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Finding Laplace transform using difference equations

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Abstract

In this paper, we extend the properties of forward and backward difference operators to continuous variables. We construct continuous solutions, with jump discontinues resulting from using floor functions, for difference equations. As an application for these properties, we find the inverse Laplace transform of functions which have the form $\frac{F(s)}{a+be^{ct}}$.

Keywords: Laplace transform, difference operators, difference equations.

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1. Introduction

Let $F(s) = (\mathcal{L}f)(s) = \int_0^\infty e^{-st}f(t)dt$, be the Laplace transform of $f(t)$. In the t-domain we have the unit step function (Heaviside function) is mapped by Laplace transform to the exponential function in the s-domain. In fact,

$$\mathcal{L}\{u(t-c)g(t-c)\} = e^{-cs}G(s). \tag{1.1}$$

The Laplace transform has many properties, see [6–9]. On the contrary, the Laplace transform has some disappointments, one of the disappointments of the Laplace transform is that the Laplace transform of a product (a quotient) of two functions does not equal the product (the quotient) of their Laplace transforms. In fact, the Laplace transform of the convolution of two functions equals the Laplace transform of the product of these functions. In [2], a formula for the Laplace transform of a product of two functions was given.

2. Difference operators

Definition 2.1. For any complex valued function $f(z)$, the forward shift opertor E_c, the backward shift opertor B_c, and the identity operator I_c are defined respectively as $(E_c f)(z) = f(z+c)$, $(B_c f)(z) = f(z-c)$, and $(I_c f)(z) = f(z)$.

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Clearly,

$$B_c E_c = E_c B_c = I_c.$$

Studying difference equations is important in Mathematics, Physics, and Engineering.

Now, for any meromorphic function of finite order f and c be a positive constant c, the backward difference operator $\nabla_{c,\alpha}$ is given as

$$(\nabla_{c,\alpha} f)(z) = f(z) - \alpha f(z-c) = ((I - \alpha B)f)(z).$$

Clearly, the well-known forward difference operator Δ and the backward difference operator ∇ are related to $\nabla_{c,\alpha}$ as

$$(\Delta f)(z) = (\nabla_{1,1} f)(z+1) = f(z+1) - f(z) = ((E_1 - I_1)f)(z),$$

and

$$(\nabla f)(z) = (\nabla_{1,1} f)(z) = f(z) - f(z-1) = ((I_1 - B_1)f)(z).$$

For further study of difference operators and their applications, see [1, 5, 9]. For example, in [4] and [3], the difference equations are used to extend the properties of differential transform.

3. Main result

Define

$$g_{c,\alpha}(t) = \sum_{k=0}^{\infty} \alpha^k f(t-kc)u(t-kc),$$

where $u(t)$ is the unit step function. As a remark, since $u(t-kc) = 1$ when $t \geqslant kc$ and zero elsewhere, it is easy to see that

$$g_{c,\alpha}(t) = \sum_{k=0}^{[t/c]} \alpha^k f(t-kc),$$

where $[x]$ is the greatest integer function or the floor function of x. Now, for $n \in \mathbb{N}$, we have

$$\begin{aligned} g_{1,\alpha}(n) &= \sum_{k=0}^{\infty} \alpha^k f(n-k)u(n-k) \\ &= \sum_{k=0}^{n} \alpha^k f(n-k) \\ &= \sum_{k=0}^{n} \alpha^{n-k} f(k) = \alpha^n \sum_{k=0}^{n} \alpha^{-k} f(k). \end{aligned}$$

The following are true about $g_{c,\alpha}(t)$.

Proposition 3.1.

1. *The Laplace transform* $G(s)$ *of* $g(t)$ *is* $G(s) = \frac{F(s)}{1-\alpha e^{-cs}}; |\alpha| < e^{cs}$.
2. $g_{c,\alpha}(t)$ *solves the difference equation*

$$y(t) - \alpha y(t-c) = f(t), \quad t \geqslant 0, \tag{3.1}$$

   i.e., $(\nabla_{c,\alpha} g)(t) = f(t)$.
3. *For* $m \in \mathbb{Z}$, $g_{c,\alpha}(t)$ *solves the difference equation* $y(t) - \alpha^m y(t-mc) = \sum_{k=0}^{m-1} \alpha^k f(t-kc)u(t-kc)$.

Proof. Using (1.1), we get that $\mathcal{L}\{\alpha^k f(t-kc)u(t-kc)\} = e^{-kcs}F(s)$. Hence, for $|\alpha| < e^{cs}$ the Laplace transform of $g_{c,\alpha}(t)$ exists and it is calculated as

$$\begin{aligned} G(s) &= \mathcal{L}\{\sum_{k=0}^{\infty} \alpha^k f(t-kc)u(t-kc)\} \\ &= F(s)\sum_{k=0}^{\infty} \alpha^k e^{-kcs} \\ &= \frac{F(s)}{1-\alpha e^{-cs}}. \end{aligned}$$

Now, to prove the second part, for $t \geqslant 0$, we have

$$\begin{aligned} g(t)-\alpha g(t-c) &= \sum_{k=0}^{\infty} \alpha^k f(t-kc)u(t-kc) - \alpha\sum_{k=0}^{\infty} \alpha^k f(t-c-kc)u(t-c-kc) \\ &= \sum_{k=0}^{\infty} \alpha^k f(t-kc)u(t-kc) - \sum_{k=0}^{\infty} \alpha^{k+1} f(t-c(k+1))u(t-c(k+1)) \\ &= \sum_{k=0}^{\infty} \alpha^k f(t-kc)u(t-kc) - \sum_{k=1}^{\infty} \alpha^k f(t-kc)u(t-kc) \\ &= f(t)u(t) = f(t). \end{aligned}$$

Moreover, for $m \in \mathbb{Z}$

$$\begin{aligned} g_{c,\alpha}(t)-\alpha^m g_{c,\alpha}(t-mc) &= \sum_{k=0}^{\infty} \alpha^k f(t-kc)u(t-kc) - \alpha^m\sum_{k=0}^{\infty} \alpha^k f(t-mc-kc)u(t-mc-kc) \\ &= \sum_{k=0}^{\infty} \alpha^k f(t-kc)u(t-kc) - \alpha^m\sum_{k=0}^{\infty} \alpha^k f(t-c(k+m))u(t-c(k+m)) \\ &= \sum_{k=0}^{\infty} \alpha^k f(t-kc)u(t-kc) - \alpha^m\sum_{k=m}^{\infty} \alpha^{k-m} f(t-kc)u(t-kc) \\ &= \sum_{k=0}^{m-1} \alpha^k f(t-kc)u(t-kc). \end{aligned}$$

□

Now, the inverse of backward difference operator for a real-valued function f, denoted by $(\nabla_{c,\alpha}^{-1} f)(t)$, satisfies $(\nabla_{c,\alpha}(\nabla_{c,\alpha}^{-1} f))(t) = f(t)$. In simple words, if $g_{c,\alpha}(t) = (\nabla_{c,\alpha}^{-1} f)(t)$ if $g_{c,\alpha}(t) - \alpha g_{c,\alpha}(t-c) = f(t)$. As a remark if $y_1(t)$ and $y_2(t)$ are solutions of (3.1) (i.e., they are two inverses of $f(t)$), then $y_1(t) - y_2(t)$ is a solution of $y(t) - \alpha y(t-c) = 0$. For example, in the case of $c = \alpha = 1$, for $n \in \mathbb{N}$, we have $y_1(n) = 1 + n$ and $y_2(n) = (-1)^n + n$ are solutions of $y(n) - y(n-1) = 2$. Therefore, y_1 and y_2 are inverses of y(n)=2 in the case of $c = \alpha = 1$.

Example 3.2. When $\alpha = 1$, we have

1. $g_{c,1}(t) = 1 + [\frac{t}{c}]$ is an inverse of $f(t) = 1$.
2. $g_{c,1}(t) = (1 + [\frac{t}{c}])(t - \frac{c}{2}[\frac{t}{c}])$ is an inverse of $f(t) = t$.
3. $g_{c,1}(t) = a^t\left(\frac{a^c - a^{-c[\frac{t}{c}]}}{a^c - 1}\right)$ is an inverse of $f(t) = a^t$.

Proof. 1.

$$\begin{aligned} g_{c,1}(t) - g_{c,1}(t-c) &= 1 + [\frac{t}{c}] - (1 + [\frac{t-c}{c}]) \\ &= 1 + [\frac{t}{c}] - (1 + [\frac{t}{c}] - 1) = 1. \end{aligned}$$

2.

$$\begin{aligned} g_{c,1}(t) - g_{c,1}(t-c) &= (1 + [\frac{t}{c}])(t - \frac{c}{2}[\frac{t}{c}]) - (1 + [\frac{t-c}{c}])(t - c - \frac{c}{2}([\frac{t-c}{c}])) \\ &= (1 + [\frac{t}{c}])(t - \frac{c}{2}[\frac{t}{c}]) - [\frac{t}{c}](t - c - \frac{c}{2}([\frac{t}{c}] - 1)) = t. \end{aligned}$$

3.

$$\begin{aligned} g_{c,1}(t) - g_{c,1}(t-c) &= a^t \left(\frac{a^c - a^{-c[\frac{t}{c}]}}{a^c - 1} \right) - a^{t-c} \left(\frac{a^c - a^{-c[\frac{t-c}{c}]}}{a^c - 1} \right) \\ &= \frac{a^t \left(a^c - a^{-c[\frac{t}{c}]} \right) - a^t \left(1 - a^{-c[\frac{t}{c}]} \right)}{a^c - 1} = a^t. \end{aligned}$$

□

4. Numerical results

Example 4.1. Using Example 3.2, the function $g(t) = 1 + [2t]$ is a solution $y(t) - y(t - \frac{1}{2}) = 1$. Also, for $m \in \mathbb{Z}$, it is a solution of $y(t) - y(t - \frac{1}{2}m) = m$.

Example 4.2. Using Proposition 3.1 with $\alpha = 1$ and Example 4.1, we get that the inverse Laplace transform of $G(s) = \frac{1}{s(1-e^{-s/2})} = \frac{\coth(s/4)+1}{2s}$ is $g(t) = 1 + [2t]$.

Example 4.3. Using Example 3.2, the function $g(t) = (1 + [t])(t - \frac{1}{2}[t])$ is a solution $y(t) - y(t-1) = t$.

Example 4.4. Using Proposition 3.1 with $\alpha = 1$ and Example 4.3, we get that the inverse Laplace transform of $G(s) = \frac{1}{s^2(1-e^{-s})}$ is $g(t) = (1 + [t])(t - \frac{1}{2}[t])$.

Example 4.5. Using Example 3.2, the function $g(t) = 2^t(2 - 2^{-[t]}) = 2^{t+1} - 2^{t-[t]}$ is a solution

$$y(t) - y(t-1) = 2^t.$$

Example 4.6. Using Proposition 3.1 with $\alpha = 1$ and Example 4.5, we get that the inverse Laplace transform of $G(s) = \frac{1}{(s-\ln 2)(1-e^{-s})}$ is $g(t) = 2^{t+1} - 2^{t-[t]}$.

Now, the following example will give us an idea how deal with a denominator in which contains more than one term

Example 4.7. Using Example 4.5, we get that the inverse Laplace transform of

$$G(s) = \frac{1}{s(1-e^{-s})(1-2e^{-s})} = \frac{-1}{s(1-e^{-s})} + \frac{2}{s(1-2e^{-s})},$$

is $g(t) = 2^{2+[t]} - [t] - 3$.

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