J. Math. Computer Sci., 23 (2021), 36–48

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

A new perturbed smooth penalty function for inequality constrained optimization

Touna Yang[a,*], Nguyen Thanh Binh[b,*], Truong Minh Chinh[c], Le Huu Tung[c], Duong Thi Hoa[d]

[a]*Department of Applied Mathematics, Northwestern Polytechnical University, Xi'an 710072, China.*

[b]*Department of Education and Training Yen Bai, Yen Bai city 32000, Vietnam.*

[c]*Viet Tri University of Industry, Viet Tri city 35000, Vietnam.*

[d]*Yen Bai Teacher's Training College, Yen Bai city 32000, Vietnam.*

Abstract

In this paper, we first propose a new smoothing approach to the nonsmooth penalty function for constrained optimization problems, and then show that an approximate optimal solution of the original problem can be obtained by solving an optimal solution of the smoothed optimization problem. Based on the perturbed smooth exact penalty function, we develop an algorithm respectively to finding an approximate optimal solution of the original constrained optimization problem and prove the convergence of the proposed algorithm. The effectiveness of the smoothed penalty function is illustrated through three examples, which show that the algorithm seems efficient.

Keywords: Exact penalty function, inequality constrained optimization, smoothing method, optimal solution.

2020 MSC: 90C30, 65K05, 57R12.

1. Introduction

Consider the following constrained optimization problem:

$$\begin{aligned} &\min\ f(x) \\ &\text{s.t.}\ \ g_i(x) \leqslant 0, \quad i \in I = \{1, 2, \dots, m\}, \\ &x \in \mathbb{R}^n, \end{aligned} \tag{P}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $g_i : \mathbb{R}^n \to \mathbb{R}$, $i \in I$, are twice continuously differentiable functions. Let $X_0 = \{x \in \mathbb{R}^n \mid g_i(x) \leqslant 0,\ i \in I\}$ be the feasible set of (P) and we assume that X_0 is not empty. This problem is widely applied in the fields such as mathematical programming, economy, transportation, network structures, etc, and it has received extensive attention on a related problem, for example, interval-valued optimization problems, equilibrium problem, bilevel programming, etc (see, e.g., [1, 8, 11, 15, 17–19]). Up

*Corresponding author

Email addresses: touna1988@yahoo.com (Touna Yang), bingbongyb@gmail.com (Nguyen Thanh Binh)

doi: 10.22436/jmcs.023.01.04

Received: 2019-12-05 *Revised:* 2020-01-24 *Accepted:* 2020-04-22

to now, some efficient methods have been studied for solving general constrained optimization problem, including the penalty function methods. The penalty function methods have been proposed in order to transform a constrained optimization problem to an unconstrained optimization problem. Since exact penalty function method in solving (P) was proposed by Zangwill [24], it has attracted researchers and lots of penalty functions have been introduced in literature.

According to Zangwill [24], the l_1 exact penalty function has been defined by

$$\omega^1_\rho(x) = f(x) + \rho \sum_{i=1}^{m} \max\{0, g_i(x)\},$$

where $\rho > 0$ is a penalty parameter. It is proved that there exists a fixed constant $\rho_0 > 0$, for any $\rho > \rho_0$, any global solution of the exact penalty problem is also a global solution of the original problem. Therefore, the exact penalty function methods have been widely used for solving constrained optimization problems (see, e.g., [2, 7, 9, 10, 16, 23, 24]).

Recently, a class of lower order penalty functions has been investigated in [20] as the following form

$$\omega^k_\rho(x) = f(x) + \rho \sum_{i=1}^{m} \max\{0, g_i(x)\}^k, \quad k \in (0,\ 1). \tag{1.1}$$

Obviously, the lower order penalty problem and the original problem have the same set of global minima when the penalty parameter is sufficiently large. If $k = 1$ the lower order penalty function $\omega^k_\rho(x)$ is reduced to the l_1 exact penalty function. However, both the penalty function $\omega^k_\rho(x)$ $(0 < k < 1)$ and the l_1 exact penalty function are non-smooth (non-Lipschitz). One of the important tools for solving these types of non-smooth problems is the smoothing approach. The smoothing approach is based on making some modification on the objective function or approximate the objective function by smooth functions. Thus smoothing approach for the exact penalty function have been proposed in the literature (see, e.g., [3–6, 12, 14, 21, 22, 25]). Pinar and Zenios [14] first proposed a smoothing method to the exact penalty functions, it was shown that an approximate solution of the original convex programming problem can be obtained by solving the smoothed penalty problem. Meng et al. [12] discussed two smoothing approximations to the lower order penalty function (1.1) with $k = \frac{1}{2}$. Wu et al. [21] proposed a quadratic smoothing approximation to the l_1 exact penalty function. It is shown that under certain conditions, any global minimizer of the smoothed penalty problem is a global minimizer of the original problem when the penalty parameter is sufficiently large. Wu et al. [20] and Binh [4] presented the ϵ-smoothing of (1.1), and obtained a modified exact penalty function under some mild conditions. Binh et al. [5] also proposed two new perturbed smoothing approach to the lower order exact penalty function, which was proved to have good prospects in solving a global approximate solution to the constrained optimization problem.

In this paper, we introduce a new smoothing method for the low order penalty function and compare it with the existing methods. With a different segmentation method, we will give a new piecewise smooth function and propose a new method to smooth the lower order penalty function (1.1).

The rest of this paper is organized as follows. In Section 2, we propose a new smoothing method and prove some results for error estimates among the optimal objective function values of the nonsmooth penalty problem, smoothed penalty problem and original constrained optimization problem. In Section 3, we construct the minimization algorithm to finding an approximate optimal solution of the inequality constrained optimization problems. In Section 4, some numerical examples are given. Finally, conclusions are discussed in Section 5.

2. A new perturbed smoothing method

For any $k \in (0,1)$, let:

$$q^k(t) = \begin{cases} 0, & \text{if } t \leqslant 0, \\ t^k, & \text{if } t \geqslant 0. \end{cases}$$

Then, we have the following low order penalty function:

$$\omega_\rho^k(x) = f(x) + \rho \sum_{i=1}^{m} q^k(g_i(x)), \tag{2.1}$$

and the corresponding penalty problem

$$\min\ \omega_\rho^k(x), \quad \text{s.t. } x \in \mathbb{R}^n. \tag{P_ρ}$$

For the penalty problem (P_ρ), in order to establish the global exact penalization, the following assumption is given in [20]. We will consider a new perturbed smoothing method under the following assumption.

Assumption 2.1.

(i) $f(x)$ *satisfies the following coercive condition:*

$$\lim_{\|x\| \to +\infty} f(x) = +\infty.$$

(ii) *The set* $G(P)$ *is a finite set, where* $G(P)$ *is the set of global solutions of* (P).

Under Assumption 2.1, problem (P) is equivalent to the following problem:

$$\begin{aligned} &\min\ f(x),\\ &\text{s.t. } g_i(x) \leqslant 0, \quad i \in I,\\ &x \in X \subset \mathbb{R}^n, \end{aligned} \tag{P'}$$

where X is a box with $\text{int}(X) \neq \emptyset$.

For any $k \in (0, 1)$, problem (P_ρ) is equivalent to the following problem:

$$\min\ \omega_\rho^k(x), \ \text{s.t. } x \in X. \tag{P'_ρ}$$

Now, we consider the perturbed smooth exact penalty function. For $\frac{1}{2} \leqslant k < 1$, $\epsilon > 0$ and $\rho > 0$, the function $q_{\epsilon,\rho}^k(t)$ is defined as:

$$q_{\epsilon,\rho}^k(t) = \begin{cases} 0, & \text{if } t \leqslant -\left(\frac{\epsilon}{m\rho}\right)^k,\\ \dfrac{km\rho}{2\epsilon}\left[t + \left(\dfrac{\epsilon}{m\rho}\right)^k\right]^2, & \text{if } -\left(\frac{\epsilon}{m\rho}\right)^k < t < 0,\\ \left(t + \dfrac{\epsilon}{m\rho}\right)^k + \dfrac{k}{2}\left(\dfrac{\epsilon}{m\rho}\right)^{2k-1} - \left(\dfrac{\epsilon}{m\rho}\right)^k, & \text{if } t \geqslant 0. \end{cases}$$

Remark 2.2. Obviously, $q_{\epsilon,\rho}^k(t)$ has the following attractive properties:

(i) $q_{\epsilon,\rho}^k(t)$ is continuously differentiable for $\frac{1}{2} \leqslant k < 1$ on $\mathbb{R}$, where

$$[q_{\epsilon,\rho}^k(t)]' = \begin{cases} 0, & \text{if } t \leqslant -\left(\frac{\epsilon}{m\rho}\right)^k,\\ \dfrac{km\rho}{\epsilon}\left[t + \left(\dfrac{\epsilon}{m\rho}\right)^k\right], & \text{if } -\left(\frac{\epsilon}{m\rho}\right)^k < t < 0,\\ k\left(t + \dfrac{\epsilon}{m\rho}\right)^{k-1}, & \text{if } t \geqslant 0; \end{cases}$$

(ii) $\lim_{\epsilon\to 0^+} q^k_{\epsilon,\rho}(t) = q^k(t)$;

(iii) $q^k(t) \leqslant q^k_{\epsilon,\rho}(t), \quad \forall t \in \mathbb{R}$.

The behavior of $q^k(t)$ and $q^k_{\epsilon,\rho}(t)$ is illustrated in Figure 1.

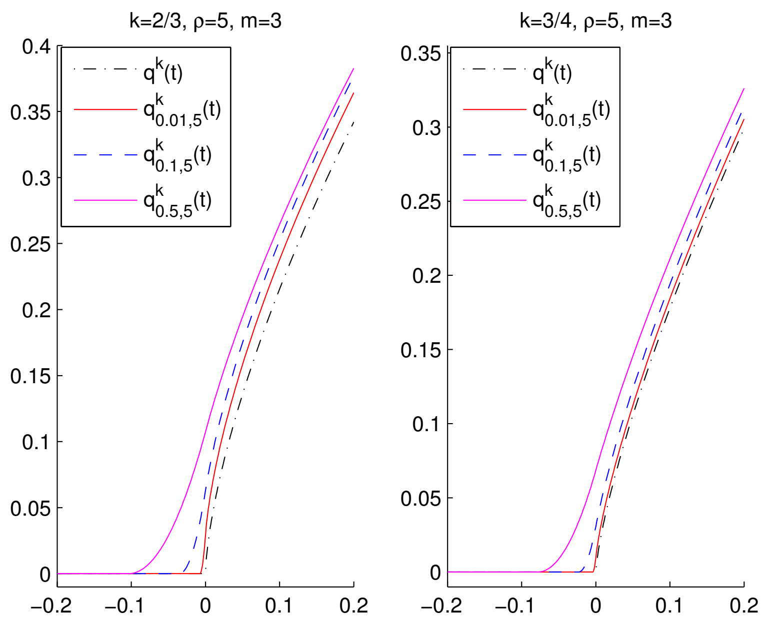

Figure 1: The behavior of $q^k(t)$ and $q^k_{\epsilon,\rho}(t)$.

Based on this, consider the perturbed smooth exact penalty function as follows:

$$\omega^k_{\epsilon,\rho}(x) = f(x) + \rho \sum_{i=1}^{m} q^k_{\epsilon,\rho}\left(g_i(x)\right),$$

where $\epsilon > 0$, $\rho > 0$. Clearly, $\omega^k_{\epsilon,\rho}(x)$ is continuously differentiable at any $x \in \mathbb{R}^n$. The corresponding smoothed optimization problem is:

$$\min\ \omega^k_{\epsilon,\rho}(x), \quad \text{s.t.} \quad x \in X. \tag{$SP_{\epsilon,\rho}$}$$

Lemma 2.3. *For any* $x \in X$, $\epsilon > 0$ *and* $\rho > 0$, *then*

$$-\frac{k}{2}\epsilon^{2k-1}(m\rho)^{2(1-k)} \leqslant \omega^k_\rho(x) - \omega^k_{\epsilon,\rho}(x) < \epsilon^k(m\rho)^{1-k}, \qquad k \in \left[\frac{1}{2}, 1\right).$$

Proof. For any $x \in X$, we have

$$\omega^k_\rho(x) - \omega^k_{\epsilon,\rho}(x) = \rho \sum_{i=1}^{m} \left(q^k(g_i(x)) - q^k_{\epsilon,\rho}(g_i(x))\right).$$

Note that

$$q^k\left(g_i(x)\right) - q^k_{\epsilon,\rho}\left(g_i(x)\right)$$

$$= \begin{cases} 0, & \text{if } g_i(x) \leqslant -\left(\frac{\epsilon}{m\rho}\right)^k, \\ -\dfrac{km\rho}{2\epsilon}\left[g_i(x) + \left(\dfrac{\epsilon}{m\rho}\right)^k\right]^2, & \text{if } -\left(\frac{\epsilon}{m\rho}\right)^k < g_i(x) < 0, \\ [g_i(x)]^k - \left[g_i(x) + \dfrac{\epsilon}{m\rho}\right]^k - \dfrac{k}{2}\left(\dfrac{\epsilon}{m\rho}\right)^{2k-1} + \left(\dfrac{\epsilon}{m\rho}\right)^k, & \text{if } g_i(x) \geqslant 0, \end{cases}$$

for any $i \in I$. If $g_i(x) \geqslant 0$, let $g_i(x) = u$. Then, we have $u \geqslant 0$. Consider the function

$$G(u) = u^k - \left(u + \frac{\epsilon}{m\rho}\right)^k, \qquad u \geqslant 0,$$

and we have

$$G'(u) = k\left[u^{k-1} - \left(u + \frac{\epsilon}{m\rho}\right)^{k-1}\right], \qquad u \geqslant 0.$$

Obviously, $G(u)$ is monotonically increasing in $u \geqslant 0$ for $\frac{1}{2} \leqslant k < 1$. One has

$$-\left(\frac{\epsilon}{m\rho}\right)^k \leqslant [g_i(x)]^k - \left[g_i(x) + \frac{\epsilon}{m\rho}\right]^k \leqslant 0.$$

It follows that

$$-\frac{k}{2}\left(\frac{\epsilon}{m\rho}\right)^{2k-1} \leqslant q^k\left(g_i(x)\right) - q^k_{\epsilon,\rho}\left(g_i(x)\right) \leqslant \left(\frac{\epsilon}{m\rho}\right)^k.$$

When $-\left(\frac{\epsilon}{m\rho}\right)^k < g_i(x) < 0$, one has

$$-\frac{k}{2}\left(\frac{\epsilon}{m\rho}\right)^{2k-1} < q^k\left(g_i(x)\right) - q^k_{\epsilon,\rho}\left(g_i(x)\right) < 0.$$

So, we have

$$-\frac{k}{2}\left(\frac{\epsilon}{m\rho}\right)^{2k-1} \leqslant q^k\left(g_i(x)\right) - q^k_{\epsilon,\rho}\left(g_i(x)\right) < \left(\frac{\epsilon}{m\rho}\right)^k, \qquad i = 1, 2, \ldots, m.$$

Adding up for all i, we obtain

$$-\frac{k}{2}\left(\frac{\epsilon}{m\rho}\right)^{2k-1} m\rho \leqslant \rho\sum_{i=1}^{m}\left(q^k(g_i(x)) - q^k_{\epsilon,\rho}(g_i(x))\right) < \left(\frac{\epsilon}{m\rho}\right)^k m\rho.$$

Therefore,

$$-\frac{k}{2}\epsilon^{2k-1}(m\rho)^{2(1-k)} \leqslant \omega^k_\rho(x) - \omega^k_{\epsilon,\rho}(x) < \epsilon^k(m\rho)^{1-k}.$$

□

Lemma 2.3 means that the gap between $\omega^k_\rho(x)$ and $\omega^k_{\epsilon,\rho}(x)$ can be arbitrarily small if the smoothing parameter ϵ is sufficiently small.

Theorem 2.4. *Let $\{\epsilon_j\} \to 0$, $\forall \epsilon_j > 0$, and x_j be a solution of $(SP_{\epsilon_j,\rho})$ for $\rho > 0, k \in \left[\frac{1}{2}, 1\right)$. Assume that x' is an accumulation point of $\{x_j\}$. Then x' is an optimal solution of (P_ρ).*

Proof. Since x_j is a solution of $(SP_{\epsilon_j,\rho})$, we have

$$\omega^k_{\epsilon_j,\rho}(x_j) \leqslant \omega^k_{\epsilon_j,\rho}(x).$$

By Lemma 2.3, we have

$$\omega^k_\rho(x_j) < \omega^k_{\epsilon_j,\rho}(x_j) + \epsilon_j^k(m\rho)^{1-k} \leqslant \omega^k_{\epsilon_j,\rho}(x) + \epsilon_j^k(m\rho)^{1-k} \leqslant \omega^k_\rho(x) + \frac{k}{2}\epsilon_j^{2k-1}(m\rho)^{2(1-k)} + \epsilon_j^k(m\rho)^{1-k}.$$

Since $\{\epsilon_j\} \to 0$ and x' is an accumulation point of $\{x_j\}$, we obtain

$$\omega^k_\rho(x') \leqslant \omega^k_\rho(x).$$

Thus x' is an optimal solution of (P_ρ). □

Theorem 2.5. *Let x^*_ρ be an optimal solution of (P'_ρ) and $x^*_{\epsilon,\rho} \in X$ be an optimal solution of $(SP_{\epsilon,\rho})$ for some $\rho > 0$ and $\epsilon > 0$. Then we have*

$$-\frac{k}{2}\epsilon^{2k-1}(m\rho)^{2(1-k)} \leqslant \omega^k_\rho(x^*_\rho) - \omega^k_{\epsilon,\rho}(x^*_{\epsilon,\rho}) < \epsilon^k(m\rho)^{1-k}, \qquad k \in \left[\frac{1}{2}, 1\right).$$

Proof. From Lemma 2.3, we obtain

$$-\frac{k}{2}\epsilon^{2k-1}(m\rho)^{2(1-k)} \leqslant \omega^k_\rho(x^*_\rho) - \omega^k_{\epsilon,\rho}(x^*_\rho) \leqslant \omega^k_\rho(x^*_\rho) - \omega^k_{\epsilon,\rho}(x^*_{\epsilon,\rho}) \leqslant \omega^k_\rho(x^*_{\epsilon,\rho}) - \omega^k_{\epsilon,\rho}(x^*_{\epsilon,\rho}) < \epsilon^k(m\rho)^{1-k}.$$

Therefore,

$$-\frac{k}{2}\epsilon^{2k-1}(m\rho)^{2(1-k)} \leqslant \omega^k_\rho(x^*_\rho) - \omega^k_{\epsilon,\rho}(x^*_{\epsilon,\rho}) < \epsilon^k(m\rho)^{1-k}.$$

□

Theorem 2.6. *Suppose that Assumption 2.1 holds, and for any $x^* \in G(P)$, there exists a $\mu \in \mathbb{R}^m_+$ such that the pair (x^*, μ^*) satisfies the second-order sufficiency condition of problem (P) (in [20]). Let x^* be an optimal solution of (P) and $x^*_{\epsilon,\rho} \in X$ be an optimal solution of $(SP_{\epsilon,\rho})$. Then there exists $\rho_0 > 0$ such that for any $\rho > \rho_0$, it holds that*

$$-\frac{k}{2}\epsilon^{2k-1}(m\rho)^{2(1-k)} \leqslant f(x^*) - \omega^k_{\epsilon,\rho}(x^*_{\epsilon,\rho}) < \epsilon^k(m\rho)^{1-k}.$$

Proof. From [20, Corollary 2.3], we have that x^* is an optimal solution of (P'_ρ). By Theorem 2.5, we have the following:

$$-\frac{k}{2}\epsilon^{2k-1}(m\rho)^{2(1-k)} \leqslant \omega^k_\rho(x^*) - \omega^k_{\epsilon,\rho}(x^*_{\epsilon,\rho}) < \epsilon^k(m\rho)^{1-k}.$$

Note that

$$\omega^k_\rho(x^*) = f(x^*) + \rho\sum_{i=1}^m q^k(g_i(x^*)).$$

Since $\sum_{i=1}^m q^k(g_i(x^*)) = 0$, we have $\omega^k_\rho(x^*) = f(x^*)$. Thus, we obtain

$$-\frac{k}{2}\epsilon^{2k-1}(m\rho)^{2(1-k)} \leqslant f(x^*) - \omega^k_{\epsilon,\rho}(x^*_{\epsilon,\rho}) < \epsilon^k(m\rho)^{1-k}.$$

□

Definition 2.7. A point $x^*_\epsilon \in X$ is called ϵ-feasible solution of (P), if it satisfies $g_i(x^*_\epsilon) \leqslant \epsilon$, $\forall i \in I$.

Theorem 2.8. *Let x^*_ρ be an optimal solution of (P'_ρ) and $x^*_{\epsilon,\rho} \in X$ be an optimal solution of ($SP_{\epsilon,\rho}$) for some $\rho > 0$ and $\epsilon > 0$. If x^*_ρ is a feasible solution of (P), and $x^*_{\epsilon,\rho}$ is an ϵ-feasible solution of (P), then we have*

$$-\frac{k}{2}\epsilon^{2k-1}(m\rho)^{2(1-k)} \leqslant f(x^*_\rho) - f(x^*_{\epsilon,\rho}) < 2^k\epsilon^k(m\rho)^{1-k} + \frac{k}{2}\epsilon^{2k-1}(m\rho)^{2(1-k)}.$$

Proof. By Theorem 2.5, we have

$$-\frac{k}{2}\epsilon^{2k-1}(m\rho)^{2(1-k)} \leqslant f(x^*_\rho) + \rho\sum_{i=1}^{m} q^k(g_i(x^*_\rho)) - \left(f(x^*_{\epsilon,\rho}) + \rho\sum_{i=1}^{m} q^k_{\epsilon,\rho}(g_i(x^*_{\epsilon,\rho}))\right) < \epsilon^k(m\rho)^{1-k}.$$

Since x^*_ρ is a feasible solutions of (P), we have $\sum_{i=1}^m q^k(g_i(x^*_\rho)) = 0$. It follows that

$$\begin{aligned}
-\frac{k}{2}\epsilon^{2k-1}(m\rho)^{2(1-k)} + \rho\sum_{i=1}^{m} q^k_{\epsilon,\rho}(g_i(x^*_{\epsilon,\rho})) &\leqslant f(x^*_\rho) - f(x^*_{\epsilon,\rho}) \\
&< \epsilon^k(m\rho)^{1-k} + \rho\sum_{i=1}^{m} q^k_{\epsilon,\rho}(g_i(x^*_{\epsilon,\rho})).
\end{aligned} \tag{2.2}$$

Note that $g_i(x^*_{\epsilon,\rho}) \leqslant \epsilon$, $\forall i \in I$. Thus, it follows from the definition of $q^k_{\epsilon,\rho}(t)$ that

$$0 \leqslant \rho\sum_{i=1}^{m} q^k_{\epsilon,\rho}(g_i(x^*_{\epsilon,\rho})) \leqslant 2^k\epsilon^k(m\rho)^{1-k} + \frac{k}{2}\epsilon^{2k-1}(m\rho)^{2(1-k)} - \epsilon^k(m\rho)^{1-k}. \tag{2.3}$$

Combining equations (2.2) and (2.3), we have

$$-\frac{k}{2}\epsilon^{2k-1}(m\rho)^{2(1-k)} \leqslant f(x^*_\rho) - f(x^*_{\epsilon,\rho}) < 2^k\epsilon^k(m\rho)^{1-k} + \frac{k}{2}\epsilon^{2k-1}(m\rho)^{2(1-k)}.$$

□

By Theorem 2.8, an optimal solution of (P) can be controlled through the smoothing parameter ϵ, and the optimal solution of ($SP_{\epsilon,\rho}$) is an approximate optimal solution of (P) if $x^*_{\epsilon,\rho}$ is an ϵ-feasible solution of (P).

Definition 2.9. A feasible solution x^* of (P) is called a KKT point, if there exists a $\mu^* \in \mathbb{R}^m$ such that the pair (x^*, μ^*) satisfies the following conditions

$$\nabla f(x^*) + \sum_{i=1}^{m} \mu^*_i \nabla g_i(x^*) = 0, \tag{2.4}$$

$$\mu^*_i g_i(x^*) = 0,\ g_i(x^*) \leqslant 0,\ \mu^*_i \geqslant 0,\quad i \in I. \tag{2.5}$$

Theorem 2.10. *Suppose the functions f, g_i $(i \in I)$ in problem (P) are convex. Let x^* be an optimal solution of (P) and $x^*_{\epsilon,\rho} \in X$ be an optimal solution of ($SP_{\epsilon,\rho}$). If $x^*_{\epsilon,\rho}$ is feasible of (P), and there exists a $\mu^* \in \mathbb{R}^m$ such that the pair $(x^*_{\epsilon,\rho}, \mu^*)$ satisfies the conditions in equations (2.4) and (2.5), then we have*

$$0 \leqslant f(x^*_{\epsilon,\rho}) - f(x^*) < \frac{k}{2}\epsilon^{2k-1}(m\rho)^{2(1-k)} + \epsilon^k(m\rho)^{1-k}.$$

Proof. Since f, g_i $(i \in I)$ are continuously differentiable and convex, we see that

$$f(x^*) \geqslant f(x^*_{\epsilon,\rho}) + \nabla f(x^*_{\epsilon,\rho})^T(x^* - x^*_{\epsilon,\rho}), \tag{2.6}$$

$$g_i(x^*) \geqslant g_i(x^*_{\epsilon,\rho}) + \nabla g_i(x^*_{\epsilon,\rho})^T(x^* - x^*_{\epsilon,\rho}),\quad i = 1, 2, \ldots, m. \tag{2.7}$$

By Equations (2.1), (2.4), (2.5), (2.6) and (2.7), we have the following:

$$\begin{aligned}
\omega_{\rho}^{k}(x^*) &= f(x^*) + \rho \sum_{i=1}^{m} q^{k}(g_i(x^*)) \\
&\geqslant f(x^*_{\epsilon,\rho}) + \nabla f(x^*_{\epsilon,\rho})^{T}(x^* - x^*_{\epsilon,\rho}) \\
&= f(x^*_{\epsilon,\rho}) - \sum_{i=1}^{m} \mu_i^* \nabla g_i(x^*_{\epsilon,\rho})^{T}(x^* - x^*_{\epsilon,\rho}) \\
&\geqslant f(x^*_{\epsilon,\rho}) - \sum_{i=1}^{m} \mu_i^* \left[g_i(x^*) - g_i(x^*_{\epsilon,\rho})\right] = f(x^*_{\epsilon,\rho}) - \sum_{i=1}^{m} \mu_i^* g_i(x^*) \geqslant f(x^*_{\epsilon,\rho}).
\end{aligned}$$

From Lemma 2.3, we obtain

$$\omega_{\rho}^{k}(x^*) < \omega_{\epsilon,\rho}^{k}(x^*) + \epsilon^{k}(m\rho)^{1-k}.$$

It follows that

$$\begin{aligned}
f(x^*_{\epsilon,\rho}) < \omega^{k}_{\epsilon,\rho}(x^*) + \epsilon^{k}(m\rho)^{1-k} &= f(x^*) + \rho \sum_{i=1}^{m} q^{k}_{\epsilon,\rho}(g_i(x^*)) + \epsilon^{k}(m\rho)^{1-k} \\
&\leqslant f(x^*) + \frac{k}{2}\epsilon^{2k-1}(m\rho)^{2(1-k)} + \epsilon^{k}(m\rho)^{1-k}.
\end{aligned} \tag{2.8}$$

Since $x^*_{\epsilon,\rho}$ is feasible of (P), which is

$$f(x^*) \leqslant f(x^*_{\epsilon,\rho}). \tag{2.9}$$

Combining equations (2.8) and (2.9), we have

$$f(x^*) \leqslant f(x^*_{\epsilon,\rho}) < f(x^*) + \frac{k}{2}\epsilon^{2k-1}(m\rho)^{2(1-k)} + \epsilon^{k}(m\rho)^{1-k},$$

which is

$$0 \leqslant f(x^*_{\epsilon,\rho}) - f(x^*) < \frac{k}{2}\epsilon^{2k-1}(m\rho)^{2(1-k)} + \epsilon^{k}(m\rho)^{1-k}.$$

□

3. Algorithm for minimization procedure

In this section, by considering the above smoothed penalty function, we propose algorithm to find an approximate optimal solution of (P), defined as Algorithm 3.1.

Algorithm 3.1 (Algorithm for solving problem (P)).

Step 1: Determine the initial point $x_1^0 \in X$ and a stopping tolerance $\epsilon > 0$. Determine $\epsilon_1 > \epsilon,\ \rho_1 > 0,\ 0 < \gamma < 1$ and $N > 1$, let $j = 1$ and go to Step 2.

Step 2: Start from the point x_j^0 and solve the following problem:

$$(SP_{\epsilon_j,\rho_j}): \quad \min_{x \in \mathbb{R}^n} \ \omega^{k}_{\epsilon_j,\rho_j}(x) = f(x) + \rho_j \sum_{i=1}^{m} q^{k}_{\epsilon_j,\rho_j}(g_i(x)).$$

Let $x^*_{\epsilon_j,\rho_j}$ be an optimal solution of (SP_{ϵ_j,ρ_j}). Here $x^*_{\epsilon_j,\rho_j}$ is obtained by the BFGS method given in [13].

Step 3: If $x^*_{\epsilon_j,\rho_j}$ is an ϵ-feasible of (P), then the algorithm stops and $x^*_{\epsilon_j,\rho_j}$ is an approximate optimal solution of problem (P). Otherwise, determine $\rho_{j+1} = N\rho_j,\ \epsilon_{j+1} = \gamma\epsilon_j,\ x^0_{j+1} = x^*_{\epsilon_j,\rho_j}$ and $j = j+1$, then go to Step 2.

Remark 3.2. Since $N > 1$, $0 < \gamma < 1$, let $N\gamma^{2k-1} < 1$, as $j \to +\infty$, the sequence $\{\epsilon_j\}$ converges to 0, the sequence $\{\rho_j\}$ converges to $+\infty$ and $\{\rho_j \epsilon_j^{2k-1}\}$ converges to 0, as $j \to +\infty$.

Theorem 3.3. *For $\frac{1}{2} \leqslant k < 1$, suppose that for any $\epsilon \in (0, \epsilon_1]$, $\rho \in [\rho_1, +\infty)$, the set*

$$\arg\min_{x \in \mathbb{R}^n} \omega^k_{\epsilon,\rho}(x) \neq \emptyset.$$

Let $\{x^*_{\epsilon_j,\rho_j}\}$ be the sequence generated by Algorithm 3.1 satisfying $N\gamma^{2k-1} < 1$. If the sequence $\{\omega^k_{\epsilon_j,\rho_j}(x^*_{\epsilon_j,\rho_j})\}$ is bounded, and Assumption 2.1 holds, then $\{x^*_{\epsilon_j,\rho_j}\}$ is bounded and the limit point of $\{x^*_{\epsilon_j,\rho_j}\}$ is a solution of (P).

Proof. First, we prove that $\{x^*_{\epsilon_j,\rho_j}\}$ is bounded. Note that

$$\omega^k_{\epsilon_j,\rho_j}(x^*_{\epsilon_j,\rho_j}) = f(x^*_{\epsilon_j,\rho_j}) + \rho_j \sum_{i=1}^{m} q^k_{\epsilon_j,\rho_j}(g_i(x^*_{\epsilon_j,\rho_j})), \quad j = 0, 1, \dots, \tag{3.1}$$

and by the definition of $p^k_{\epsilon,\rho}(t)$, we have

$$\rho_j \sum_{i=1}^{m} q^k_{\epsilon_j,\rho_j}(g_i(x^*_{\epsilon_j,\rho_j})) \geqslant 0. \tag{3.2}$$

Suppose on the contrary that the sequence $\{x^*_{\epsilon_j,\rho_j}\}$ is unbounded. Without any loss of generality $\|x^*_{\epsilon_j,\rho_j}\| \to +\infty$ as $j \to +\infty$. Then, $\lim_{j \to +\infty} f(x^*_{\epsilon_j,\rho_j}) = +\infty$ by Assumption 2.1, and from equations (3.1) and (3.2), we have

$$\omega^k_{\epsilon_j,\rho_j}(x^*_{\epsilon_j,\rho_j}) \geqslant f(x^*_{\epsilon_j,\rho_j}) \to +\infty, \quad j = 0, 1, \dots,$$

which contradicts with the sequence $\{\omega^k_{\epsilon_j,\rho_j}(x^*_{\epsilon_j,\rho_j})\}$ being bounded. Thus, $\{x^*_{\epsilon_j,\rho_j}\}$ is bounded.

Next, we prove that the limit point of $\{x^*_{\epsilon_j,\rho_j}\}$ is the solution of (P). Without loss of generality, we assume $x^*_{\epsilon_j,\rho_j} \to x^*$ as $j \to \infty$. To prove x^* is an optimal solution of (P), it is sufficient to show that $x^* \in X_0$, and $f(x^*) \leqslant f(x)$, $\forall x \in X_0$.

(i) To show that $x^* \in X_0$, we outline a proof by contradiction. Suppose $x^* \notin X_0$. Then, there exist $\theta_0 > 0$ and the subset $J \subset N$, such that $g_{i'}(x^*_{\epsilon_j,\rho_j}) \geqslant \theta_0 > \epsilon_j$ for any $j \in J$ and some $i' \in I$, where N is the natural number set.

From the definition of $q^k_{\epsilon,\rho}(t)$ and $x^*_{\epsilon_j,\rho_j}$ is the optimal solution according j-th values of the parameters ϵ_j, ρ_j for any $x \in X_0$, we have

$$\begin{aligned} f(x^*_{\epsilon_j,\rho_j}) + \rho_j \left[\left(\theta_0 + \frac{\epsilon_j}{m\rho_j} \right)^k + \frac{k}{2} \left(\frac{\epsilon_j}{m\rho_j} \right)^{2k-1} - \left(\frac{\epsilon_j}{m\rho_j} \right)^k \right] &\leqslant \omega^k_{\epsilon_j,\rho_j}(x^*_{\epsilon_j,\rho_j}) \\ &\leqslant \omega^k_{\epsilon_j,\rho_j}(x) \leqslant f(x) + \frac{k}{2} m\rho_j \left(\frac{\epsilon_j}{m\rho_j} \right)^{2k-1}. \end{aligned}$$

It follows that

$$f(x^*_{\epsilon_j,\rho_j}) + \rho_j \left[\left(\theta_0 + \frac{\epsilon_j}{m\rho_j} \right)^k - \left(\frac{\epsilon_j}{m\rho_j} \right)^k \right] \leqslant f(x) + (m-1) \frac{k\rho_j \epsilon_j^{2k-1}}{2(m\rho_j)^{2k-1}},$$

which contradicts with $\rho_j \to +\infty$, $\epsilon_j \to 0$, and $\rho_j \epsilon_j^{2k-1} \to 0$, as $j \to \infty$. Then, we have that $x^* \in X_0$.

(ii) For any $x \in X_0$, it holds that

$$f(x^*_{\epsilon_j,\rho_j}) \leqslant \omega^k_{\epsilon_j,\rho_j}(x^*_{\epsilon_j,\rho_j}) \leqslant \omega^k_{\epsilon_j,\rho_j}(x) \leqslant f(x) + \frac{km\rho_j \epsilon_j^{2k-1}}{2(m\rho_j)^{2k-1}}.$$

Letting $j \to \infty$ yields that

$$f(x^*) \leqslant f(x).$$

□

4. Numerical examples

In this section, we apply the Algorithm 3.1 to test problems. In each example, we let $\epsilon = 10^{-6}$ is expected to get an ϵ-solution of (P) with Algorithm 3.1, j be the number of iterations, $x^*_{\epsilon_j,\rho_j}$ be the optimal solution of the j-th iteration, $f(x^*_{\epsilon_j,\rho_j})$ be the objective value at $x^*_{\epsilon_j,\rho_j}$, $g_i(x^*_{\epsilon_j,\rho_j})$, $i \in I$ is a constrain value at $x^*_{\epsilon_j,\rho_j}$, and the numerical results are presented in the tables as following.

Example 4.1. Consider the following problem ([21], Example 3.3)

$$\begin{aligned} \min\ & f(x) = -2x_1 - 6x_2 + x_1^2 - 2x_1x_2 + 2x_2^2 \\ \text{s.t. } & g_1(x) = x_1 + x_2 - 2 \leqslant 0, \\ & g_2(x) = -x_1 + 2x_2 - 2 \leqslant 0, \\ & x_1,\ x_2 \geqslant 0. \end{aligned} \tag{4.1}$$

Let $x_1^0 = (1,\ 1),\ \rho_1 = 2,\ N = 8,\ \epsilon_1 = 0.1,\ \gamma = 0.01$. With different k, the results of Algorithm 3.1 for solving problem 4.1 are shown in Tables 1, 2, and 3.

Table 1: Results of Algorithm 3.1 with $k = \frac{2}{3}$ for problem 4.1.

j	ρ_j	ϵ_j	$x^*_{\epsilon_j,\rho_j}$	$f(x^*_{\epsilon_j,\rho_j})$	$g_1(x^*_{\epsilon_j,\rho_j})$	$g_2(x^*_{\epsilon_j,\rho_j})$
1	2	0.1	(2.000000, 1.993789)	-11.987500	1.993789	-0.012422
2	16	0.001	(1.217802, 1.337227)	-8.656528	0.555029	-0.543348
3	128	0.00001	(0.800000, 1.200000)	-7.200000	-0.000000	-0.400000

Table 2: Results of Algorithm 3.1 with $k = \frac{3}{5}$ for problem 4.1.

j	ρ_j	ϵ_j	$x^*_{\epsilon_j,\rho_j}$	$f(x^*_{\epsilon_j,\rho_j})$	$g_1(x^*_{\epsilon_j,\rho_j})$	$g_2(x^*_{\epsilon_j,\rho_j})$
1	2	0.1	(4.059684, 2.997685)	-15.991521	5.057369	-0.064313
2	16	0.001	(0.820632, 1.322433)	-7.575229	0.143065	-0.175767
3	128	0.00001	(0.800002, 1.199998)	-7.199999	-0.000000	-0.400006

Table 3: Results of Algorithm 3.1 with $k = \frac{6}{7}$ for problem 4.1.

j	ρ_j	ϵ_j	$x^*_{\epsilon_j,\rho_j}$	$f(x^*_{\epsilon_j,\rho_j})$	$g_1(x^*_{\epsilon_j,\rho_j})$	$g_2(x^*_{\epsilon_j,\rho_j})$
1	2	0.1	(2.826217, 2.402502)	-14.115896	3.228719	-0.021214
2	16	0.001	(0.808600, 1.191516)	-7.199961	0.000116	-0.425569
3	128	0.00001	(0.799693, 1.200307)	-7.199999	-0.000000	-0.399080

The results in Tables 1-3 show that, the convergence Algorithm 3.1 and the obtained approximate optimal solutions are similar. In [21], the obtained approximate optimal solution is $x^* = (0.8000,\ 1.2000)$ with objective function value $f(x^*) = -7.2000$. Numerical results obtained by our algorithm are similar to the results in [21].

Example 4.2. Consider the following problem ([12], Example 4.2)

$$\begin{aligned} \min\ & f(x) = x_1^2 + x_2^2 + 2x_3^2 + x_4^2 - 5x_1 - 5x_2 - 21x_3 + 7x_4 \\ \text{s.t. } & g_1(x) = 2x_1^2 + x_2^2 + x_3^2 + 2x_1 + x_2 + x_4 - 5 \leqslant 0, \\ & g_2(x) = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_1 - x_2 + x_3 - x_4 - 8 \leqslant 0, \\ & g_3(x) = x_1^2 + 2x_2^2 + x_3^2 + 2x_4^2 - x_1 - x_4 - 10 \leqslant 0. \end{aligned} \tag{4.2}$$

For $k = \frac{2}{3}$, let $x_1^0 = (5, 5, 5, 5)$, $\rho_1 = 10$, $N = 8$, $\epsilon_1 = 0.1$, $\gamma = 0.01$. The results of Algorithm 3.1 for solving problem 4.2 are shown in Table 4.

Table 4: Results of Algorithm 3.1 with $x_1^0 = (5, 5, 5, 5)$ for problem 4.2.

j	ρ_j	ϵ_j	$x^*_{\epsilon_j,\rho_j}$	$f(x^*_{\epsilon_j,\rho_j})$	$g_1(x^*_{\epsilon_j,\rho_j})$	$g_2(x^*_{\epsilon_j,\rho_j})$	$g_3(x^*_{\epsilon_j,\rho_j})$
1	10	0.1	(0.042482, 0.996409, 1.957354, -1.068548)	-43.979820	-0.159503	0.039639	-0.871644
2	80	0.001	(0.169560, 0.835531, 2.008634, -0.964876)	-44.233826	-0.000004	-0.000004	-1.883130

For $k = \frac{1}{2}$, let $x_1^0 = (7, 7, 7, 7)$, $\rho_1 = 10$, $N = 9$, $\epsilon_1 = 0.01$, $\gamma = 0.1$. The results of Algorithm 3.1 for solving problem 4.2 are shown in Table 5.

Table 5: Results of Algorithm 3.1 with $x_1^0 = (7, 7, 7, 7)$ for problem 4.2.

j	ρ_j	ϵ_j	$x^*_{\epsilon_j,\rho_j}$	$f(x^*_{\epsilon_j,\rho_j})$	$g_1(x^*_{\epsilon_j,\rho_j})$	$g_2(x^*_{\epsilon_j,\rho_j})$	$g_3(x^*_{\epsilon_j,\rho_j})$
1	10	0.01	(0.228405, 0.827990, 2.195021, -1.162955)	-47.791667	0.729868	1.666709	-0.119102
2	90	0.001	(0.168417, 0.836036, 2.009003, -0.964650)	-44.233824	-0.000001	-0.000002	-1.880298

For $k = \frac{3}{4}$, let $x_1^0 = (1, 1, 1, 1)$, $\rho_1 = 10$, $N = 8$, $\epsilon_1 = 0.1$, $\gamma = 0.1$. The results of Algorithm 3.1 for solving problem 4.2 are shown in Table 6.

Table 6: Results of Algorithm 3.1 with $x_1^0 = (1, 1, 1, 1)$ for problem 4.2.

j	ρ_j	ϵ_j	$x^*_{\epsilon_j,\rho_j}$	$f(x^*_{\epsilon_j,\rho_j})$	$g_1(x^*_{\epsilon_j,\rho_j})$	$g_2(x^*_{\epsilon_j,\rho_j})$	$g_3(x^*_{\epsilon_j,\rho_j})$
1	10	0.1	(0.164525, 0.852259, 2.013672, -0.954093)	-44.276228	0.062573	-0.001386	-1.855209
2	80	0.01	(0.169557, 0.835528, 2.008628, -0.964875)	-44.233725	-0.000041	-0.000041	-1.883165

The results in Tables 4-6 show that, the convergence of Algorithm 3.1 and the obtained approximate optimal solutions are similar. By Table 4, an approximate optimal solution to problem 4.2 is obtained after 2 iterations with objective function value $f(x^*) = -44.233826$. In [4], the obtained approximate optimal solution is $x^* = (0.170446, 0.834248, 2.008753, -0.964559)$ with function value $f(x^*) = -44.233627$. In the paper [12], the obtained approximate optimal solution is $x^* = (0.169234, 0.835656, 2.008690, -0.964901)$ with function value $f(x^*) = -44.233582$. Numerical results obtained by our algorithm are slightly better than the results in [4, 12].

Example 4.3. Consider the following problem ([21], Example 3.2)

$$
\begin{aligned}
\min\quad & f(x) = -x_1 - x_2 \\
\text{s.t.}\quad & g_1(x) = -2x_1^4 + 8x_1^3 - 8x_1^2 + x_1 - 2 \leqslant 0, \\
& g_2(x) = -4x_1^4 + 32x_1^3 - 88x_1^2 + 96x_1 + x_2 - 36 \leqslant 0, \\
& 0 \leqslant x_1 \leqslant 3, \\
& 0 \leqslant x_2 \leqslant 4.
\end{aligned} \tag{4.3}
$$

Let $k = \frac{3}{4}$, $\rho_1 = 8$, $N = 6$, $\epsilon_1 = 0.4$, $\gamma = 0.1$. With different starting points, the results of Algorithm 3.1 for solving problem 4.3 are shown in Tables 7, 8, and 9.

Table 7: Results of Algorithm 3.1 with $x_1^0 = (0, 3)$ for problem 4.3.

j	ρ_j	ϵ_j	$x^*_{\epsilon_j,\rho_j}$	$f(x^*_{\epsilon_j,\rho_j})$	$g_1(x^*_{\epsilon_j,\rho_j})$	$g_2(x^*_{\epsilon_j,\rho_j})$
1	8	0.4	(2.112066, 3.900188)	-6.012254	0.000021	0.000028
2	48	0.04	(2.112096, 3.900107)	-6.012203	-0.000012	-0.000001

Table 8: Results of Algorithm 3.1 with $x_1^0 = (2, 1)$ for problem 4.3.

j	ρ_j	ϵ_j	$x^*_{\epsilon_j,\rho_j}$	$f(x^*_{\epsilon_j,\rho_j})$	$g_1(x^*_{\epsilon_j,\rho_j})$	$g_2(x^*_{\epsilon_j,\rho_j})$
1	8	0.4	(2.112028, 3.900313)	-6.012341	0.000063	0.000085
2	48	0.04	(2.112101, 3.900094)	-6.012195	-0.000018	-0.000004

Table 9: Results of Algorithm 3.1 with $x_1^0 = (3, 1)$ for problem 4.3.

j	ρ_j	ϵ_j	$x^*_{\epsilon_j,\rho_j}$	$f(x^*_{\epsilon_j,\rho_j})$	$g_1(x^*_{\epsilon_j,\rho_j})$	$g_2(x^*_{\epsilon_j,\rho_j})$
1	8	0.4	(2.112028, 3.900313)	-6.012341	0.000063	0.000085
2	48	0.04	(2.112087, 3.900025)	-6.012112	-0.000002	-0.000099

The results in Tables 7-9 show that, the convergence Algorithm 3.1 and the obtained approximate optimal solutions are similar. That is to say, the numerical results of Algorithm 3.1 does not depend on the starting point x^0 for this example. By Table 7, an approximate optimal solution to problem 4.3 is obtained after 2 iterations with objective function value $f(x^*) = -6.012203$. In [21], the obtained global solution is $x^* = (2.3295, 3.1784)$ with objective function value $f(x^*) = -5.5080$. In the paper [22], the obtained approximate optimal solution is $x^* = (2.112103, 3.900086)$ with objective function value $f(x^*) = -6.012190$. Numerical results obtained by our algorithm are much better than the results in [21] and find the correct solutions as in [22].

5. Conclusions

In this paper, we proposed a new perturbed smooth penalty function for inequality constrained optimization. Furthermore, we proved that the algorithm based on the smoothed penalty functions is globally convergent under mild conditions. The numerical results given in Section 4 show that the Algorithm 3.1 has a good convergence for an approximate optimal solution.

Acknowledgment

We would like to thank the anonymous referees for their valuable comments which helped to improve this paper. This work was supported by the National Natural Science Foundation of China (Grant Nos. 11772254, 11742013).

References

[1] I. Ahmad, A. Ayswal, J. Banerjee, *On interval-valued optimization problems with generalized invex functions*, J. Inequal. Appl., **2013** (2013), 14 pages. 1

[2] M. S. Bazaraa, J. J. Goode, *Sufficient conditions for a globally exact penalty function without convexity*, Math. Programming Stud., **19** (1982), 1–15. 1

[3] N. T. Binh, *Smoothing approximation to l_1 exact penalty function for constrained optimization problems*, J. Appl. Math. Inform., **33** (2015), 387–399. 1

[4] N. T. Binh, *Smoothed lower order penalty function for constrained optimization problems*, IAENG Int. J. Appl. Math., **46** (2016), 76–81. 1, 4

[5] N. T. Binh, Y. Q. Bai, X. Yan, T. Yang, *Perturbed smoothing approach to the lower order exact penalty functions for nonlinear inequality constrained optimization*, Tamkang J. Math., **50** (2019), 37–60. 1

[6] C. H. Chen, O. L. Mangasarian, *Smoothing methods for convex inequalities and linear complementarity problems*, Math. Programming, **71** (1995), 51–69. 1

[7] G. Di Pillo, L. Grippo, *An exact penalty function method with global convergence properties for nonlinear programming problems*, Math. Programming, **36** (1986), 1–18. 1

[8] J. B. Etoa Etoa, *Solving quadratic convex bilevel programming problems using a smoothing method*, Appl. Math. Comput., **217** (2011), 6680–6690. 1

[9] S. P. Han, O. L. Mangasrian, *Exact penalty function in nonlinear programming*, Math. Programming, **17** (1979), 251–269. 1

[10] J. B. Lasserre, *A globally convergent algorithm for exact penalty functions*, European J. Oper. Res., **7** (1981), 389–395. 1

[11] H. Li, L. Zhang, Y.-C. Jiao, *An interactive approach based on a discrete differential evolution algorithm for a class of integer bilevel programming problems*, Internat. J. Systems Sci., **47** (2016), 2330–2341. 1

[12] Z. Q. Meng, C. Y. Dang, X. Q. Yang, *On the smoothing of the square-root exact penalty function for inequality constrained optimization*, Comput. Optim. Appl., **35** (2006), 375–398. 1, 4.2, 4

[13] J. Nocedal, S. J. Wright, *Numerical Optimization*, Springer-Verlag, New York, (1999). 3.1

[14] M. C. Pinar, S. A. Zenios, *On smoothing exact penalty function for convex constrained optimization*, SIAM J. Optim., **4** (1994), 486–511. 1

[15] L. Q. Qi, X. J. Tong, Y. J. Wang, *Computing power system parameters to maximize the small signal stability margin based on min-max models*, Optim. Eng., **10** (2009), 465–476. 1

[16] E. Rosenberg, *Exact penalty functions and stability in locally Lipschitz programming*, Math. Programming, **30** (1984), 340–356. 1

[17] M. Ruiz Galán, *Elementary convex techniques for equilibrium, minimax and variational problems*, Optim. Lett., **12** (2018), 137–154. 1

[18] Y. H. Sun, L. S. Wang, *Optimality conditions and duality in nondifferentiable interval-valued programming*, J. Ind. Manag. Optim., **9** (2013), 131–142.

[19] C. W. Wang, Y. J. Wang, *A superlinearly convergent projection method for constrained systems of nonlinear equations*, J. Glob. Optim., **44** (2009), 283–296. 1

[20] Z. Y. Wu, F. S. Bai, X. Q. Yang, L. S. Zhang, *An exact lower order penalty function and its smoothing in nonlinear programming*, Optimization, **53** (2004), 51–68. 1, 2, 2.6, 2

[21] Z. Y. Wu, H. W. J. Lee, F. S. Bai, L. S. Zhang, *Quadratic smoothing approximation to l_1 exact penalty function in global optimization*, J. Ind. Manag. Optim., **1** (2005), 533–547. 1, 4.1, 4, 4.3, 4

[22] T. Yang, N. T. Binh, T. M. Thang, D. T. Hoa, *A new smoothing nonlinear penalty function for constrained optimization*, Math. Comput. Appl., **22** (2017), 14 pages. 1, 4

[23] C. J. Yu, K. L. Teo, L. S. Zhang, Y. Q. Bai, *A new exact penalty function method for continuous inequality constrained optimization problems*, J. Ind. Manag. Optim., **6** (2010), 895–910. 1

[24] W. I. Zangwill, *Non-linear programming via penalty functions*, Management Sci., **13** (1967), 344–358. 1, 1

[25] S. A. Zenios, M. C. Pinar, R. S. Dembo, *A smooth penalty function algorithm for network-structured problems*, Eur. J. Oper. Res., **83** (1995), 220–236. 1