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Multiple criteria decision making based on bipolar picture fuzzy sets and extended TOPSIS

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Abstract

The notion of bipolar fuzzy sets (B_pFSs) has got much attention from the experts or decision-makers (DMs). B_pFSs have ample information in the form of two degrees called the positive belonging degree ($P_\nu BD$) and a negative belonging degree ($N_\nu BD$). In this article, we introduced the concept of bipolar picture fuzzy sets (BP_cFSs) by connecting the concepts of B_pFSs and picture fuzzy sets (P_cFSs). Firstly, we presented the concept, operational rules, score, and accuracy functions of BP_cFSs. Secondly, a distance measure is formulated for the BP_cFSs and then implemented for the extension of TOPSIS. Thirdly, a multiple criteria decision making (MCDM) model is proposed to handle the uncertain MCDM problems. Lastly, a practical example related to the sum of money's investment is exemplified to validate and effectiveness of the proposed model.

Keywords: Picture fuzzy sets, fuzzy sets, BP_cFSs, linear programming model.

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1. Introduction

In 2013, Coung [4] introduced the generalization of fuzzy sets (FSs) [20] by presenting the idea of picture fuzzy sets (P_cFSs). P_cFSs consists of three well-known degrees, membership degree (MD), non-membership degree (NMD) and neutral degree (ND) so that $0 \leqslant \text{MD} + \text{NMD} + \text{ND} \leqslant 1$. Cuong and Kreinovich [3] established various operational laws of P_cFSs to handle vague information perfectly. The notion of bipolar fuzzy sets (B_pFSs) [21, 22] have come to account as a superior device to portray the vagueness in the decision-making process. B_pFSs contain two elements called, the positive membership degree ($P_\nu MD$) and the negative membership degree ($N_\nu MD$) to represent the bipolar fuzzy (B_pF) information and the range both the degrees always lie in $[-1, 1]$. Currently, B_pFSs have been utilized in various fields of research [7, 9, 23–25]. Gul [5] presented several arithmetic and geometric operators for bipolar fuzzy information. Wei et al. [18] presented the concept of hesitant B_pFSs and its operational laws to deal with B_pF elements. Lu et al. [10] introduced the idea of bipolar 2-tuple linguistic fuzzy sets ($B_p2TLFSs$). Further, Xu and Wei [19] suggested the dual B_pFSs and established many arithmetic laws to fuse the dual bipolar fuzzy data. Moreover, plenty of research work has been done on the B_pFSs for example, Hashim

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et al. [6] presented the idea of neutrosophic bipolar fuzzy sets and developed an algorithm to find the best medicine for some particular diseases, Riaz and Tehrim [12] gave the concept of cubic bipolar fuzzy sets (CBFSs), a generalization of B_pFSs and implemented it in group decision making with the help of geometric aggregation operators.

The linear programming (LP) model introduced by Vanderbei [15], permits some target function to be minimized or maximized inside the system of given situational limitations. LP is a computational technique that enables DMs to solve the problems which they face in decision-making model. It encourages the DMs to deal with constrained ideal conditions that they need to make the best of their resources. Various experts utilized LP [1, 2, 8, 13, 16] in MCDM in different fields. Recently, Sindhu et al. [14] implemented the LP methodology with extended TOPSIS for picture fuzzy sets.

From the above discussion, it can be noticed that P_cFSs and B_pFSs are getting a lot of attention from the DMs and are playing an important role in the decision-making process. However, all these are concerned with discrete information due to which a chance of loss of information is present. In order to reduce the chance of loss of information, we presented the concept of BP_cFSs that consists of P_νMD and N_νMD in terms of fuzzy numbers. BP_cFSs also have a lot of information that helps the DMs to reach the best decisions in the MCDM problems. The weights of criteria appear to specify that the DMs identify the significance of people's views and their influence on attaining the objective. Allocation of weights to the criteria epitomizes the importance of each decision criterion relative to each other. We apply the technique for order preference by similarity to ideal solution (TOPSIS) to get the objective function and then find out the weights of criteria under some constraints by using the LP model in this article.

The remaining part of the article is organized as follows. Section 2 briefly shows the basics like FSs and P_cFSs to reach the notion BP_cFSs and the LP model that will be used to compute the weights of criteria. In Section 3, we introduced the concept of BP_cFSs, operational laws distance, and similarity measures of BP_cFSs. Based on the TOPSIS, an MCDM model is proposed in Section 4. In Section 5, the developed MCDM model is then applied to a practical example to select the best alternative. For the validity, effectiveness, and stability of the proposed MCDM model, we performed the sensitivity analysis in Section 6. Lastly, conclusions are drawn in Sections 7.

2. Preliminaries

A brief introduction of the notions FSs, P_cFSs, B_pFSs BP_cFSs and the LP model is presented in this section.

Definition 2.1 ([20]). Let $X = \{x_1, x_2, \dots, x_n\}$ be a discourse set,a fuzzy set (FS) F on X is represented in terms of a functions $m : X \to [0, 1]$ such as

$$F = \{\langle x_i, m_F(x_i)\rangle \,|x_i \in X\}.$$

Definition 2.2 ([4]). Let $X = \{x_1, x_2, \dots, x_n\}$ be a fixed set, a picture fuzzy set P_c on X is defined as:

$$P_c = \{\langle x_i, \alpha_{P_c}(x_i), \gamma_{P_c}(x_i), \beta_{P_c}(x_i)\rangle \,|x_i \in X, i = 1, 2, \dots, n\},$$

where $\alpha_{P_c}(x_i)$, $\beta_{P_c}(x_i)$, $\gamma_{P_c}(x_i) \in [0, 1]$ are called the acceptance membership, neutral and rejection membership degrees of $x_i \in X$ to the set P_c, respectively and $\alpha_{P_c}(x_i)$, $\gamma_{P_c}(x_i)$ and $\beta_{P_c}(x_i)$ fulfill the condition: $0 \leqslant \alpha_{P_c}(x_i) + \gamma_{P_c}(x_i) + \beta_{P_c}(x_i) \leqslant 1$, for all $x_i \in X$. Also $\zeta_{P_c}(x_i) = 1 - \alpha_{P_c}(x_i) - \gamma_{P_c}(x_i) - \beta_{P_c}(x_i)$, then $\zeta_{P_c}(x_i)$ is said to be a degree of refusal membership of $x_i \in X$ in P_c. For our convenience, we can write $P_k = (\alpha^k_{P_c}(x_i), \beta^k_{P_c}(x_i), \gamma^k_{P_c}(x_i))$ as the picture fuzzy numbers $(P_c\text{FNs})$ over a set P_c, where k is positive integer.

Definition 2.3 ([17]). Let $P = (\alpha_{P_c}(x_i), \gamma_{P_c}(x_i), \beta_{P_c}(x_i))$, $P_1 = (\alpha^1_{P_c}(x_i), \gamma^1_{P_c}(x_i), \beta^1_{P_c}(x_i))$, and $P_2 = (\alpha^2_{P_c}(x_i), \gamma^2_{P_c}(x_i), \beta^2_{P_c}(x_i))$ be three P_cFNs, then arithmetic operations are listed as follows:

1. $P_1 \oplus P_2 = (\alpha^1_{P_c} + \alpha^2_{P_c} - \alpha^1_{P_c} \times \alpha^2_{P_c}, \gamma^1_{P_c} \times \gamma^2_{P_c}, \beta^1_{P_c} \times \beta^2_{P_c})$;

2. $P_1 \otimes P_2 = (\alpha^1_{P_c} \times \alpha^2_{P_c}, \gamma^1_{P_c} + \gamma^2_{P_c} - \gamma^1_{P_c} \times \gamma^2_{P_c}, \beta^1_{P_c} + \beta^2_{P_c} - \beta^1_{P_c} \times \beta^2_{P_c})$;
3. $\lambda P = (1-(1-\alpha_{P_c})^\lambda, \gamma^\lambda_{P_c}, \beta^\lambda_{P_c})$, where, $\lambda > 0$;
4. $P^\lambda_p = (\alpha^\lambda_{P_c}, 1-(1-\gamma_{P_c})^\lambda, 1-(1-\beta_{P_c})^\lambda)$, where, $\lambda > 0$.

Definition 2.4 ([21, 22]). Suppose that X is a discourse set such that $X = \{x_1, x_2, \dots, x_n\}$, then a bipolar fuzzy set B_p on X is described as follows:

$$B_p = \{\left\langle x_i, (\alpha^+_{B_p(x_i)}, \beta^-_{B_p}(x_i))\right\rangle | x_i \in X, i = 1, 2, \dots, n\},$$

where $\alpha^+_{B_p(x_i)} : X \to [0,1]$, $\beta^-_{B_p}(x_i) : X \to [-1,0]$ are named as $P_\nu BD$ and $N_\nu BD$ of $x_i \in X$ to B_p, respectively.

Definition 2.5 ([5]). Let B_p, B^1_p and B^2_p be any three B_pFSs on $X = \{x_1, x_2, \dots, x_n\}$, then some aggregation operators are defined as:

1. $B^1_p \oplus B^2_p = (\alpha^+_1 + \alpha^+_2 - \alpha^+_1 \times \alpha^+_2, -|\beta^-_1| \times |\beta^-_2|)$;
2. $B^1_p \otimes B^2_p = (|\alpha^-_1| \times |\alpha^-_2|, \beta^+_1 + \beta^+_2 - \beta^+_1 \times \beta^+_2)$;
3. $\kappa B_p = (1-(1-\alpha^+)^\kappa, -|\beta^-|)$, where, $\kappa > 0$;
4. $B^\kappa_p = (\alpha^+)^\kappa, -1 + |1+\beta^-|^\kappa$, where, $\kappa > 0$;
5. $B^c_p = (1-\alpha^+, |\beta^- - 1|$.

Definition 2.6 ([15]). Vanderbei defined the LP model as follows:

$$\begin{aligned}
\text{Maximize:}\quad & Z = c_1y_1 + c_2y_2 + c_3y_3 + \cdots + c_ny_n,\\
\text{Subject to:}\quad & a_{11}y_1 + a_{12}y_2 + a_{13}y_3 + \cdots + a_{1n}y_n \leqslant b_1,\\
& a_{21}y_1 + a_{22}y_2 + a_{23}y_3 + \cdots + a_{2n}y_n \leqslant b_2,\\
& \vdots\\
& a_{m1}y_1 + a_{m2}y_2 + a_{m3}y_3 + \cdots + a_{mn}y_n \leqslant b_m,\\
& y_1, y_2, \dots, y_n \geqslant 0,
\end{aligned}$$

where m, n represent the cardinalities of constraints and decision variables $(y_1, y_2, \dots, y_n)$, respectively. The solution $(y_1, y_2, \dots, y_n)$ is known as viable if it satisfies all the provided restrictions. LP model is used to compute the optimal solution of $y_1, y_2, \dots, y_n$ to maximize the linearly objective function Z.

3. Bipolar picture fuzzy sets ($B_{P_c}FSs$)

In this section, we introduced the notion of $B_{P_c}FSs$ by combining both B_pFSs andP_cFSs. Also, various operational laws are established and then a novel distance measure is proposed for bipolar picture fuzzy numbers ($B_{P_c}FNs$).

Definition 3.1. Suppose that $X = \{x_1, x_2, \dots, x_n\}$ is a discourse, then the $B_{P_c}FSs$ B_{P_c} on X is presented as:

$$B_{P_c} = \left\{\left\langle x_i, (\tilde{B_{P_c}}^+(x_i), \tilde{B_{P_c}}^-(x_i))\right\rangle | x_i \in X, i = 1, 2, \dots, n\right\},$$

here $\tilde{B_{P_c}}^+(x_i) = (\alpha^+_{B_{P_c}}(x_i), \gamma^+_{B_{P_c}}(x_i), \beta^+_{B_{P_c}}(x_i))$, $\tilde{P}^-_c(x_i) = (\alpha^-_{B_{P_c}}(x_i), \gamma^-_{B_{P_c}}(x_i), \beta^-_{B_{P_c}}(x_i))$ satisfy the following condition:

$$0 \leqslant (\alpha^+_{B_{P_c}}(x_i) + \gamma^+_{B_{P_c}}(x_i) + \beta^+_{B_{P_c}}(x_i)) \leqslant 1 \text{ and } -1 \leqslant (\alpha^-_{B_{P_c}}(x_i) + \gamma^-_{B_{P_c}}(x_i) + \beta^-_{B_{P_c}}(x_i)) \leqslant 0 \text{ for all } x_i \in X.$$

For simplicity, the pair, $\tilde{p_k}(x) = (\tilde{B_{P_c}}^{k+}(x), \tilde{B_{P_c}}^{k-}(x))$ is known as bipolar picture fuzzy number (BP_cFN) denoted by $\tilde{p_k} = (\tilde{B_{P_c}}^{k+}, \tilde{B_{P_c}}^{k-})$, satisfying the conditions: $(\alpha^{k+}_{B_{P_c}}, \gamma^{k+}_{B_{P_c}}, \beta^{k+}_{B_{P_c}}) \in [0,1]$, $(\alpha^{k-}_{B_{P_c}}, \gamma^{k-}_{B_{P_c}}, \beta^{k-}_{B_{P_c}}) \in [-1,0]$, $0 \leqslant \alpha^{k+}_{B_{P_c}} + \gamma^{k+}_{B_{P_c}} + \beta^{k+}_{B_{P_c}} \leqslant 1$ and $-1 \leqslant \alpha^{k-}_{B_{P_c}}, \gamma^{k-}_{B_{P_c}}, \beta^{k-}_{B_{P_c}} \leqslant 0$.

Definition 3.2. Let $\tilde{p} = (\alpha^{+}_{B_{Pc}}, \gamma^{+}_{B_{Pc}}, \beta^{+}_{B_{Pc}}, \alpha^{-}_{B_{Pc}}, \gamma^{-}_{B_{Pc}}, \beta^{-}_{B_{Pc}})$, $\tilde{p_1} = (\alpha^{1+}_{B_{Pc}}, \gamma^{1+}_{B_{Pc}}, \beta^{1+}_{B_{Pc}}, \alpha^{1-}_{B_{Pc}}, \gamma^{1-}_{B_{Pc}}, \beta^{1-}_{B_{Pc}})$ and $\tilde{p_2} = (\alpha^{2+}_{B_{Pc}}, \gamma^{2+}_{B_{Pc}}, \beta^{2+}_{B_{Pc}}, \alpha^{2-}_{B_{Pc}}, \gamma^{2-}_{B_{Pc}}, \beta^{2-}_{B_{Pc}})$ be three BB_{Pc}FNs, then the operational rules are penned as:

1. $\tilde{p_1} \oplus \tilde{p_2} = ((\alpha^{1+}_{B_{Pc}} + \alpha^{2+}_{B_{Pc}} - \alpha^{1+}_{B_{Pc}} \cdot \alpha^{2+}_{B_{Pc}}, \gamma^{1+}_{B_{Pc}} \cdot \gamma^{2+}_{B_{Pc}}, \beta^{1+}_{B_{Pc}} \cdot \beta^{2+}_{B_{Pc}}), -(\alpha^{1-}_{B_{Pc}} + \alpha^{2-}_{B_{Pc}} - \alpha^{1-}_{B_{Pc}} \cdot \alpha^{2-}_{B_{Pc}}), -|\gamma^{1-}_{B_{Pc}}| \cdot |\gamma^{2-}_{B_{Pc}}|, -|\beta^{1-}_{B_{Pc}}| \cdot |\beta^{2-}_{B_{Pc}}|)$;
2. $\tilde{p_1} \otimes \tilde{p_2} = (\alpha^{1+}_{B_{Pc}} \cdot \alpha^{2+}_{B_{Pc}}, \gamma^{1+}_{B_{Pc}} + \gamma^{2+}_{B_{Pc}} - \gamma^{1+}_{B_{Pc}} \cdot \gamma^{2+}_{B_{Pc}}, \beta^{1+}_{B_{Pc}} + \beta^{2+}_{B_{Pc}} - \beta^{1+}_{B_{Pc}} \cdot \beta^{2+}_{B_{Pc}}), -(|\alpha^{1-}_{B_{Pc}}| \cdot |\alpha^{2-}_{B_{Pc}}|, -(\gamma^{1-}_{B_{Pc}} + \gamma^{2-}_{B_{Pc}} - \gamma^{1-}_{B_{Pc}} \cdot \gamma^{2-}_{B_{Pc}}), -(\beta^{1-}_{B_{Pc}} + \beta^{2-}_{B_{Pc}} - \beta^{1-}_{B_{Pc}} \cdot \beta^{2-}_{B_{Pc}}))$;
3. $\lambda\tilde{p} = ((1-(1-\alpha^{+}_{B_{Pc}})^{\lambda}, (\gamma^{+})^{\lambda}_{B_{Pc}}, (\beta^{+})^{\lambda}_{B_{Pc}}), -(1-|(1-\alpha^{-}_{B_{Pc}})|^{\lambda}, -|(\gamma^{-})^{\lambda}_{B_{Pc}}|, -|(\beta^{-})^{\lambda}_{B_{Pc}}|))$, where, $\lambda > 0$;
4. $\tilde{p}^{\lambda} = ((\alpha^{+}_{B_{Pc}})^{\lambda}, 1-(1-\gamma^{+}_{B_{Pc}})^{\lambda}, 1-(1-\beta^{+}_{B_{Pc}})^{\lambda}), -|(\alpha^{-}_{B_{Pc}})^{\lambda}|, -(1-|(1-\gamma^{-}_{B_{Pc}})^{\lambda}|, -(1-|(1-\beta^{-}_{B_{Pc}})^{\lambda}|))$, where, $\lambda > 0$.

Definition 3.3. Let l and q be two B_{pc}FNs of the B_{pc}FSs L and Q, respectively defined on a discourse set $X = \{x_1, x_2, \ldots, x_n\}$, then the distance $D_{pc}(L, Q)$ is defined as:

$$D_{pc}(L,Q) = \frac{1}{n}\sum_{i=1}^{n}\left(\begin{array}{c}\left[\begin{array}{c}|\alpha^{l+}_{B_{Pc}}(x_i) - \alpha^{q+}_{B_{Pc}}(x_i)| + |\gamma^{l+}_{B_{Pc}}(x_i) - \gamma^{q+}_{B_{Pc}}(x_i)| + |\beta^{l+}_{B_{Pc}}(x_i) - \beta^{q+}_{B_{Pc}}(x_i)| \\ +|\alpha^{l-}_{B_{Pc}}(x_i) - \alpha^{q-}_{B_{Pc}}(x_i)| + |\gamma^{l-}_{B_{Pc}}(x_i) - \gamma^{q-}_{B_{Pc}}(x_i)| + |\beta^{l-}_{B_{Pc}}(x_i) - \beta^{q-}_{B_{Pc}}(x_i)|\end{array}\right] \\ +\max\left[\begin{array}{c}|\alpha^{l+}_{B_{Pc}}(x_i) - \alpha^{q+}_{B_{Pc}}(x_i)|, |\gamma^{l+}_{B_{Pc}}(x_i) - \gamma^{q+}_{B_{Pc}}(x_i)|, |\beta^{l+}_{B_{Pc}}(x_i) - \beta^{q+}_{B_{Pc}}(x_i)| \\ , |\alpha^{l-}_{B_{Pc}}(x_i) - \alpha^{q-}_{B_{Pc}}(x_i)|, |\gamma^{l-}_{B_{Pc}}(x_i) - \gamma^{q-}_{B_{Pc}}(x_i)|, |\beta^{l-}_{B_{Pc}}(x_i) - \beta^{q-}_{B_{Pc}}(x_i)|\end{array}\right]\end{array}\right).$$

Theorem 3.4. *Suppose that,* D_{pc} *is a mapping* $D_{pc} : B_{Pc}FSs(X) \times B_{Pc}FSs(X) \longrightarrow [0, 1]$, *then* $D_{pc}(L, Q)$ *is a distance measure if the following four conditions hold:*

1. $0 \leqslant D_{pc}(L, Q) \leqslant 1$;
2. $D_{pc}(L, Q) = 0$ *iff* $L = Q$;
3. $D_{pc}(L, Q) = D_{pc}(L, Q)$;
4. $D_{pc}(L, R) \geqslant D_{pc}(L, Q)$ *and* $D_{pc}(L, R) \geqslant D_{pc}(Q, R)$, *for any* $L, Q, R \in B_{Pc}FSs(X)$.

Proof. Since the proofs of $1-3$ are obvious, thereby, we need to prove the last condition 4. For any $l = (\alpha^{l+}_{P_c}, \gamma^{l+}_{P_c}, \beta^{l+}_{P_c}, \alpha^{l-}_{P_c}, \gamma^{l-}_{P_c}, \beta^{l-}_{P_c}) \in L$, $q = (\alpha^{q+}_{P_c}, \gamma^{q+}_{P_c}, \beta^{q+}_{P_c}, \alpha^{q-}_{P_c}, \gamma^{q-}_{P_c}, \beta^{q-}_{P_c}) \in Q$, and $r = (\alpha^{r+}_{P_c}, \gamma^{r+}_{P_c}, \beta^{r+}_{P_c}, \alpha^{r-}_{P_c}, \gamma^{r-}_{P_c}, \beta^{r-}_{P_c}) \in R$ be three B_{Pc}FNs defined on X so that, $L \subseteq Q \subseteq R$, then based on Definition 3.3, we get

$$|\alpha^{l+}_{P_c}(x_i) - \alpha^{r+}_{P_c}(x_i)| \geqslant |\alpha^{l+}_{P_c}(x_i) - \alpha^{q+}_{P_c} x_i)|, \tag{3.1}$$
$$|\alpha^{l+}_{P_c}(x_i) - \beta^{r+}_{P_c}(x_i)| \geqslant |\alpha^{l+}_{P_c}(x_i) - \beta^{q+}_{P_c}(x_i)|, \tag{3.2}$$
$$|\alpha^{l+}_{P_c}(x_i) - \gamma^{r+}_{P_c}(x_i)| \geqslant |\alpha^{l+}_{P_c}(x_i) - \gamma^{q+}_{P_c}(x_i)|, \tag{3.3}$$
$$|\alpha^{l-}_{P_c}(x_i) - \alpha^{r-}_{P_c}(x_i)| \geqslant |\alpha^{l-}_{P_c}(x_i) - \alpha^{q-}_{P_c} x_i)|, \tag{3.4}$$
$$|\alpha^{l-}_{P_c}(x_i) - \beta^{r-}_{P_c}(x_i)| \geqslant |\alpha^{l-}_{P_c}(x_i) - \beta^{q-}_{P_c}(x_i)|, \tag{3.5}$$
$$|\alpha^{l-}_{P_c}(x_i) - \gamma^{r-}_{P_c}(x_i)| \geqslant |\alpha^{l-}_{P_c}(x_i) - \gamma^{q-}_{P_c}(x_i)|. \tag{3.6}$$

By adding Eqs. (3.1)-(3.3) and (3.4)-(3.6), we get

$$\begin{aligned} &|\alpha^{l+}_{P_c}(x_i) - \alpha^{r+}_{P_c}(x_i)| + |\alpha^{l+}_{P_c}(x_i) - \beta^{r+}_{P_c}(x_i)| + |\alpha^{l+}_{P_c}(x_i) - \gamma^{r+}_{P_c}(x_i)| \\ &\quad \geqslant |\alpha^{l+}_{P_c}(x_i) - \alpha^{q+}_{P_c} x_i)| + |\alpha^{l+}_{P_c}(x_i) - \beta^{q+}_{P_c}(x_i)||\alpha^{l+}_{P_c}(x_i) - \gamma^{q+}_{P_c}(x_i)|, \\ &|\alpha^{l-}_{P_c}(x_i) - \alpha^{r-}_{P_c}(x_i)| + |\alpha^{l-}_{P_c}(x_i) - \beta^{r-}_{P_c}(x_i)| + |\alpha^{l-}_{P_c}(x_i) - \gamma^{r-}_{P_c}(x_i)| \\ &\quad \geqslant |\alpha^{l-}_{P_c}(x_i) - \alpha^{q-}_{P_c} x_i)| + |\alpha^{l-}_{P_c}(x_i) - \beta^{q-}_{P_c}(x_i)||\alpha^{l-}_{P_c}(x_i) - \gamma^{q-}_{P_c}(x_i)|. \end{aligned} \tag{3.7}$$

By adding Eqs. in (3.7), we have

$$|\alpha^{l+}_{P_c}(x_i) - \alpha^{r+}_{P_c}(x_i)| + |\alpha^{l+}_{P_c}(x_i) - \beta^{r+}_{P_c}(x_i)| + |\alpha^{l+}_{P_c}(x_i) - \gamma^{r+}_{P_c}(x_i)|$$

$$
\begin{aligned}
&+|\alpha^{l-}_{P_c}(x_i)-\alpha^{r-}_{P_c}(x_i)|+|\alpha^{l-}_{P_c}(x_i)-\beta^{r-}_{P_c}(x_i)|+|\alpha^{l-}_{P_c}(x_i)-\gamma^{r-}_{P_c}(x_i)|\\
\geqslant\ &|\alpha^{l+}_{P_c}(x_i)-\alpha^{q+}_{P_c}x_i)|+|\alpha^{l+}_{P_c}(x_i)-\beta^{q+}_{P_c}(x_i)||\alpha^{l+}_{P_c}(x_i)-\gamma^{q+}_{P_c}(x_i)|\\
&+|\alpha^{l-}_{P_c}(x_i)-\alpha^{q-}_{P_c}x_i)|+|\alpha^{l-}_{P_c}(x_i)-\beta^{q-}_{P_c}(x_i)||\alpha^{l-}_{P_c}(x_i)-\gamma^{q-}_{P_c}(x_i)|.
\end{aligned}
$$

Let

$$
\begin{aligned}
U &= |\alpha^{l+}_{P_c}(x_i)-\alpha^{r+}_{P_c}(x_i)|+|\alpha^{l+}_{P_c}(x_i)-\beta^{r+}_{P_c}(x_i)|+|\alpha^{l+}_{P_c}(x_i)-\gamma^{r+}_{P_c}(x_i)|\\
&\quad+|\alpha^{l-}_{P_c}(x_i)-\alpha^{r-}_{P_c}(x_i)|+|\alpha^{l-}_{P_c}(x_i)-\beta^{r-}_{P_c}(x_i)|+|\alpha^{l-}_{P_c}(x_i)-\gamma^{r-}_{P_c}(x_i)|,\\
V &= |\alpha^{l+}_{P_c}(x_i)-\alpha^{q+}_{P_c}x_i)|+|\alpha^{l+}_{P_c}(x_i)-\beta^{q+}_{P_c}(x_i)||\alpha^{l+}_{P_c}(x_i)-\gamma^{q+}_{P_c}(x_i)|\\
&\quad+|\alpha^{l-}_{P_c}(x_i)-\alpha^{q-}_{P_c}x_i)|+|\alpha^{l-}_{P_c}(x_i)-\beta^{q-}_{P_c}(x_i)||\alpha^{l-}_{P_c}(x_i)-\gamma^{q-}_{P_c}(x_i)|,\\
&\Rightarrow U\geqslant V,
\end{aligned}
$$

then

$$
\begin{aligned}
&U+\max[|\alpha^{l+}_{P_c}(x_i)-\alpha^{r+}_{P_c}(x_i)|,|\alpha^{l+}_{P_c}(x_i)-\beta^{r+}_{P_c}(x_i)|,|\alpha^{l+}_{P_c}(x_i)-\gamma^{r+}_{P_c}(x_i)|]\\
&\quad+\max[|\alpha^{l-}_{P_c}(x_i)-\alpha^{r-}_{P_c}(x_i)|,|\alpha^{l-}_{P_c}(x_i)-\beta^{r-}_{P_c}(x_i)|,|\alpha^{l-}_{P_c}(x_i)-\gamma^{r-}_{P_c}(x_i)|]\\
&\geqslant V+\max[|\alpha^{l+}_{P_c}(x_i)-\alpha^{q+}_{P_c}x_i)|,|\alpha^{l+}_{P_c}(x_i)-\beta^{q+}_{P_c}(x_i)||\alpha^{l+}_{P_c}(x_i)-\gamma^{q+}_{P_c}(x_i)|\\
&\quad+\max[|\alpha^{l-}_{P_c}(x_i)-\alpha^{q-}_{P_c}x_i)|],|\alpha^{l-}_{P_c}(x_i)-\beta^{q-}_{P_c}(x_i)||\alpha^{l-}_{P_c}(x_i)-\gamma^{q-}_{P_c}(x_i)|],\\
&\Rightarrow D_{pc}(L,R)\geqslant D_{pc}(L,Q),
\end{aligned}
$$

and similarly we can prove that, $D_{pc}(L,R)\geqslant D_{pc}(Q,R)$. □

Generally, weights of the criteria have a great influence on the results of the decision making process, therefore, a weighted distance measure between two B_{Pc}FSs is developed on the basis of Definition 3.3 as following.

Definition 3.5. Let L and Q be two B_{Pc}FSs defined on a discourse set $X=\{x_1,x_2,\dots,x_n\}$ and w_j be the weights of the m criteria such that $\sum_{j=1}^m w_j=1$. Then the weighted distance measure D^w_{pc}(L,Q) is defined in the following way,

$$
D^w_{pc}(L,Q)=\sum_{i=1}^n w_j\left(\begin{array}{c}\left[\begin{array}{c}|\alpha^{l+}_{B_{Pc}}(x_i)-\alpha^{q+}_{B_{Pc}}(x_i)|+|\gamma^{l+}_{B_{Pc}}(x_i)-\gamma^{q+}_{B_{Pc}}(x_i)|+|\beta^{l+}_{B_{Pc}}(x_i)-\beta^{q+}_{B_{Pc}}(x_i)|\\+|\alpha^{l-}_{B_{Pc}}(x_i)-\alpha^{q-}_{B_{Pc}}(x_i)|+|\gamma^{l-}_{B_{Pc}}(x_i)-\gamma^{q-}_{B_{Pc}}(x_i)|+|\beta^{l-}_{B_{Pc}}(x_i)-\beta^{q-}_{B_{Pc}}(x_i)|\end{array}\right]\\+\max\left[\begin{array}{c}|\alpha^{l+}_{B_{Pc}}(x_i)-\alpha^{q+}_{B_{Pc}}(x_i)|,|\gamma^{l+}_{B_{Pc}}(x_i)-\gamma^{q+}_{B_{Pc}}(x_i)|,|\beta^{l+}_{B_{Pc}}(x_i)-\beta^{q+}_{B_{Pc}}(x_i)|\\,|\alpha^{l-}_{B_{Pc}}(x_i)-\alpha^{q-}_{B_{Pc}}(x_i)|,|\gamma^{l-}_{B_{Pc}}(x_i)-\gamma^{q-}_{B_{Pc}}(x_i)|,|\beta^{l-}_{B_{Pc}}(x_i)-\beta^{q-}_{B_{Pc}}(x_i)|\end{array}\right]\end{array}\right).
$$

Theorem 3.6. *Suppose that $X=\{x_1,x_2,\dots,x_n\}$ is discourse set, then the weighted distance measure D^W_{pc} between the two B_{Pc}FSs satisfies the following properties:*

1. $0\leqslant D^W_{pc}(L,Q)\leqslant 1$;
2. $D^W_{pc}(L,Q)=0$ *iff* $L=Q$;
3. $D^W_{pc}(L,Q)=D^W_{pc}(L,Q)$;
4. $D^W_{pc}(L,R)\geqslant D^W_{pc}(L,Q)$ *and* $D^W_{pc}(L,R)\geqslant D^W_{pc}(Q,R)$, *for any* $L,Q,R\in B_{Pc}FSs(X)$.

Proof. The proof of this Theorem can be completed on the same steps as Theorem 3.4. □

Definition 3.7. Let L and Q be two B_{Pc}FSs defined on a discourse set $X=\{x_1,x_2,\dots,x_n\}$. Then a similarity measure $\tilde{S}_{pc}(L,Q)$ based on Definition 3.5 is defined as:

$$
\tilde{S}_{pc}(L,Q)=1-D^w_{pc}(L,Q).
$$

Definition 3.8. Suppose that $X = \{x_1, x_2, \dots, x_n\}$ is discourse set, then the weighted distance measure D^W_{pc} between the two B_{Pc}FSs satisfies the following properties:

1. $0 \leqslant \tilde{S}_{pc}(L, Q) \leqslant 1$;
2. $\tilde{S}_{pc}(L, Q) = 1$ iff $L = Q$;
3. $\tilde{S}_{pc}(L, Q) = \tilde{S}_{pc}(Q, L)$;
4. $\tilde{S}_{pc}(L, R) \geqslant \tilde{S}_{pc}(L, Q)$ and $\tilde{S}_{pc}(L, R) \geqslant \tilde{S}_{pc}(Q, R)$, for any $L, Q, R \in B_{Pc}FSs(X)$.

4. Bipolar picture fuzzy TOPSIS (B_pF-TOPSIS) for MCDM

In this section, we proposed an MCDM model for B_pF information based on TOPSIS, named B_pF-TOPSIS and LP technique is implemented to evaluate the weights of criteria, under various constraints. A linear objective function of weights is computed with the help of the first four steps of TOPSIS and then used the remaining steps to recognize the best alternative. Let $B = \{B_1, B_2, \dots, B_n\}$ be a discrete set of alternatives, and $S = \{S_1, S_2, \dots, S_m\}$ be the collection of criteria with $w = \{w_1, w_2, \dots, w_m\}$, where $\sum_{j=1}^{m} w_j = 1$ is the weight vector of the criteria S_j where $j = 1, 2, 3, \dots, m$. A B_pF decision matrix (B_pFDM) is represented by $\tilde{B}_p = [\Delta_{ij}]_{n\times m}$ with α_{ij} as a degree of positive acceptance, γ_{ij} degree of positive neutral and β_{ij} degree of positive rejection while, α_{ij} as a degree of negative acceptance, γ_{ij} degree of negative neutral and β_{ij} degree of negative rejection of $B_i (i = 1, 2, \dots, n)$, respectively. The proposed B_pF-TOPSIS consists of the following steps.

Step 1. Form a B_pFDM, $\tilde{B}_p = [\Delta_{ij}]_{n\times m}$ based on the information provided by the DMs.

Step 2. Find out the bipolar picture fuzzy positive ideal solution (B_pFPIS) denoted by Δ^+ and bipolar picture fuzzy negative ideal solution (B_pFNIS) represented by Δ^-, respectively for beneficial criteria,

$$\Delta^+ = \left((\max_j(\alpha^+_{ij}), \max_j(\gamma^+_{ij}), \max_j(\beta^+_{ij})), \min_j(\alpha^-_{ij}), \min_j(\gamma^-_{ij}), \min_j(\beta^-_{ij})) \right),$$
$$\Delta^- = \left((\min_j(\alpha^+_{ij}), \min_j(\gamma^+_{ij}), \min_j(\beta^+_{ij})), \max_j(\alpha^-_{ij}), \max_j(\gamma^-_{ij}), \max_j(\beta^-_{ij})) \right).$$

Step 3. Based on Definition 3.7, calculate the degree of weighted similarity $\tilde{S}^+_{p_i}$ between B_pFPIS Δ^+ and each alternative as well as the degree of weighted similarity $\tilde{S}^-_{p_i}$ between B_pFNIS Δ^- by using the Eqs. below, respectively:

$$\tilde{S}^+_{Pci}(B_i, \Delta^+) = 1 - D^w_{Pc}(B_i, \Delta^+), \tag{4.1}$$
$$\tilde{S}^-_{Pci}(B_i, \Delta^-) = 1 - D^w_{Pc}(B_i, \Delta^-), \tag{4.2}$$

where, $1 \leqslant i \leqslant n$.

Step 4. Based on Eqs. (4.1) and (4.2) established an model to find the objective function Z to compute the weights of criteria under the given constraints,

$$Z = \sum_{i=1}^{n} (\tilde{S}^+_{Pci}(B_i, \Delta^+) - \tilde{S}^-_{Pci}(B_i, \Delta^-)). \tag{4.3}$$

Step 5. Based on LP model described in Section 2, compute the weights w_j of the criteria U_j where $j = 1, 2, 3, \dots, m$ such that the objective function Z is maximized.

Step 6. Calculate the degree of similarity $\tilde{S}^+_{Pci}$ and $\tilde{S}^-_{Pci}$ among each alternative and the elements obtained in B_PFPIS Δ^+ and B_PFNIS Δ^-, respectively.

Step 7. Compute the relative closeness R_{Ci} of alternative B_i with respect to the B_PFPIS Δ^+ as:

$$R_{Ci} = \frac{\tilde{S}^+_{Pci}}{\tilde{S}^+_{Pci} + \tilde{S}^-_{Pci}}. \tag{4.4}$$

The larger the value of the relative closeness R_{Ci} of the alternatives with regard to the B_PFPIS $\tilde{S}^+_{Pci}$ means that, we get the best alternative from different alternative B_i, where $1 \leqslant i \leqslant n$.

5. Practical example

In this section, an example of the MCDM problem of alternatives is used as the illustration of the application of the proposed MCDM model. Consider an organization that needs to recruit the technical staff to manage the technical issues of the organization. In order to resolve the issue, DM arrange the interview of the five short-listed candidates (alternatives), $B = \{B_1, B_2, \ldots, B_5\}$ under the following four beneficial criteria $Q = \{Q_1, Q_2, Q_3, Q_4\}$ such that: Q_1 (advancement in technology), Q_2 (market potential), Q_3 (the ability of vendors) and Q_4 (formation of employment and the innovations in technology and of science). The five possible alternatives are to be evaluated by using the bipolar picture fuzzy decision matrix B_pFDM, $\tilde{B}_p = [\Delta_{ij}]_{5\times4}$ presented in Table 1.

Step 1. Information provided by the DM is written as BP_cFDM , $B_{pc} = [b_{ij}]_{5\times4}$.

Table 1: B_pFDM, $\tilde{B}_p$ provided by the DM.

	Q_1	Q_2	Q_3	Q_4
B_1	$(0.5, 0.3, 0.10, -0.2, -0.1, -0.5)$	$(0.8, 0.1, 0.1, -0.3, -0.4, -0.2)$	$(0.4, 0.3, 0.1, -0.5, -0.3, -0.0)$	$(0.9, 0.0, 0.1, -0.4, -0.3, -0.2)$
B_2	$(0.7, 0.1, 0.2, -0.3, -0.4, -0.1)$	$(0.1, 0.6, 0.2, -0.2, -0.3, -0.3)$	$(0.6, 0.3, 0.1, -0.4, -0.5, -0.1)$	$(0.7, 0.1, 0.1, -0.4, -0.3, -0.1)$
B_3	$(0.8, 0.0, 0.2, -0.3, -0.2, -0.1)$	$(0.8, 0.1, 0.0, -0.5, -0.3, -0.2)$	$(0.1, 0.8, 0.1, -0.2, -0.1, -0.6)$	$(0.6, 0.2, 0.1, -0.4, -0.3, -0.2)$
B_4	$(0.8, 0.0, 0.2, -0.3, -0.4, -0.3)$	$(0.7, 0.1, 0.2, -0.3, -0.2, -0.5)$	$(0.3, 0.5, 0.2, -0.1, -0.4, -0.3)$	$(0.7, 0.2, 0.1, -0.4, -0.3, -0.1)$
B_5	$(0.5, 0.4, 0.0, -0.4, -0.3, -0.2)$	$(0.6, 0.2, 0.1, -0.3, -0.7, 0.0)$	$(0.6, 0.2, 0.2, -0.3, -0.4, -0.3)$	$(0.1, 0.7, 0.2, -0.3, -0.2, -0.1)$

Step 2. The B_pFPIS denoted by Δ^+ and B_pFNIS represented by Δ^- are: $\Delta^+_{pc} = ((0.8, 0.4, 0.2, -0.4 - 0.4, -0.5), (0.8, 0.6, 0.2, -0.5, -0.4, -0.5), (0.6, 0.8, 0.2, -0.5, -0.4, -0.6), (0.9, 0.2, 0.4, -0.5, -0.5, -0.2))$, $\Delta^-_{pc} = ((0.5, 0.1, 0.0, -0.2 - 0.1, -0.1), (0.1, 0.1, 0.0, -0.2, -0.2, 0.0), (0.1, 0.2, 0.1, -0.1, -0.1, -0.0), (0.1, 0.0, 0.1, -0.3, -0.2, -0.1))$.

Step 3. Evaluate the degree of weighted similarity $\tilde{S}^+_{p_i}$ between B_pFPIS Δ^+ and each alternative as well as the degree of weighted similarity $\tilde{S}^-_{p_i}$ between B_pFNIS Δ^- by using the Eqs. (4.1) and (4.2).

Step 4. Based on Eq. (4.3), we get the linear objective function Z as:

$$Z = 0.8800w_1 + 1.6933w_2 + 1.6800w_3 + 0.8893w_4.$$

Step 5. Based on LP model as described in Section 2, compute the weights $w_j (j = 1, 2, 3, 4)$ of criteria with distinct limitation given below:

$$\begin{aligned}
\max Z = {} & 0.8800w_1 + 1.6933w_2 + 1.6800w_3 + 0.8893w_4,\\
& 0.9000w_1 + 0.5000w_2 + 0.1000w_3 + 0.6000w_4 \geqslant 0.2000,\\
& 0.9000w_1 + 0.5000w_2 + 0.1000w_3 + 0.6000w_4 \leqslant 0.3500,\\
& 0.3000w_1 + 0.1100w_2 + 0.7000w_3 + 0.5000w_4 \geqslant 0.0500,\\
& 0.3000w_1 + 1.1000w_2 + 0.7000w_3 + 0.5000w_4 \leqslant 0.0550,\\
& 0.2000w_1 + 0.5000w_2 + 0.2000w_3 + 0.4000w_4 \geqslant 0.0300,\\
& 0.2000w_1 + 0.5000w_2 + 0.2000w_3 + 0.4000w_4 \leqslant 0.0350,\\
& 1.0000w_1 + 1.0000w_2 + 1.0000w_3 + 1.0000w_4 = 1,\\
& 0.1000 \leqslant w_1 \leqslant 0.20000,\\
& 0.2500 \leqslant w_2 \leqslant 0.3000,\\
& 0.3500 \leqslant w_3 \leqslant 0.4000,\\
& 0.1500 \leqslant w_4 \leqslant 0.2000,
\end{aligned}$$

we get, $w_1 = 0.1000; w_2 = 0.4000; w_3 = 0.3500$ and $w_4 = 0.1500$.

Step 6. On the basis of weights of criteria as obtained in Step 5, compute the degree of similarity $\tilde{S}^+_{Pci}$ and $\tilde{S}^-_{Pci}$ amongst each alternative and the elements obtained in B_PFPIS Δ^+ and B_PFNIS Δ^-, respectively, we get: $\tilde{S}^{+}_{\mathrm{Pc}1} = 0.6230$; $\tilde{S}^{+}_{\mathrm{Pc}2} = 0.6230$; $\tilde{S}^{+}_{\mathrm{Pc}3} = 0.6500$ $\tilde{S}^{+}_{\mathrm{Pc}4} = 0.6660$; $\tilde{S}^{+}_{\mathrm{Pc}5} = 0.6410$, and $\tilde{S}^{-}_{\mathrm{Pc}1} = 0.6610$; $\tilde{S}^{-}_{\mathrm{Pc}1} = 0.6700$; $\tilde{S}^{-}_{\mathrm{Pc}1} = 0.6420$; $\tilde{S}^{-}_{\mathrm{Pc}1} = 0.6590$; $\tilde{S}^{-}_{\mathrm{Pc}1} = 0.6520$.

Step 7. From Eq. (4.4), we obtain values of relative closeness R_{Ci} of each alternative B_i with respect to the B_PFPIS Δ^+ as:

$$R_{C1} = 0.4852; \quad R_{C2} = 0.4818; \quad R_{C3} = 0.5031; \quad R_{C4} = 0.5026; \quad R_{C5} = 0.4957.$$

It reveals that, $R_{C3} \succ R_{C4} \succ R_{C5} \succ R_{C1} \succ R_{C2}, \Rightarrow B_3 \succ B_4 \succ B_5 \succ B_1 \succ B_2$ that is, B_3 is the best option or alternative.

6. Sensitivity analysis

In order to see the validity and stability of the proposed B_{P_c}F-TOPSIS, a weighted sensitivity analysis is performed [11]. According to Mareschal [11], mostly MCDM techniques require the quantitative weights of the criteria which are sometimes difficult to get because we cannot be sure that the DMs have provided the precise weights to the criteria. Thereby, it is important to compute what changes occur by altering the weights of criteria. If there are minor or no changes happened then we are more confident about the results. In light of our performed sensitivity analysis, we examined the four criteria individually by increasing the weights from 2 to 10 percent randomly. We see that there is no minor change that happened in the arrangement of the criteria which represents that our B_{P_c}F-TOPSIS MCDM model is effective and stronger.

Table 2: Results obtained by proposed MCDM model.

Relative closeness	Original Values	2 percent increase	5 percent increase	10 percent increase
R_{C1}	0.4852	0.4847	0.4840	0.4828
R_{C2}	0.4818	0.4813	0.4804	0.4789
R_{C3}	0.5031	0.5032	0.5033	0.5036
R_{C4}	0.5026	0.5027	0.5028	0.5031
R_{C5}	0.4957	0.4956	0.4954	0.4951
Alternatives	$B_3 \succ B_4 \succ B_5 \succ B_1 \succ B_2$	$B_3 \succ B_4 \succ B_5 \succ B_1 \succ B_2$	$B_3 \succ B_4 \succ B_5 \succ B_1 \succ B_2$	$B_3 \succ B_4 \succ B_5 \succ B_1 \succ B_2$

7. Conclusions

We introduced the concept of bipolar picture fuzzy sets, operational rules, and extended the TOPSIS named B_{P_c}F-TOPSIS in this article. On the basis of the novel distance measure, an MCDM model (B_{P_c}F-TOPSIS) is developed to select the best alternative. A sensitivity analysis is performed to strengthen our MCDM approach. In the future, we shall establish aggregation operators like Bonferroni, and Hamy mean for B_{P_c}FSs and implement these operators to solve the MCDM problems. Also, we shall present the concept of interval-valued bipolar picture fuzzy sets (IVB_{P_c}FSs) and operational laws. Further, we shall apply the IVB_{P_c}FSs in various group decision-making problems, like signature theory, signal processing, and operations management.

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