k-2)}, c \neq 1-k, c \neq 2-k\ \forall k \in \mathbb{N}\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}\Gamma_1(c-n) =&\Gamma_1 - \frac{x}{b-1}\sum_{k=1}^{n}\Gamma_1(a+1, b-1, c+k-1; x, y)\\ &-aby\sum_{k=1}^{n}\frac{\Gamma_1(a+1, b+1, c-k-1; x, y)}{(c-k)(c-k-1)}, c \neq k, c \neq k+1\ \forall k \in \mathbb{N}.\end{aligned} \tag{2.9}$$

Proof. Using the definition of the Γ_1 and the equality

$$(c+1)_{m-n} = \left(1+\frac{m}{c}-\frac{n}{c}\right)(c)_{m-n}, c \neq 0,$$

we obtain the contiguous function

$$\begin{aligned}\Gamma_1(c+1) =&\Gamma_1 + \frac{ax}{b-1}\Gamma_1(a+1, b-1, c+1; x, y)\\ &-\frac{by}{c(c-1)}\Gamma_1(a, b+1, c-1; x, y), b \neq 1, c \neq 0, c \neq 1.\end{aligned} \tag{2.10}$$

By iterating this method on the above contiguous relation for Γ_1 with the parameter $c+n$ for n times, we obtain (2.8).

Replacing c by $c-1$ in the contiguous relation (2.10), we get

$$\begin{aligned}\Gamma_1(c-1) =&\Gamma_1 - \frac{ax}{b-1}\Gamma_1(a+1, b-1, c; x, y)\\ &+\frac{by}{(c-1)(c-2)}\Gamma_1(a, b+1, c-2; x, y), b, c \neq 1, b \neq 2.\end{aligned} \tag{2.11}$$

If we compute the Horn's function Γ_1 with the parameter $c-n$ by contiguous relation (2.11) for n times, we obtain (2.9). □

Now, we apply differential operators $\theta_x = x\frac{\partial}{\partial x}$ and $\theta_y = y\frac{\partial}{\partial y}$ and state the following theorem.

Theorem 2.5. *Differential recursion formula for the function* Γ_1 *is as follows*

$$\Gamma_1(a+1) = \left(1+\frac{\theta_x}{a}\right)\Gamma_1, a \neq 0. \tag{2.12}$$

Proof. Definie the differential operators

$$\theta_x x^m = m x^m, \text{ and } \theta_y y^n = n y^n.$$

By using the above differential operators and the transformation

$$(a+1)_m = \left(1+\frac{m}{a}\right)(a)_m, \ a \neq 0,$$

we get

$$\begin{aligned}
\Gamma_1(a+1) &= \sum_{m,n=0}^{\infty} \frac{(a+1)_m (b)_{n-m} (c)_{m-n}}{m!n!} x^m y^n \\
&= \sum_{m,n=0}^{\infty} \left(1+\frac{m}{a}\right) \frac{(a)_m (b)_{n-m} (c)_{m-n}}{m!n!} x^m y^n \\
&= \sum_{m,n=0}^{\infty} \frac{(a)_m (b)_{n-m} (c)_{m-n}}{m!n!} x^m y^n + \frac{1}{a} \sum_{m,n=0}^{\infty} \frac{m(a)_m (b)_{n-m} (c)_{m-n}}{m!n!} x^m y^n \\
&= \Gamma_1 + \frac{1}{a}\theta_x \Gamma_1, \ a \neq 0.
\end{aligned}$$

Hence, we obtain the differential recursion formula (2.12). □

Theorem 2.6. *The differential recursion formulas hold true for the Horn's function* Γ_1*:*

$$\Gamma_1(b+1) = \left(1+\frac{\theta_y}{b}-\frac{\theta_x}{b}\right)\Gamma_1, b \neq 0 \tag{2.13}$$

and

$$\Gamma_1(c+1) = \left(1+\frac{\theta_x}{c}-\frac{\theta_y}{c}\right)\Gamma_1, c \neq 0. \tag{2.14}$$

Proof. By defining the transformation

$$(b+1)_{n-m} = \left(1+\frac{n-m}{b}\right)(b)_{n-m}, b \neq 0$$

and using the differential operators for the Horn's function Γ_1, we get

$$\begin{aligned}
\Gamma_1(b+1) &= \sum_{m,n=0}^{\infty} \frac{(a)_m (b+1)_{n-m} (c)_{m-n}}{m!n!} x^m y^n \\
&= \sum_{m,n=0}^{\infty} \left(1+\frac{n-m}{b}\right) \frac{(a)_m (b)_{n-m} (c)_{m-n}}{m!n!} x^m y^n
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{m,n=0}^{\infty} \frac{(a)_m (b)_{n-m} (c)_{m-n}}{m!n!} x^m y^n + \frac{1}{b} \sum_{m,n=0}^{\infty} \frac{n(a)_m (b)_{n-m} (c)_{m-n}}{m!n!} x^m y^n \\
&\quad - \frac{1}{b} \sum_{m,n=0}^{\infty} \frac{m(a)_m (b)_{n-m} (c)_{m-n}}{m!n!} x^m y^n \\
&= \Gamma_1 + \frac{1}{b}\theta_y \Gamma_1 - \frac{1}{b}\theta_x \Gamma_1, b \neq 0.
\end{aligned}
$$

Thus, we obtain (2.13).

Using the relation

$$
(c+1)_{m-n} = \left(1 + \frac{m-n}{c}\right)(c)_{m-n}, c \neq 0
$$

and (1.4), after simplification we get (2.14). □

Theorem 2.7. *The derivative formulas hold true for Horn's function* Γ_1*,*

$$
\frac{\partial^r}{\partial x^r}\Gamma_1 = \frac{(-1)^r (a)_r (c)_r}{(1-b)_r}\Gamma_1(a+r, b-r, c+r; x, y), b \neq 1, 2, 3, \dots \tag{2.15}
$$

and

$$
\frac{\partial^r}{\partial y^r}\Gamma_1 = \frac{(-1)^r (b)_r}{(1-c)_r}\Gamma_1(a, b+r, c-r; x, y), c \neq 1, 2, 3, \dots. \tag{2.16}
$$

Proof. Differentiating (1.4) with respect to x, we get

$$
\frac{\partial}{\partial x}\Gamma_1 = \frac{ac}{b-1}\Gamma_1(a+1, b-1, c+1; x, y), b \neq 1.
$$

Repeating the above process, we eventually arrive at

$$
\begin{aligned}
\frac{\partial^r}{\partial x^r}\Gamma_1 &= \frac{a(a+1)\dots(a+r-1)c(c+1)\dots(c+r-1)}{(b-1)(b-2)\dots(b-r)}\Gamma_1(a+r, b-r, c+r; x, y) \\
&= \frac{(-1)^r (a)_r (c)_r}{(1-b)_r}\Gamma_1(a+r, b-r, c+r; x, y), b \neq 1, 2, 3, \dots.
\end{aligned}
$$

The differential recursion formulas (2.16) can be proved in similar manner (2.15). □

Theorem 2.8. *The infinite summation formulas of Horn's function* Γ_1 *hold true:*

$$
\sum_{k=0}^{\infty} \frac{(a)_k}{k!}\Gamma_1(a+k)t^k = (1-t)^{-a}\Gamma_1\left(a, b, c; \frac{x}{1-t}, y\right), |t| < 1, \tag{2.17}
$$

$$
\sum_{k=0}^{\infty} \frac{(b)_k}{k!}\Gamma_1(b+k)t^k = (1-t)^{-b}\Gamma_1\left(a, b, c; x(1-t), \frac{y}{1-t}\right), |t| < 1, \tag{2.18}
$$

and

$$
\sum_{k=0}^{\infty} \frac{(c)_k}{k!}\Gamma_1(c+k)t^k = (1-t)^{-c}\Gamma_1\left(a, b, c; \frac{x}{1-t}, y(1-t)\right), |t| < 1. \tag{2.19}
$$

Proof. Using (1.4), (1.3), and (1.6), we obtain (2.17), (2.18), and (2.19). □

3. Recursion formulas and differential recursion formulas for Horn's function Γ_2

In this section, we derive the recursion formulas, differential recursion formulas, generating relations and integral representations for Horn's function Γ_2 with five theorems as follows. First, we present the recursion formulas of Horn's function Γ_2.

Theorem 3.1. *For $n \in \mathbb{N}$, the recursion formulas of Γ_2 with parameters b and c are*

$$\begin{aligned}
\Gamma_2(b+n) =&\Gamma_2 + \frac{y}{c-1}\sum_{k=1}^{n}\Gamma_2(b+k, c-1; x, y)\\
&- cx\sum_{k=1}^{n}\frac{\Gamma_2(b+k-2, c+1; x, y)}{(b+k-1)(b+k-2)}, c\neq 1, b\neq 1-k, b\neq 2-k, \forall k\in\mathbb{N},\\
\Gamma_2(b-n) =&\Gamma_2 - \frac{y}{c-1}\sum_{k=1}^{n}\Gamma_2(b-k+1, c-1; x, y)\\
&- cx\sum_{k=1}^{n}\frac{\Gamma_2(b-k-1, c+1; x, y)}{(b-k)(b-k-1)}, c\neq 1, b\neq k, b\neq 1+k, \forall k\in\mathbb{N},\\
\Gamma_2(c+n) =&\Gamma_2 + \frac{x}{b-1}\sum_{k=1}^{n}\Gamma_2(b-1, c+k; x, y)\\
&- by\sum_{k=1}^{n}\frac{\Gamma_2(b+1, c+k-2; x, y)}{(c+k-1)(c+k-2)}, b\neq 1, c\neq 1-k, c\neq 2-k, \forall k\in\mathbb{N},\\
\Gamma_2(c-n) =&\Gamma_2 - \frac{x}{b-1}\sum_{k=1}^{n}\Gamma_2(b-1, c+k-1; x, y)\\
&- by\sum_{k=1}^{n}\frac{\Gamma_2(b+1, c-k-1; x, y)}{(c-k)(c-k-1)}, b\neq 1, c\neq k, c\neq 1+k, \forall k\in\mathbb{N}.
\end{aligned} \tag{3.1}$$

Proof. By the similar method in the Theorems 2.3-2.4, we obtain the recursion formulas for Horn's's function Γ_2, (3.1). □

Theorem 3.2. *The derivative formulas for the Horn's function Γ_2 hold true:*

$$\Gamma_2(b+1) = \left(1+\frac{\theta_y}{b}-\frac{\theta_x}{b}\right)\Gamma_2, b\neq 0$$

and

$$\Gamma_2(c+1) = \left(1+\frac{\theta_x}{c}-\frac{\theta_y}{c}\right)\Gamma_2, c\neq 0.$$

Proof. The proof of the current theorem is similar to the proof of Theorem 2.6. □

Theorem 3.3. *For the Horn's function Γ_2, the differential recurrence relations hold true*

$$\frac{\partial^r}{\partial x^r}\Gamma_2 = \frac{(-1)^r(c)_r}{(1-b)_r}\Gamma_2(b-r, c+r; x, y), b\neq 1,2,3,\dots \tag{3.2}$$

and

$$\frac{\partial^r}{\partial y^r}\Gamma_2 = \frac{(-1)^r(b)_r}{(1-c)_r}\Gamma_2(b+r, c-r; x, y), c\neq 1,2,3,\dots. \tag{3.3}$$

Proof. We obtain results (3.2) and (3.3) as same way as the proof of (2.15) and (2.16). □

In a similar manner in Theorem 2.8, we can obtain the next results.

Theorem 3.4. *The infinite summation formulas for Horn's function Γ_2 hold true:*

$$\sum_{k=0}^{\infty}\frac{(b)_k}{k!}\Gamma_2(b+k)t^k = (1-t)^{-b}\Gamma_2\left(b, c; x(1-t), \frac{y}{1-t}\right), |t|<1$$

and

$$\sum_{k=0}^{\infty}\frac{(c)_k}{k!}\Gamma_2(c+k)t^k=(1-t)^{-c}\Gamma_2\Big(b,c;\frac{x}{1-t},y(1-t)\Big),|t|<1.$$

4. Integral operators of Γ_1 and Γ_2

Here we apply integral operator of Γ_1 and Γ_2 and state the following theorem.

Theorem 4.1. *For $a\neq 1,2$, $b\neq 1,2$ and $c\neq 1,2$, the integration formula of the function Γ_1 holds true:*

$$\begin{aligned}\hat{I}^2\,\Gamma_1=&\frac{b(b+1)}{x^2(a-1)(a-2)(c-1)(c-2)}\,\Gamma_1(a-2,b+2,c-2;x,y)+\frac{2}{xy(a-1)}\,\Gamma_1(a-1,b,c;x,y)\\&+\frac{c(c+1)}{y^2(b-1)(b-2)}\,\Gamma_1(a,b-2,c+2;x,y),x,y\neq 0.\end{aligned}\tag{4.1}$$

Proof. Let $\hat{I}$ acts on the Horn's function Γ_1, then we have

$$\begin{aligned}\hat{I}\,\Gamma_1&=\sum_{m,n=0}^{\infty}\Big(\frac{1}{m+1}+\frac{1}{n+1}\Big)\frac{(a)_m(b)_{n-m}(c)_{m-n}}{m!n!}x^my^n\\&=\sum_{m=1,n=0}^{\infty}\frac{(a)_m(b)_{n-m}(c)_{m-n}}{(m+1)!n!}x^my^n+\sum_{m=0,n=1}^{\infty}\frac{(a)_m(b)_{n-m}(c)_{m-n}}{m!(n+1)!}x^my^n\\&=\sum_{m,n=0}^{\infty}\frac{(a)_{m-1}(b)_{n-m+1}(c)_{m-n-1}}{m!n!}x^{m-1}y^n+\sum_{m,n=0}^{\infty}\frac{(a)_m(b)_{n-m-1}(c)_{m-n+1}}{m!n!}x^my^{n-1}\\&=\frac{b}{x(a-1)(c-1)}\sum_{m,n=0}^{\infty}\frac{(a-1)_m(b+1)_{n-m}(c-1)_{m-n}}{m!n!}x^my^n\\&\quad+\frac{c}{y(b-1)}\sum_{m,n=0}^{\infty}\frac{(a)_m(b-1)_{n-m}(c+1)_{m-n}}{m!n!}x^my^n\\&=\frac{b}{x(a-1)(c-1)}\,\Gamma_1(a-1,b+1,c-1;x,y)+\frac{c}{y(b-1)}\,\Gamma_1(a,b-1,c+1;x,y),x,y\neq 0,a,c,b\neq 1.\end{aligned}$$

We can writte $\hat{I}=\hat{I}_x+\hat{I}_y$, where $\hat{I}_x=\frac{1}{x}\int_0^x dx$ and $\hat{I}_y=\frac{1}{y}\int_0^y dy$, then the operator $\hat{I}^2$ is such that

$$\hat{I}^2=\hat{I}\hat{I}=(\hat{I}_x)^2+2\hat{I}_x\hat{I}_y+(\hat{I}_x)^2=\frac{1}{x^2}\int_0^x\int_0^x dxdx+\frac{2}{xy}\int_0^y\int_0^x dxdy+\frac{1}{y^2}\int_0^y\int_0^y dydy.$$

By using the above integral operator $\hat{I}^2$, we obtain the relation (4.1). □

Theorem 4.2. *The integration formula of the function Γ_1 holds true:*

$$\hat{I}^r\,\Gamma_1=\frac{(-1)^r}{x^ry^r(1-a)_r}\prod_{k=1}^{r}(\theta_x+\theta_y-k+1)\Gamma_1(a-r),\ x,y\neq 0,a\neq 1,2,3,\dots.\tag{4.2}$$

Proof. With the help of the integral operator $\hat{I}$ and differential operators, we get the formula

$$\begin{aligned}\hat{I}\,\Gamma_1&=\sum_{m,n=1}^{\infty}\frac{(m+n+2)(a)_m(b)_{n-m}(c)_{m-n}}{(m+1)!(n+1)!}x^my^n\\&=\sum_{m,n=0}^{\infty}\frac{(m+n)(a)_{m-1}(b)_{n-m}(c)_{m-n}}{m!n!}x^{m-1}y^{n-1}\\&=\frac{\theta_x+\theta_y}{xy(a-1)}\,\Gamma_1(a-1),\ x,y\neq 0,a\neq 1.\end{aligned}$$

Iterating this integral operator $\hat{I}$ and differential operators on Γ_1 for r-times, we get the formula (4.2). □

Theorem 4.3. *For Horn's function* Γ_1*, we have the integral operators* $\hat{I}_x^r$ *and* $\hat{I}_y^r$*:*

$$\hat{I}_x^r\,\Gamma_1 = \frac{(b)_r}{x^r(1-a)_r(1-c)_r}\,\Gamma_1(a-r,b+r,c-r;x,y),\ x\neq 0, a,c\neq 1,2,3,\ldots, \tag{4.3}$$

$$\hat{I}_y^r\,\Gamma_1 = \frac{(-1)^r(c)_r}{y^r(1-b)_r}\,\Gamma_1(a,b-r,c+r;x,y),\ y\neq 0, b\neq 1,2,3,\ldots \tag{4.4}$$

and

$$\left(\hat{I}_x\hat{I}_y\right)^r\,\Gamma_1 = \frac{(-1)^r}{x^ry^r(1-a)_r}\,\Gamma_1(a-r),\ x,y\neq 0, a\neq 1,2,3,\ldots. \tag{4.5}$$

Proof. Also, we get relations concerning $\hat{I}_x$ and $\hat{I}_y$ individually, we get

$$\hat{I}_x\,\Gamma_1 = \frac{b}{x(a-1)(c-1)}\,\Gamma_1(a-1,b+1,c-1;x,y), x\neq 0, a,c\neq 1.$$

By applying the operation of the above relation for r times, we obtain the desired (4.3).

Similarly, for I_y, applying operation in the proof of the relation (4.3), we obtain (4.4) and (4.5). □

Theorem 4.4. *For* $b\neq 1,2$ *and* $c\neq 1,2$*, the integration formula of the function* Γ_2 *holds true:*

$$\begin{aligned}\hat{I}^2\,\Gamma_2 =& \frac{b(b+1)}{x^2(c-1)(c-2)}\,\Gamma_2(b+2,c-2;x,y)+\frac{2}{xy}\,\Gamma_2\\ &+\frac{c(c+1)}{y^2(b-1)(b-2)}\,\Gamma_2(b-2,c+2;x,y),\ x,y\neq 0.\end{aligned}$$

Theorem 4.5. *The integration formula of the function* Γ_2 *holds true:*

$$\hat{I}^r\,\Gamma_2 = \frac{1}{x^ry^r}\prod_{k=1}^{r}(\theta_x+\theta_y-k+1)\Gamma_2,\ x,y\neq 0.$$

Theorem 4.6. *For Horn's function* Γ_2*, we have the integral operators* $\hat{I}_x^r$ *and* $\hat{I}_y^r$*:*

$$\begin{aligned}\hat{I}_x^r\,\Gamma_2 &= \frac{(b)_r}{x^r(1-c)_r}\,\Gamma_2(b+r,c-r;x,y),\ x\neq 0, c\neq 1,2,3,\ldots,\\ \hat{I}_y^r\,\Gamma_2 &= \frac{(-1)^r(c)_r}{y^r(1-b)_r}\,\Gamma_2(b-r,c+r;x,y),\ y\neq 0, b\neq 1,2,3,\ldots,\\ \left(\hat{I}_x\hat{I}_y\right)^r\,\Gamma_2 &= \frac{(-1)^r}{x^ry^r}\,\Gamma_2, x,y\neq 0.\end{aligned}$$

5. Concluding remarks and observations

As a direct consequence of the several recursion formulas and differential recursion formulas of Horn's functions Γ_1 and Γ_2 which we have established here, the infinite summation formulas, integral operators and integral representations for Γ_1 and Γ_2 have been discussed. Our analytic expressions can be used as a benchmark for an accuracy of a different approximation techniques designed especially for an investigation of radiation field problems.

Acknowledgment

The authors would like to thank the anonymous referees for their valuable comments and suggestions which improved the quality and clarity of the paper.

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