



On approximate solutions for fractional system of differential equations with Caputo-Fabrizio fractional operator



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Abstract

In this paper, we apply the Daftardar-Jafari method (DJM) and Sumudu decomposition method (SDM) to obtain the approximate analytical solutions of the fractional system of differential equations with Caputo-Fabrizio fractional derivative operator. The results obtained by DJM are compared with the results obtained by SDM. The results reveal that the suggested algorithms are very effective and simple and can be applied for linear and nonlinear problems in sciences and engineering.

Keywords: Daftardar-Jafari method, Sumudu transform, Adomian decomposition method, Caputo-Fabrizio fractional operator.

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1. Introduction

The concept of fractional calculus can be traced back to the letter of Leibniz to L'Hospital in 1695. Mathematician Euler discovered the gamma function in 1729, which established the foundation of fractional derivative. Compared with the classical integer-order models, fractional derivatives provide a more profound and comprehensive explanation about memory, heredity, non-locality of complex phenomena and processes. At present, the most commonly used definitions of fractional derivatives are the Grunwald-Letnikov derivative, the Riemann-Liouville derivative, the Riesz derivative, and the Caputo derivative [22]. Some new fractional derivatives, such as the Caputo-Fabrizio derivative and the Atangana-Baleanu fractional derivative, have tremendously promoted the capability of modeling complex physical phenomena and processes. The connatural behavior has been analyzed for evolution equations generated by three fractional derivatives namely the Riemann-Liouville, Caputo-Fabrizio, and Atangana-Baleanu fractional derivatives [2, 3, 17].

Several analytical and numerical techniques were successfully applied to deal with fractional partial differential equations, the homotopy perturbation method [10, 23], the series expansion method [18, 25], Adomian decomposition methods [5, 27], the variational iteration method [16, 20], reduce differential transform method [12, 14], homotopy perturbation transform methods [4, 9, 21], and other analytical approaches that could be of interest for the reader are presented in [6, 7, 11, 13, 15, 19, 24, 26].

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In this paper, we use the DJM and SDM to solve nonlinear fractional system of partial differential equations involving the fractional operator of Caputo-Fabrizio type. The paper has been organized as follows. The basic definitions of fractional calculus are given in Section 2, analysis of the methods used is given in Section 3 and Section 4, An illustrative example that show the effectiveness of the proposed method is given in Section 5, and finally the conclusion is given in Section 6.

2. Preliminaries of fractional calculus

Definition 2.1 ([1, 8, 9, 24]). The Caputo-Fabrizio fractional operator is defined for $n - 1 < \alpha \leq n$ as:

$${}^{CF}D_t^\alpha u(t) = \frac{(2-\alpha)M(\alpha)}{2(1-\alpha)} \int_0^t \exp\left[-\frac{\alpha(t-s)}{1-\alpha}\right] u^{(n)}(s) ds, \quad t \geq 0,$$

where $M(\alpha)$ is a normalization function such that $M(0) = M(1) = 1$ and $u^{(n)}$ is the derivative of integer n order of u , $n = 1, 2, \dots, \in \mathbb{N}$.

The following are the basic properties of the operator ${}^{CF}D_t^\alpha$:

1. ${}^{CF}D_t^\alpha u(t) = u(t)$, where $\alpha = 0$;
2. ${}^{CF}D_t^\alpha [u(t) + v(t)] = {}^{CF}D_t^\alpha u(t) + {}^{CF}D_t^\alpha v(t)$;
3. ${}^{CF}D_t^\alpha (c) = 0$, where c is constant.

Definition 2.2 ([1, 8]). The fractional Caputo-Fabrizio fractional integral operator of order $0 < \alpha \leq 1$ and $t > 0$ is given by:

$${}^{CF}I_t^\alpha u(t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} u(t) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t u(s) ds.$$

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2. ${}^{CF}I_t^\alpha [u(t) + v(t)] = {}^{CF}I_t^\alpha u(t) + {}^{CF}I_t^\alpha v(t)$;
3. ${}^{CF}I_t^\alpha [{}^{CF}D_t^\alpha u(t)] = u(t) - u(0)$.

Definition 2.3. The Sumudu transform is defined over the set of function

$$A = \left\{ u(t) \mid \exists M, \tau_1, \tau_2 > 0, |u(t)| < Me^{t/\tau_j}, \text{ if } t \in (-1)^j \times [0, \infty) \right\},$$

by the following formula

$$S\{u(t)\} = \int_0^\infty u(wt)e^{-t} dt, \quad w \in (\tau_1, \tau_2).$$

3. Analysis of DJ method

Let us consider the following nonlinear partial differential equation in the Caputo-Fabrizio sense:

$${}^{CF}D_t^{n+\alpha} u(x, t) + R[u(x, t)] + N[u(x, t)] = g(x, t) \quad (3.1)$$

with initial conditions

$$\frac{\partial^k u(x, 0)}{\partial t^k} = \phi_k(x), \quad k = 0, 1, \dots, m-1,$$

where ${}^{CF}D_t^{n+\alpha} u(x, t)$ is Caputo-Fabrizio operator of $u(x, t)$, $m-1 < n+\alpha \leq m$, $m \in \mathbb{N}$, R is a linear operator, N is a nonlinear operator and g is a source term.

Taking integral of Caputo-Fabrizio to both sides of (3.1) we get

$${}^{CF}I_t^\alpha [{}^{CF}D_t^{n+\alpha} u(x, t)] + {}^{CF}I_t^\alpha R[u(x, t)] + {}^{CF}I_t^\alpha N[u(x, t)] = {}^{CF}I_t^\alpha [g(x, t)].$$

Then, we obtain

$$u(x, t) = \sum_{k=0}^{m-1} u^{(k)}(x, 0) \frac{x^k}{k!} + {}^{CF}I_t^\alpha [g(x, t)] - {}^{CF}I_t^\alpha R[u(x, t)] - {}^{CF}I_t^\alpha N[u(x, t)]. \quad (3.2)$$

We are looking for a solution $u(x, t)$ of Eq. (3.2) having the series form:

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t). \quad (3.3)$$

The non-linear operator N can be decomposed as

$$N \left[\sum_{n=0}^{\infty} u_n(x, t) \right] = N[u_0] + \sum_{n=1}^{\infty} \left(N \left[\sum_{i=0}^n u_i \right] - N \left[\sum_{i=0}^{n-1} u_i \right] \right). \quad (3.4)$$

In view of (3.3) and (3.4), Eq. (3.2) is equivalent to

$$\begin{aligned} \sum_{n=0}^{\infty} u_n &= \sum_{k=0}^{m-1} u^{(k)}(x, 0) \frac{x^k}{k!} + {}^{CF}I_t^\alpha [g(x, t)] - {}^{CF}I_t^\alpha R \left[\sum_{n=0}^{\infty} u_n \right] \\ &\quad - {}^{CF}I_t^\alpha N[u_0(x, t)] - {}^{CF}I_t^\alpha \left(\sum_{n=1}^{\infty} N \left[\sum_{i=0}^n u_i \right] - N \left[\sum_{i=0}^{n-1} u_i \right] \right). \end{aligned}$$

Moreover, the relation is defined with recurrence so that

$$\begin{aligned} u_0(x, t) &= \sum_{k=0}^{m-1} u^{(k)}(x, 0) \frac{x^k}{k!} + {}^{CF}I_t^\alpha [g(x, t)], \\ u_1(x, t) &= -{}^{CF}I_t^\alpha R[u_0] - {}^{CF}I_t^\alpha N[u_0(x, t)], \\ u_{n+1}(x, t) &= -{}^{CF}I_t^\alpha R[u_n] - {}^{CF}I_t^\alpha \left(N \left[\sum_{i=0}^n u_i \right] - N \left[\sum_{i=0}^{n-1} u_i \right] \right), n = 1, 2, \dots \end{aligned} \quad (3.5)$$

Then k -term approximate solution of Eq. (3.1) is given by:

$$u(x, t) = u_0(x, t) + u_1(x, t) + u_2(x, t) + u_3(x, t) + \dots$$

4. Analysis of Sumudu decomposition method

Applying the Sumudu transform to (3.1) yields

$$\mathbf{S} \{ {}^{CF}D_t^{n+\alpha} u(x, t) \} + \mathbf{S} \{ R[u(x, t)] \} + \mathbf{S} \{ N[u(x, t)] \} = \mathbf{S} \{ g(x, t) \}.$$

By applying the Sumudu transform differentiation property, we have

$$\frac{1}{(1 - \alpha + \alpha\sigma)(\sigma^m)} \left[\mathbf{S} \{ u(x, t) \} - \sum_{k=0}^{m-1} w^k \phi_k(x) \right] = \mathbf{S} \{ g(x, t) \} - \mathbf{S} \{ R[u(x, t)] \} - \mathbf{S} \{ N[u(x, t)] \}$$

or

$$\begin{aligned} \mathbf{S} \{ u(x, t) \} &= \sum_{k=0}^{m-1} w^k \phi_k(x) + (1 - \alpha + \alpha\sigma) \sigma^m \mathbf{S} \{ g(x, t) \} \\ &\quad - (1 - \alpha + \alpha\sigma) \sigma^m \mathbf{S} \{ R[u(x, t)] \} - (1 - \alpha + \alpha\sigma) \sigma^m \mathbf{S} \{ N[u(x, t)] \}. \end{aligned} \quad (4.1)$$

Taking the inverse of Sumudu transform of (4.1), we obtain

$$\begin{aligned} u(x, t) = \mathbf{S}^{-1} \left(\sum_{k=0}^{m-1} w^k \phi_k(x) \right) + \mathbf{S}^{-1} [(1 - \alpha + \alpha\sigma)\sigma^m \mathbf{S}\{g(x, t)\}] \\ - \mathbf{S}^{-1} [(1 - \alpha + \alpha\sigma)\sigma^m \mathbf{S}\{R[u(x, t)]\}] - \mathbf{S}^{-1} [(1 - \alpha + \alpha\sigma)\sigma^m \mathbf{S}\{N[u(x, t)]\}]. \end{aligned} \quad (4.2)$$

Suppose that $u(x, t)$ is a solution of (4.2), which we express as

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t), \quad (4.3)$$

the nonlinear term can be decomposed as

$$N[u(x, t)] = \sum_{n=0}^{\infty} A_n, \quad (4.4)$$

where

$$A_n = \frac{1}{n!} \frac{\partial^n}{\partial \xi^n} \left[N \left(\sum_{i=0}^n \xi^i u_i(x, t) \right) \right]_{\xi=0}, \quad n = 0, 1, 2, \dots$$

Substituting (4.3) and (4.4) into (4.2) gives us the result that

$$\begin{aligned} u(x, t) = \mathbf{S}^{-1} \left(\sum_{k=0}^{m-1} w^k \phi_k(x) \right) + \mathbf{S}^{-1} [(1 - \alpha + \alpha\sigma)\sigma^m \mathbf{S}\{g(x, t)\}] \\ - \mathbf{S}^{-1} \left[(1 - \alpha + \alpha\sigma)\sigma^m \mathbf{S} \left\{ R \left[\sum_{n=0}^{\infty} u_n(x, t) \right] \right\} \right] - \mathbf{S}^{-1} \left[(1 - \alpha + \alpha\sigma)\sigma^m \mathbf{S} \left\{ \sum_{n=0}^{\infty} A_n \right\} \right]. \end{aligned} \quad (4.5)$$

When we compare the left and right hand sides of (4.5) we obtain

$$\begin{aligned} u_0(x, t) &= \mathbf{S}^{-1} \left(\sum_{k=0}^{m-1} w^k \phi_k(x) \right) + \mathbf{S}^{-1} [(1 - \alpha + \alpha\sigma)\sigma^m \mathbf{S}\{g(x, t)\}], \\ u_1(x, t) &= -\mathbf{S}^{-1} [(1 - \alpha + \alpha\sigma)\sigma^m \mathbf{S}\{R[u_0(x, t)]\}] - \mathbf{S}^{-1} [(1 - \alpha + \alpha\sigma)\sigma^m \mathbf{S}\{A_0\}], \\ u_2(x, t) &= -\mathbf{S}^{-1} [(1 - \alpha + \alpha\sigma)\sigma^m \mathbf{S}\{R[u_1(x, t)]\}] - \mathbf{S}^{-1} [(1 - \alpha + \alpha\sigma)\sigma^m \mathbf{S}\{A_1\}], \\ &\vdots \end{aligned}$$

The recursive relation in its general form is

$$\begin{aligned} u_0(x, t) &= \mathbf{S}^{-1} \left(\sum_{k=0}^{m-1} w^k \phi_k(x) \right) + \mathbf{S}^{-1} [(1 - \alpha + \alpha\sigma)\sigma^m \mathbf{S}\{g(x, t)\}], \\ u_n(x, t) &= -\mathbf{S}^{-1} [(1 - \alpha + \alpha\sigma)\sigma^m \mathbf{S}\{R[u_{n-1}(x, t)]\}] - \mathbf{S}^{-1} [(1 - \alpha + \alpha\sigma)\sigma^m \mathbf{S}\{A_{n-1}\}]. \end{aligned}$$

The approximate solution is given by

$$u(x, t) = u_0(x, t) + u_2(x, t) + u_3(x, t) + \dots$$

5. Illustrative paradigm

Example 1. Consider the nonlinear system of time-fractional differential equation in the Caputo-Fabrizio operator:

$$\begin{aligned} {}^{CF}D_t^\alpha(x, y, t) &= -u + v_y w_x - w_y v_x, \\ {}^{CF}D_t^\beta(x, y, t) &= v - u_y w_x - w_y u_x, \\ {}^{CF}D_t^\varepsilon(x, y, t) &= w - v_y u_x - u_y v_x, \end{aligned} \quad (5.1)$$

where $0 < \alpha, \beta, \varepsilon \leq 1$ and the initial conditions are

$$u(x, y, 0) = e^{x+y}, \quad v(x, y, 0) = e^{x-y}, \quad w(x, y, 0) = e^{y-x}.$$

Below we present the DJM.

Taking ${}^{CF}I_t^\alpha$, ${}^{CF}I_t^\beta$, and ${}^{CF}I_t^\varepsilon$ to both sides of (5.1), respectively, we get

$$\begin{aligned} u(x, y, t) &= u(x, y, 0) - {}^{CF}I_t^\alpha [u] + {}^{CF}I_t^\alpha \left[\frac{\partial v}{\partial y} \frac{\partial w}{\partial x} - \frac{\partial w}{\partial y} \frac{\partial v}{\partial x} \right], \\ v(x, y, t) &= v(x, y, 0) + {}^{CF}I_t^\beta [v] - {}^{CF}I_t^\beta \left[\frac{\partial u}{\partial y} \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} \frac{\partial u}{\partial x} \right], \\ w(x, y, t) &= w(x, y, 0) + {}^{CF}I_t^\varepsilon [w] - {}^{CF}I_t^\varepsilon \left[\frac{\partial v}{\partial y} \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} \right]. \end{aligned}$$

Thus according to Eq. (3.5), the approximate solution can be obtained:

$$\begin{aligned} u_0(x, y, t) &= u(x, y, 0), \quad v_0(x, y, t) = v(x, y, 0), \quad w_0(x, y, t) = w(x, y, 0), \\ u_1(x, y, t) &= -{}^{CF}I_t^\alpha [u_0] + {}^{CF}I_t^\alpha \left[\frac{\partial v_0}{\partial y} \frac{\partial w_0}{\partial x} - \frac{\partial w_0}{\partial y} \frac{\partial v_0}{\partial x} \right], \\ v_1(x, y, t) &= {}^{CF}I_t^\beta [v_0] - {}^{CF}I_t^\beta \left[\frac{\partial u_0}{\partial y} \frac{\partial w_0}{\partial x} + \frac{\partial w_0}{\partial y} \frac{\partial u_0}{\partial x} \right], \\ w_1(x, y, t) &= {}^{CF}I_t^\varepsilon [w_0] - {}^{CF}I_t^\varepsilon \left[\frac{\partial v_0}{\partial y} \frac{\partial u_0}{\partial x} + \frac{\partial u_0}{\partial y} \frac{\partial v_0}{\partial x} \right], \\ u_2(x, y, t) &= -{}^{CF}I_t^\alpha [u_1] + {}^{CF}I_t^\alpha \left[\frac{\partial(v_0 + v_1)}{\partial y} \frac{\partial(w_0 + w_1)}{\partial x} - \frac{\partial v_0}{\partial y} \frac{\partial w_0}{\partial x} \right. \\ &\quad \left. - {}^{CF}I_t^\alpha \left[\frac{\partial(w_0 + w_1)}{\partial y} \frac{\partial(v_0 + v_1)}{\partial x} + \frac{\partial w_0}{\partial y} \frac{\partial v_0}{\partial x} \right] \right], \\ v_2(x, y, t) &= {}^{CF}I_t^\beta [v_1] - {}^{CF}I_t^\beta \left[\frac{\partial(u_0 + u_1)}{\partial y} \frac{\partial(w_0 + w_1)}{\partial x} - \frac{\partial u_0}{\partial y} \frac{\partial w_0}{\partial x} \right] \\ &\quad + {}^{CF}I_t^\beta \left[\frac{\partial(w_0 + w_1)}{\partial y} \frac{\partial(u_0 + u_1)}{\partial x} + \frac{\partial w_0}{\partial y} \frac{\partial u_0}{\partial x} \right], \\ w_2(x, y, t) &= {}^{CF}I_t^\varepsilon [w_1] - {}^{CF}I_t^\varepsilon \left[\frac{\partial(v_0 + v_1)}{\partial y} \frac{\partial(u_0 + u_1)}{\partial x} - \frac{\partial v_0}{\partial y} \frac{\partial u_0}{\partial x} \right] \\ &\quad - {}^{CF}I_t^\varepsilon \left[\frac{\partial(u_0 + u_1)}{\partial y} \frac{\partial(v_0 + v_1)}{\partial x} + \frac{\partial u_0}{\partial y} \frac{\partial v_0}{\partial x} \right] \\ &\quad \vdots \end{aligned}$$

By the above algorithms, we obtain:

$$\begin{aligned} u_0(x, y, t) &= e^{x+y}, \quad v_0(x, y, t) = e^{x-y}, \quad w_0(x, y, t) = e^{y-x}, \\ u_1(x, y, t) &= -{}^{CF}I_t^\alpha [e^{x+y}] + {}^{CF}I_t^\alpha [e^{x-y}e^{y-x} - e^{y-x}e^{x-y}] = -e^{x+y}(1 - \alpha + \alpha t), \\ v_1(x, y, t) &= {}^{CF}I_t^\beta [e^{x-y}] - {}^{CF}I_t^\beta [-e^{x+y}e^{y-x} + e^{y-x}e^{x+y}] = e^{x-y}(1 - \beta + \beta t), \\ w_1(x, y, t) &= {}^{CF}I_t^\varepsilon [e^{x-y}] - {}^{CF}I_t^\varepsilon [-e^{x+y}e^{x-y} + e^{x-y}e^{x+y}] = e^{y-x}(1 - \varepsilon + \varepsilon t), \\ u_2(x, y, t) &= -{}^{CF}I_t^\alpha [-e^{x+y}(1 - \alpha + \alpha t)] + {}^{CF}I_t^\alpha [(2 - \beta + \beta t)(2 - \varepsilon + \varepsilon t) - (2 - \varepsilon + \varepsilon t)(2 - \beta + \beta t)] \\ &\quad - {}^{CF}I_t^\alpha [e^{x-y}e^{y-x} - e^{y-x}e^{x-y}] \\ &= (1 - \alpha)e^{x+y}(1 - \alpha + \alpha t) + \alpha e^{x+y}(t - \alpha t + \frac{1}{2}\alpha t^2), \\ v_2(x, y, t) &= {}^{CF}I_t^\beta [e^{x-y}(1 - \beta + \beta t)] - {}^{CF}I_t^\beta [-(\alpha t - \alpha)(2 - \varepsilon + \varepsilon t) + (2 - \varepsilon + \varepsilon t)(\alpha t - \alpha)] \end{aligned}$$

$$\begin{aligned}
 & + {}^{\text{CF}} I_t^\beta [-e^{x+y} e^{y-x} + e^{y-x} e^{x+y}] \\
 & = (1 - \beta) e^{x-y} (1 - \beta + \beta t) + \beta e^{x-y} (t - \beta t + \frac{1}{2} \beta t^2), \\
 w_2(x, y, t) & = {}^{\text{CF}} I_t^\varepsilon [e^{y-x} (1 - \varepsilon + \varepsilon t)] - {}^{\text{CF}} I_t^\varepsilon [-(\alpha t - \alpha)(2 - \beta + \beta t) + (2 - \beta + \beta t)(\alpha t - \alpha)] \\
 & + {}^{\text{CF}} I_t^\varepsilon [-e^{x+y} e^{x-y} + e^{x-y} e^{x+y}] \\
 & = (1 - \varepsilon) e^{y-x} (1 - \varepsilon + \varepsilon t) + \varepsilon e^{y-x} (t - \varepsilon t + \frac{1}{2} \varepsilon t^2) \\
 & \vdots
 \end{aligned}$$

Then the approximate solution of (5.1) is

$$\begin{aligned}
 u(x, y, t) & = e^{x+y} - e^{x+y} (1 - \alpha + \alpha t) + (1 - \alpha) e^{x+y} (1 - \alpha + \alpha t) + \alpha e^{x+y} (t - \alpha t + \frac{1}{2} \alpha t^2) + \dots, \\
 v(x, y, t) & = e^{x-y} + e^{x-y} (1 - \beta + \beta t) + (1 - \beta) e^{x-y} (1 - \beta + \beta t) + \beta e^{x-y} (t - \beta t + \frac{1}{2} \beta t^2) + \dots, \quad (5.2) \\
 w(x, y, t) & = e^{y-x} + e^{y-x} (1 - \varepsilon + \varepsilon t) + (1 - \varepsilon) e^{y-x} (1 - \varepsilon + \varepsilon t) + \varepsilon e^{y-x} (t - \varepsilon t + \frac{1}{2} \varepsilon t^2) + \dots.
 \end{aligned}$$

As a next step, we apply the SADM. Taking the Sumudu transform on both sides of (5.1), we have

$$\begin{aligned}
 \mathbf{S}\{u(x, y, t)\} & = u(x, y, 0) + (1 - \alpha + \alpha \sigma) \mathbf{S} \left\{ -u + \frac{\partial v}{\partial y} \frac{\partial w}{\partial x} - \frac{\partial w}{\partial y} \frac{\partial v}{\partial x} \right\}, \\
 \mathbf{S}\{v(x, y, t)\} & = v(x, y, 0) + (1 - \beta + \beta \sigma) \mathbf{S} \left\{ v - \frac{\partial u}{\partial y} \frac{\partial w}{\partial x} - \frac{\partial w}{\partial y} \frac{\partial u}{\partial x} \right\}, \\
 \mathbf{S}\{w(x, y, t)\} & = w(x, y, 0) + (1 - \varepsilon + \varepsilon \sigma) \mathbf{S} \left\{ w - \frac{\partial v}{\partial y} \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} \right\},
 \end{aligned}$$

or

$$\begin{aligned}
 \mathbf{S}\{u(x, y, t)\} & = e^{x+y} + (1 - \alpha + \alpha \sigma) \mathbf{S} \left\{ -u + \frac{\partial v}{\partial y} \frac{\partial w}{\partial x} - \frac{\partial w}{\partial y} \frac{\partial v}{\partial x} \right\}, \\
 \mathbf{S}\{v(x, y, t)\} & = e^{x-y} + (1 - \beta + \beta \sigma) \mathbf{S} \left\{ v - \frac{\partial u}{\partial y} \frac{\partial w}{\partial x} - \frac{\partial w}{\partial y} \frac{\partial u}{\partial x} \right\}, \quad (5.3) \\
 \mathbf{S}\{w(x, y, t)\} & = e^{y-x} + (1 - \varepsilon + \varepsilon \sigma) \mathbf{S} \left\{ w - \frac{\partial v}{\partial y} \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} \right\}.
 \end{aligned}$$

Operating with the ST on both sides of (5.3) gives

$$\begin{aligned}
 u(x, y, t) & = e^{x+y} + \mathbf{S}^{-1} \left[(1 - \alpha + \alpha \sigma) \mathbf{S} \left\{ -u + \frac{\partial v}{\partial y} \frac{\partial w}{\partial x} - \frac{\partial w}{\partial y} \frac{\partial v}{\partial x} \right\} \right], \\
 v(x, y, t) & = e^{x-y} + \mathbf{S}^{-1} \left[(1 - \beta + \beta \sigma) \mathbf{S} \left\{ v - \frac{\partial u}{\partial y} \frac{\partial w}{\partial x} - \frac{\partial w}{\partial y} \frac{\partial u}{\partial x} \right\} \right], \quad (5.4) \\
 w(x, y, t) & = e^{y-x} + \mathbf{S}^{-1} \left[(1 - \varepsilon + \varepsilon \sigma) \mathbf{S} \left\{ w - \frac{\partial v}{\partial y} \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} \right\} \right].
 \end{aligned}$$

Now, we represent solution as an infinite series given below

$$u(x, y, t) = \sum_{n=0}^{\infty} u_n(x, y, t), \quad v(x, y, t) = \sum_{n=0}^{\infty} v_n(x, y, t), \quad w(x, y, t) = \sum_{n=0}^{\infty} w_n(x, y, t), \quad (5.5)$$

and the nonlinear terms can be decomposed as

$$v_y w_x - w_y v_x = \sum_{n=0}^{\infty} A_n, \quad u_y w_x - w_y u_x = \sum_{n=0}^{\infty} B_n, \quad v_y u_x - u_y v_x = \sum_{n=0}^{\infty} C_n, \quad (5.6)$$

where

$$\begin{aligned} A_0 &= \frac{\partial v_0}{\partial y} \frac{\partial w_0}{\partial x} - \frac{\partial w_0}{\partial y_0} \frac{\partial v_0}{\partial x}, \\ A_1 &= \frac{\partial v_0}{\partial y} \frac{\partial w_1}{\partial x} + \frac{\partial v_1}{\partial y} \frac{\partial w_0}{\partial x} - \frac{\partial w_0}{\partial y} \frac{\partial v_1}{\partial x} - \frac{\partial w_1}{\partial y} \frac{\partial v_0}{\partial x}, \\ &\vdots \\ B_0 &= -\frac{\partial u_0}{\partial y} \frac{\partial w_0}{\partial x} - \frac{\partial w_0}{\partial y} \frac{\partial u_0}{\partial x}, \\ B_1 &= -\frac{\partial u_0}{\partial y} \frac{\partial w_1}{\partial x} - \frac{\partial u_1}{\partial y} \frac{\partial w_0}{\partial x} - \frac{\partial w_0}{\partial y} \frac{\partial u_1}{\partial x} - \frac{\partial w_1}{\partial y} \frac{\partial u_0}{\partial x}, \\ &\vdots \\ C_0 &= -\frac{\partial v_0}{\partial y} \frac{\partial u_0}{\partial x} - \frac{\partial u_0}{\partial y} \frac{\partial v_0}{\partial x}, \\ C_1 &= -\frac{\partial v_0}{\partial y} \frac{\partial u_1}{\partial x} - \frac{\partial v_1}{\partial y} \frac{\partial u_0}{\partial x} - \frac{\partial u_0}{\partial y} \frac{\partial v_1}{\partial x} - \frac{\partial u_1}{\partial y} \frac{\partial v_0}{\partial x}, \\ &\vdots \end{aligned}$$

Substituting (5.5) and (5.6) in (5.4), we get

$$\begin{aligned} \sum_{n=0}^{\infty} u_n(x, y, t) &= e^{x+y} + \mathbf{S}^{-1} \left[(1 - \alpha + \alpha\sigma) \mathbf{S} \left\{ -\sum_{n=0}^{\infty} u_n + \sum_{n=0}^{\infty} A_n \right\} \right], \\ \sum_{n=0}^{\infty} v_n(x, y, t) &= e^{x-y} + \mathbf{S}^{-1} \left[(1 - \beta + \beta\sigma) \mathbf{S} \left\{ \sum_{n=0}^{\infty} v_n - \sum_{n=0}^{\infty} B_n \right\} \right], \\ \sum_{n=0}^{\infty} w_n(x, y, t) &= e^{y-x} + \mathbf{S}^{-1} \left[(1 - \varepsilon + \varepsilon\sigma) \mathbf{S} \left\{ \sum_{n=0}^{\infty} w_n - \sum_{n=0}^{\infty} C_n \right\} \right]. \end{aligned} \quad (5.7)$$

On comparing both sides of the (5.7), we get

$$\begin{aligned} u_0(x, y, t) &= e^{x+y}, \quad v_0(x, y, t) = e^{x-y}, \quad w_0(x, y, t) = e^{y-x}, \\ u_1(x, y, t) &= \mathbf{S}^{-1} [(1 - \alpha + \alpha\sigma) \mathbf{S} \{-u_0 + A_0\}] \\ &= \mathbf{S}^{-1} [(1 - \alpha + \alpha\sigma) \mathbf{S} \{-e^{x+y} + (-e^{x-y} e^{y-x} + e^{x-y} e^{y-x})\}] \\ &= \mathbf{S}^{-1} [(1 - \alpha + \alpha\sigma) \mathbf{S} \{-e^{x+y}\}] \\ &= -e^{x+y} \mathbf{S}^{-1} [(1 - \alpha + \alpha\sigma)] = -e^{x+y} (1 - \alpha + \alpha t), \\ v_1(x, y, t) &= \mathbf{S}^{-1} [(1 - \beta + \beta\sigma) \mathbf{S} \{v_0 + B_0\}] \\ &= \mathbf{S}^{-1} [(1 - \beta + \beta\sigma) \mathbf{S} \{e^{x-y} + (e^{x+y} e^{y-x} - e^{x+y} e^{y-x})\}] \\ &= \mathbf{S}^{-1} [(1 - \beta + \beta\sigma) \mathbf{S} \{e^{x-y}\}] \\ &= e^{x-y} \mathbf{S}^{-1} [(1 - \beta + \beta\sigma)] = e^{x-y} (1 - \beta + \beta t), \\ w_1(x, y, t) &= \mathbf{S}^{-1} [(1 - \varepsilon + \varepsilon\sigma) \mathbf{S} \{w_0 + C_0\}] \end{aligned}$$

$$\begin{aligned}
&= \mathbf{S}^{-1} [(1 - \varepsilon + \varepsilon\sigma)\mathbf{S}\{e^{y-x} + (e^{x+y}e^{x-y} - e^{x+y}e^{x-y})\}] \\
&= \mathbf{S}^{-1} [(1 - \varepsilon + \varepsilon\sigma)\mathbf{S}\{e^{y-x}\}] \\
&= e^{y-x}\mathbf{S}^{-1} [(1 - \varepsilon + \varepsilon\sigma)] = e^{y-x}(1 - \varepsilon + \varepsilon t), \\
u_2(x, y, t) &= \mathbf{S}^{-1} [(1 - \alpha + \alpha\sigma)\mathbf{S}\{-u_1 + A_1\}\mathbf{S}^{-1}] \\
&= \mathbf{S}^{-1} [(1 - \alpha + \alpha\sigma)\mathbf{S}\{e^{x+y}(1 - \alpha + \alpha t)\}] \\
&= e^{x+y}\mathbf{S}^{-1} [(1 - \alpha + \alpha\sigma)^2] = e^{x+y} \left[(1 - 2\alpha + \alpha^2) + (2\alpha - 2\alpha^2)t + \frac{1}{2}\alpha^2 t^2 \right], \\
v_2(x, y, t) &= \mathbf{S}^{-1} [(1 - \beta + \beta\sigma)\mathbf{S}\{v_1 + B_1\}] \\
&= \mathbf{S}^{-1} [(1 - \beta + \beta\sigma)\mathbf{S}\{e^{x-y}(1 - \beta + \beta t)\}] \\
&= e^{x-y}\mathbf{S}^{-1} [(1 - \beta + \beta\sigma)^2] = e^{x-y} \left[(1 - 2\beta + \beta^2) + (2\beta - 2\beta^2)t + \frac{1}{2}\beta^2 t^2 \right], \\
w_2(x, y, t) &= \mathbf{S}^{-1} [(1 - \varepsilon + \varepsilon\sigma)\mathbf{S}\{w_1 + C_1\}] \\
&= \mathbf{S}^{-1} [(1 - \varepsilon + \varepsilon\sigma)\mathbf{S}\{e^{y-x}(1 - \varepsilon + \varepsilon t)\}] \\
&= e^{y-x}\mathbf{S}^{-1} [(1 - \varepsilon + \varepsilon\sigma)^2] \\
&= e^{y-x} \left[(1 - 2\varepsilon + \varepsilon^2) + (2\varepsilon - 2\varepsilon^2)t + \frac{1}{2}\varepsilon^2 t^2 \right].
\end{aligned}$$

Then the approximate solution of (5.1) is

$$\begin{aligned}
u(x, y, t) &= e^{x+y} - e^{x+y}(1 - \alpha + \alpha t) + e^{x+y} \left[(1 - 2\alpha + \alpha^2) + (2\alpha - 2\alpha^2)t + \frac{1}{2}\alpha^2 t^2 + \dots \right], \\
v(x, y, t) &= e^{x-y} + e^{x-y}(1 - \beta + \beta t) + e^{x-y} \left[(1 - 2\beta + \beta^2) + (2\beta - 2\beta^2)t + \frac{1}{2}\beta^2 t^2 + \dots \right], \\
w(x, y, t) &= e^{y-x} + e^{y-x}(1 - \varepsilon + \varepsilon t) + e^{y-x} \left[(1 - 2\varepsilon + \varepsilon^2) + (2\varepsilon - 2\varepsilon^2)t + \frac{1}{2}\varepsilon^2 t^2 + \dots \right].
\end{aligned} \tag{5.8}$$

From Eqs. (5.2) and (5.8), the approximate solution of the given problem (5.1), by using DJM is the same result as that obtained by SDM.

If we put $\alpha \rightarrow 1$, $\beta \rightarrow 1$, and $\varepsilon \rightarrow 1$ in Eq. (5.2), we reproduce the solution of the problem as follows

$$\begin{aligned}
u(x, y, t) &= e^{x+y} \left(1 - t + \frac{t^2}{2!} - \dots \right), \\
v(x, y, t) &= e^{x-y} \left(1 + t + \frac{t^2}{2!} + \dots \right), \\
w(x, y, t) &= e^{y-x} \left(1 + t + \frac{t^2}{2!} + \dots \right).
\end{aligned}$$

This solution is equivalent to the exact solution in closed form:

$$u(x, y, t) = e^{x+y-t}, \quad v(x, y, t) = e^{x-y+t}, \quad w(x, y, t) = e^{y-x+t}.$$

6. Conclusions

In this work, the DJM and SDM have been successfully applied to finding the approximate solution of nonlinear system of partial differential equations with Caputo-Fabrizio fractional operator. The example shows that the results of DJM are in excellent agreement with the results given by SDM. The methods are very powerful and efficient in finding analytical as well as numerical solutions for wide classes of linear and non-linear fractional differential equations.

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