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Some fractional dynamic inequalities on time scales of Hardy's type

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Abstract

In this paper, we prove some new fractional dynamic inequalities on time scales of Hardy's type due to Yang and Hwang. The results will be proved by employing the chain rule, Hölder's inequality, and integration by parts on fractional time scales. Several well-known dynamic inequalities on time scales will be obtained as special cases from our results.

Keywords: Hardy's inequality, Yang and Hwang's inequality, Copson's inequality, Hölder's inequality, time scales, conformable fractional calculus.

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1. Introduction

In 1920, Hardy [19] proved the discrete inequality

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}a(i)\right)^{p}\leqslant\left(\frac{p}{p-1}\right)^{p}\sum_{n=1}^{\infty}a^{p}(n),\quad p>1, \tag{1.1}$$

where $a(n)$ is a positive sequence defined for all $n\geqslant 1$. In 1925 Hardy [20], by using the calculus of variations, proved the continuous inequality of (1.1) which has the form

$$\int_{0}^{\infty}\left(\frac{1}{x}\int_{0}^{x}f(s)ds\right)^{p}dx\leqslant\left(\frac{p}{p-1}\right)^{p}\int_{0}^{\infty}f^{p}(x)dx, \tag{1.2}$$

for a given positive function f, which is integrable over $(0,x)$, f^{p} is convergent and integrable over $(0,\infty)$ and $p>1$. In (1.1) and (1.2), the constant $(p/(p-1))^{p}$ is a sharp constant. The generalizations of (1.2) have been proved by Hardy [21] who showed that when $p>1$, then

$$\int_{0}^{\infty}x^{-c}\left(\int_{0}^{x}f(s)ds\right)^{p}dx\leqslant\left(\frac{p}{c-1}\right)^{p}\int_{0}^{\infty}x^{p-c}f^{p}(x)dx,\ \text{for } c>1, \tag{1.3}$$

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and

$$\int_0^\infty x^{-c}\left(\int_x^\infty f(s)ds\right)^p dx \leqslant \left(\frac{p}{1-c}\right)^p \int_0^\infty x^{p-c} f^p(x)dx, \text{ for } c<1. \tag{1.4}$$

The constants $(p/(c-1))^p$ and $(p/(1-c))^p$ in (1.3) and (1.4) are the best possible. Copson [15] considered the adjoint operator and proved that if $f(x)>0$, $p>1$ and $f^p(x)$ is integrable on the interval $(0,\infty)$, then $\int_x^\infty \left(\frac{f(s)}{s}\right) ds$ converges if $x>0$ and

$$\int_0^\infty \left(\int_x^\infty \frac{f(s)}{s}ds\right)^p dx \leqslant p^p \int_0^\infty f^p(x)dx,$$

where p^p is the best possible. In [16] Copson extended the inequalities due to Hardy and proved that if $p>1$, $a(i)\geqslant 0$, $\lambda(i)\geqslant 0$, $\forall i\geqslant 1$, $\Omega(n)=\sum_{i=1}^n \lambda(i)$, and $c>1$, then

$$\sum_{n=1}^\infty \frac{\lambda(n)}{\Omega^c(n)}\left(\sum_{i=1}^n a(i)\lambda(i)\right)^p \leqslant \left(\frac{p}{c-1}\right)^p \sum_{n=1}^\infty \lambda(n)\Omega^{p-c}(n)a^p(n), \tag{1.5}$$

and if $p>1$ and $0\leqslant c<1$, then

$$\sum_{n=1}^\infty \frac{\lambda(n)}{\Omega^c(n)}\left(\sum_{i=n}^\infty a(i)\lambda(i)\right)^p \leqslant \left(\frac{p}{1-c}\right)^p \sum_{n=1}^\infty \lambda(n)\Omega^{p-c}(n)a^p(n). \tag{1.6}$$

The continuous versions of the inequalities of (1.5) and (1.6) were proved by Copson in [17, Theorems 1 and 3]. In particular, he proved that if $p\geqslant 1$, $c>1$, and $\Omega(s)=\int_0^s \lambda(t)dt$, then

$$\int_0^\infty \frac{\lambda(s)}{\Omega^c(s)}\Theta^p(s)ds \leqslant \left(\frac{p}{c-1}\right)^p \int_0^\infty \frac{\lambda(s)}{\Omega^{c-p}(s)} f^p(s)ds,$$

where $\Theta(s)=\int_0^s \lambda(t)f(t)dt$, and if $p>1$, $0\leqslant c<1$, then

$$\int_0^\infty \frac{\lambda(s)}{\Omega^c(s)}\overline{\Theta}^p(s)ds \leqslant \left(\frac{p}{1-c}\right)^p \int_0^\infty \frac{\lambda(s)}{\Omega^{c-p}(s)} f^p(s)ds,$$

where $\overline{\Theta}(s)=\int_s^\infty \lambda(t)f(t)dt$. An interesting variant of Copson inequalities (1.5) and (1.6) was proved by Leindler in [28] and Bennett in [8]. Leindler proved that if $\Omega^*(n)=\sum_{i=n}^\infty \lambda(i)<\infty$, $p>1$, and $0\leqslant c<1$, then

$$\sum_{n=1}^\infty \frac{\lambda(n)}{(\Omega^*(n))^c}\left(\sum_{i=1}^n a(i)\lambda(i)\right)^p \leqslant \left(\frac{p}{1-c}\right)^p \sum_{n=1}^\infty \lambda(n)(\Omega^*(n))^{p-c}a^p(n),$$

and Bennett in [8] showed that if $1<c\leqslant p$, then

$$\sum_{n=1}^\infty \frac{\lambda(n)}{(\Omega^*(n))^c}\left(\sum_{i=n}^\infty a(i)\lambda(i)\right)^p \leqslant \left(\frac{p}{c-1}\right)^p \sum_{n=1}^\infty \lambda(n)(\Omega^*(n))^{p-c}a^p(n).$$

Levinson [29] proved that, if $\phi(u)$ is a real - valued positive convex function for $u>0$, $p>1$, $f(t)>0$, $\lambda(t)>0$ for $t>0$, and there exists a constant $K>0$ such that

$$p-1+\frac{\lambda'(t)\Upsilon(t)}{\lambda^2(t)}\geqslant \frac{p}{K}, \quad \text{for all } t>0,$$

then

$$\int_0^\infty \phi\left(\frac{1}{\Upsilon(t)}\int_0^t \lambda(s)f(s)ds\right)dt \leqslant K^p \int_0^\infty \phi\left(f(t)\right)dt, \tag{1.7}$$

where $\Upsilon(t) = \int_0^t \lambda(s)ds$. Yang and Hwang [41] generalized the inequality (1.7) and proved that, if $p > 1$, $\lambda(t)$, $q(t)$, $f(t)$ are nonnegative functions and there exists a constant $K > 0$ such that

$$p - 1 + \frac{q'(t)\Upsilon(t)}{q^2(t)\lambda(t)} \geqslant \frac{p}{K}, \quad \text{for all } t > 0,$$

then

$$\int_0^\infty \lambda(t)\phi\left(\frac{\Theta(t)}{\Upsilon(t)}\right)^p dt \leqslant K^p \int_0^\infty \lambda(t) f^p(t)dt, \tag{1.8}$$

where

$$\Upsilon(t) := \int_0^t \lambda(s)q(s)ds, \quad \text{and} \quad \Theta(t) := \int_0^t \lambda(s)q(s)f(s)ds.$$

In last decades, the study of dynamic equations and inequalities on time scales has become an important major field in pure and applied mathematics, we refer to the books [4, 31]. The book [4] contains the time scales versions of several inequalities of Hardy's type their extensions. The idea of time scales goes back to Stefan Hilger [22] who begin the study of dynamic equations on time scales. The two books by Bohner and Peterson in [12, 13] summarized and organized most calculus of time scales. The most three popular examples of calculus on time scales are difference calculus, differential calculus, and quantum calculus (see Kac and Cheung [25]), i.e., when $\mathbb{T} = \mathbb{N}$, $\mathbb{T} = \mathbb{R}$, and $\mathbb{T} = q^{\mathbb{N}_0} = \{q^t : t \in \mathbb{N}_0\}$, where $q > 1$.

In recent years, some authors studied the fractional inequalities by using fractional Caputo and Riemann-Liouville derivatives, we refer the reader to the papers [9, 24, 42] and the references cited therein. In [1, 26] the authors extended the calculus of fractional order to conformable calculus. Very recently, some authors have extended classical inequalities by using conformable fractional calculus such as Opial's inequality [37, 38], Hermite-Hadamard's inequality [14, 27, 40], Chebyshev's inequality [6] and Steffensen's inequality [39]. In [1, 26] the authors extended the calculus of fractional order to conformable calculus and gave new definitions of the derivatives and integrals. In [7, 30], the authors combined a conformable fractional calculus and a time scale calculus and obtained new fractional calculus on time scales.

Our aim in this paper is to employ these new theory to prove some fractional dynamic inequalities on time scales. The new inequalities contain the classical Hardy, Copson, Yang and Hwang inequalities as special cases. The paper is organized as follows. In Section 2, we present some preliminaries about fractional calculus on time scales and in Section 3, we will prove the main results.

2. Preliminaries and basic lemmas

In this section, we present the basics of fractional calculus on time scales that will be needed throughout the paper. The results are adapted from [7, 12, 13, 30]. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. We assume throughout that $\mathbb{T}$ has the topology that it inherits from the standard topology on $\mathbb{R}$. We define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$, as

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\},$$

while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$, is defined by:

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\},$$

for any $t \in \mathbb{T}$ and the notation $f^\sigma(t)$ refer to $f(\sigma(t))$, i.e., $f^\sigma = f \circ \sigma$. Finally, the graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) := \sigma(t) - t.$$

Definition 2.1 (Conformable α-fractional derivative)**.** Let the function $f : \mathbb{T} \to \mathbb{R}$ and $\alpha \in (0,1]$. Then, for $t > 0$, we define $T_\alpha^\Delta(f)(t)$ to be the number (provided it exists) with the property that, given any $\epsilon > 0$, there is a neighborhood U of t such that for all $t \in U$,

$$\left|[f^\sigma(t) - f(s)]t^{1-\alpha} - T_\alpha^\Delta(f(t))(\sigma(t) - s)\right| \leqslant \epsilon\, |\sigma(t) - s)|\,.$$

$T_\alpha^\Delta(f(t))$ is called the conformable α-fractional derivative of f of order α at t on $\mathbb{T}$, and we define the conformable fractional derivative on $\mathbb{T}$ at 0 as

$$T_\alpha^\Delta(f(0)) = \lim_{t\to 0} T_\alpha^\Delta(f(t)).$$

The conformable fractional derivative has the following properties.

Theorem 2.2. *Let u, $v : \mathbb{T} \to \mathbb{R}$ be conformable fractional derivative of order $\alpha \in (0,1]$. Then the following properties are hold.*

(i) *The $u + v : \mathbb{T} \to \mathbb{R}$ is conformable fractional derivative and*

$$T_\alpha^\Delta(u + v) = T_\alpha^\Delta(u) + T_\alpha^\Delta(v).$$

(ii) *For any $\lambda \in \mathbb{R}$, then $\lambda u : \mathbb{T} \to \mathbb{R}$ is α-fractional differentiable and*

$$T_\alpha^\Delta(\lambda u) = \lambda T_\alpha^\Delta(u).$$

(iii) *If u and v are α-fractional differentiable, then the product $uv : \mathbb{T} \to \mathbb{R}$ is α-fractional differentiable and*

$$T_\alpha^\Delta(uv) = T_\alpha^\Delta(u)v + (u \circ \sigma)T_\alpha^\Delta(v) = T_\alpha^\Delta(u)(v \circ \sigma) + uT_\alpha^\Delta(v).$$

(iv) *If u is α-fractional differentiable, then $1/u$ is α-fractional differentiable with*

$$T_\alpha^\Delta\left(\frac{1}{u}\right) = -\frac{T_\alpha^\Delta(u)}{u(u \circ \sigma)}.$$

(v) *If u and v are α-fractional differentiable, then u/v is α-fractional differentiable with*

$$T_\alpha^\Delta(u/v) = \frac{vT_\alpha^\Delta(u) - uT_\alpha^\Delta(v)}{v(v \circ \sigma)}, \tag{2.1}$$

valid at all points $t \in \mathbb{T}^k$ for which $v(t)(v(\sigma(t)) \neq 0$.

Lemma 2.3 (Chain rule A)**.** *Let $v : \mathbb{T} \to \mathbb{R}$ be continuous and α-fractional differentiable at $t \in \mathbb{T}$, for $\alpha \in (0,1]$, and $u : \mathbb{R} \to \mathbb{R}$ be continuously differentiable. Then there exists $d \in [t, \sigma(t)]$ with*

$$T_\alpha^\Delta(u \circ v)(t) = u^{'}(v(d))T_\alpha^\Delta(v(t)). \tag{2.2}$$

Lemma 2.4 (Chain rule B)**.** *Let $u : \mathbb{R} \to \mathbb{R}$ be continuously differentiable, and $v : \mathbb{T} \to \mathbb{R}$ be α-fractional differentiable for $\alpha \in (0,1]$. Then $(u \circ v) : \mathbb{T} \to \mathbb{R}$ is also α-fractional differentiable, and we have*

$$T_\alpha^\Delta(u \circ v)(s) = \left(\int_0^1 u^{'}\left(v(s) + h\mu(s)s^{\alpha-1}T_\alpha^\Delta(v(s))\right) dh\right) T_\alpha^\Delta(v(s)).$$

Definition 2.5 (Conformable fractional integral)**.** For $0 < \alpha \leqslant 1$, the α-fractional integral of f is defined by

$$\int f(s)\Delta_\alpha s = \int f(s)s^{\alpha-1}\Delta s.$$

The conformable fractional integral satisfies the following properties.

Theorem 2.6. *Let* $a, b, c \in \mathbb{T}$, $\lambda \in \mathbb{R}$ *and let* $u, v : \mathbb{T} \to \mathbb{R}$. *Then*

(i) $\int_a^b [u(s)+v(s)]\Delta_\alpha s = \int_a^b u(s)\Delta_\alpha s + \int_a^b v(s)\Delta_\alpha s$;
(ii) $\int_a^b \lambda u(s)\Delta_\alpha s = \lambda \int_a^b u(s)\Delta_\alpha s$;
(iii) $\int_a^b u(s)\Delta_\alpha s = -\int_b^a u(s)\Delta_\alpha s$;
(iv) $\int_a^b u(s)\Delta_\alpha s = \int_a^c u(s)\Delta_\alpha s + \int_c^b u(s)\Delta_\alpha s$;
(v) $\int_a^a u(s)\Delta_\alpha s = 0$.

Lemma 2.7 (Integration by parts)**.** *Let* $\mathbb{T}$ *be a time scale,* $a, b \in \mathbb{T}$ *where* $b > a$. *Let* u, v *be conformable* α*-fractional differentiable,* $\alpha \in (0, 1]$. *Then the formula of integration by parts is given by*

$$\int_a^b u(s)T_\alpha^\Delta v(s)\Delta_\alpha s = [u(s)v(s)]_a^b - \int_a^b v^\sigma(s)T_\alpha^\Delta u(s)\Delta_\alpha s. \tag{2.3}$$

Lemma 2.8 (Hölder's inequality)**.** *Let* $\mathbb{T}$ *be a time scale,* $a, b \in \mathbb{T}$ *and* $\alpha \in (0, 1]$ *and let* $u, v : \mathbb{T} \to \mathbb{R}$. *Then*

$$\int_a^b |u(s)v(s)|\Delta_\alpha s \leqslant \left[\int_a^b |u(s)|^p \Delta_\alpha s\right]^{\frac{1}{p}} \left[\int_a^b |v(s)|^q \Delta_\alpha s\right]^{\frac{1}{q}}, \tag{2.4}$$

where $p > 1$ *and* $1/p + 1/q = 1$.

3. Main results

In this section, we will prove our main results by employing Hölder's inequality, chain rule, and integration by parts for fractional on time scales. Throughout this paper (without mentioning it) we assume that the integrals in the statements of the theorems are assumed to exist and finite. For simplicity, we define the operators

$$\Upsilon(t) := \int_a^t \lambda(s)q^\sigma(s)\Delta_\alpha s, \quad \text{and} \quad \Theta(t) := \int_a^t \lambda(s)q(s)f(s)\Delta_\alpha s,$$

where

$$\Theta(\infty) < \infty, \quad \text{and} \quad \int_a^\infty \frac{\lambda(s)}{(\Upsilon^\sigma(s))^{\gamma-\alpha+1}}\Delta_\alpha s < \infty.$$

Theorem 3.1. *Let* $\mathbb{T}$ *be a time scale with* $a \in [0, \infty)_{\mathbb{T}}$, $1 < \gamma \leqslant p$, *and* $\alpha \in (0, 1]$, *and* $q(t)$ *be an increasing function on* $[a, \infty)_{\mathbb{T}}$. *Furthermore, assume that there exists a constant* $K > 0$ *such that*

$$\gamma - \alpha + \frac{T_\alpha^\Delta q(t)\Upsilon^\sigma(t)\Theta^p(t)}{\lambda(t)\,(q^\sigma(t))^2\,(\Theta^\sigma(t))^p} \geqslant \frac{p}{K}, \quad \text{for } t \in [a, \infty)_{\mathbb{T}}. \tag{3.1}$$

Then

$$\int_a^\infty \frac{\lambda(t)}{(\Upsilon^\sigma(t))^{\gamma-\alpha+1}}\,(\Theta^\sigma(t))^p\,\Delta_\alpha t \leqslant K^p \int_a^\infty \frac{(\Upsilon^\sigma(t))^{(\gamma-\alpha+1)(p-1)}}{\Upsilon^{p(\gamma-\alpha)}(t)}\lambda(t)f^p(t)\Delta_\alpha t. \tag{3.2}$$

Proof. By employing the formula of integration by parts (2.3) on the term

$$\int_a^\infty \frac{\lambda(t)}{(\Upsilon^\sigma(t))^{\gamma-\alpha+1}}\,(\Theta^\sigma(t))^p\,\Delta_\alpha t,$$

with $\nu^\sigma(t) = (\Theta^\sigma(t))^p / q^\sigma(t)$ and $T_\alpha^\Delta u(t) = \frac{\lambda(t)q^\sigma(t)}{(\Upsilon^\sigma(t))^{\gamma-\alpha+1}}$, we have that

$$\int_a^\infty \frac{\lambda(t)}{(\Upsilon^\sigma(t))^{\gamma-\alpha+1}} (\Theta^\sigma(t))^p \Delta_\alpha t = u(t)\frac{\Theta^p(t)}{q(t)}\Big|_a^\infty + \int_a^\infty -u(t)T_\alpha^\Delta\left(\frac{\Theta^p(t)}{q(t)}\right)\Delta_\alpha t, \tag{3.3}$$

where

$$-u(t) = \int_t^\infty \frac{\lambda(s)q^\sigma(s)}{(\Upsilon^\sigma(s))^{\gamma-\alpha+1}}\Delta_\alpha s = \int_t^\infty T_\alpha^\Delta \Upsilon(s)\left(\Upsilon^\sigma(s)\right)^{\alpha-\gamma-1}\Delta_\alpha s.$$

Since $T_\alpha^\Delta\Upsilon(t) = \lambda(t)q^\sigma(t) \geqslant 0$, and by using chain rule (2.2), we obtain that

$$-T_\alpha^\Delta\left(\Upsilon^{\alpha-\gamma}(t)\right) = -(\alpha-\gamma)\Upsilon^{\alpha-\gamma-1}(d)T_\alpha^\Delta\Upsilon(t), \text{ where } d\in[t,\sigma(t)] = \frac{(\gamma-\alpha)\,T_\alpha^\Delta\Upsilon(t)}{\Upsilon^{\gamma-\alpha+1}(d)} \geqslant \frac{(\gamma-\alpha)\,T_\alpha^\Delta\Upsilon(t)}{(\Upsilon^\sigma(t))^{\gamma-\alpha+1}}.$$

Then we have

$$T_\alpha^\Delta\Upsilon(t)\left(\Upsilon^\sigma(t)\right)^{\alpha-\gamma-1} \leqslant \frac{-1}{\gamma-\alpha}T_\alpha^\Delta\left(\Upsilon^{\alpha-\gamma}(t)\right),$$

and thus

$$-u(t) = \int_t^\infty \frac{\lambda(s)q^\sigma(s)}{(\Upsilon^\sigma(s))^{\gamma-\alpha+1}}\Delta_\alpha s \leqslant \frac{-1}{\gamma-\alpha}\int_t^\infty T_\alpha^\Delta\left(\Upsilon^{\alpha-\gamma}(s)\right)\Delta_\alpha s \leqslant \frac{\Upsilon^{\alpha-\gamma}(t)}{\gamma-\alpha}. \tag{3.4}$$

By using the quotient rule (2.1), we see that

$$T_\alpha^\Delta\left(\frac{\Theta^p(t)}{q(t)}\right) = \frac{q(t)T_\alpha^\Delta\Theta^p(t) - \Theta^p(t)T_\alpha^\Delta q(t)}{q(t)q^\sigma(t)}, \tag{3.5}$$

and by using chain rule (2.2), we have that

$$T_\alpha^\Delta\left(\Theta^p(t)\right) = p\Theta^{p-1}(d)T_\alpha^\Delta\Theta(t), \text{ where } d\in[t,\,\sigma(t)],$$

since $T_\alpha^\Delta\Theta(t) = \lambda(t)q(t)f(t) \geqslant 0$ and $d \leqslant \sigma(t)$ we have

$$T_\alpha^\Delta\left(\Theta^p(t)\right) = p\Theta^{p-1}(d)T_\alpha^\Delta\Theta(t) \leqslant p\lambda(t)q(t)f(t)\left(\Theta^\sigma(t)\right)^{p-1}. \tag{3.6}$$

From (3.5) and (3.6), we have that

$$T_\alpha^\Delta\left(\frac{\Theta^p(t)}{q(t)}\right) = \frac{q(t)T_\alpha^\Delta\Theta^p(t) - \Theta^p(t)T_\alpha^\Delta q(t)}{q(t)q^\sigma(t)} \leqslant \frac{p\lambda(t)q(t)f(t)\left(\Theta^\sigma(t)\right)^{p-1}}{q^\sigma(t)} - \frac{\Theta^p(t)T_\alpha^\Delta q(t)}{q(t)q^\sigma(t)}. \tag{3.7}$$

Since $\Theta(a) = 0$, $u(\infty) = 0$, and from (3.4), (3.7), and (3.3) we have

$$\begin{aligned}\int_a^\infty \frac{\lambda(t)}{(\Upsilon^\sigma(t))^{\gamma-\alpha+1}}(\Theta^\sigma(t))^p\Delta_\alpha t &\leqslant \int_a^\infty \frac{\Upsilon^{\alpha-\gamma}(t)}{\gamma-\alpha}\left(\frac{p\lambda(t)q(t)f(t)\left(\Theta^\sigma(t)\right)^{p-1}}{q^\sigma(t)}\frac{\Theta^p(t)T_\alpha^\Delta q(t)}{q(t)q^\sigma(t)}\right)\Delta_\alpha t\\ &\leqslant \frac{p}{\gamma-\alpha}\int_a^\infty \frac{\lambda(t)q(t)f(t)\left(\Theta^\sigma(t)\right)^{p-1}}{\Upsilon^{\gamma-\alpha}(t)q^\sigma(t)}\Delta_\alpha t \frac{1}{\gamma-\alpha}\int_a^\infty \frac{\Theta^p(t)T_\alpha^\Delta q(t)}{\Upsilon^{\gamma-\alpha}(t)q(t)q^\sigma(t)}\Delta_\alpha t,\end{aligned}$$

since $T_\alpha^\Delta\Upsilon(t) = \lambda(t)q^\sigma(t) \geqslant 0$, and $d \leqslant \sigma(t)$, then we have (note that $q(t)$ is an increasing function on $[a,\infty)_{\mathbb{T}}$)

$$\int_a^\infty \frac{\lambda(t)}{(\Upsilon^\sigma(t))^{\gamma-\alpha+1}}(\Theta^\sigma(t))^p\Delta_\alpha t \leqslant \frac{p}{\gamma-\alpha}\int_a^\infty \frac{\lambda(t)f(t)\left(\Theta^\sigma(t)\right)^{p-1}}{\Upsilon^{\gamma-\alpha}(t)}\Delta_\alpha t - \frac{1}{\gamma-\alpha}\int_a^\infty \frac{\Theta^p(t)T_\alpha^\Delta q(t)}{\left(\Upsilon^\sigma(t)\right)^{\gamma-\alpha}\left(q^\sigma(t)\right)^2}\Delta_\alpha t,$$

hence

$$\int_a^\infty \frac{\lambda(t)}{(\Upsilon^\sigma(t))^{\gamma-\alpha+1}}(\Theta^\sigma(t))^p\left[\gamma-\alpha+\frac{T_\alpha^\Delta q(t)\Upsilon^\sigma(t)\Theta^p(t)}{\lambda(t)\left(q^\sigma(t)\right)^2\left(\Theta^\sigma(t)\right)^p}\right]\Delta_\alpha t \leqslant p\int_a^\infty \frac{\lambda(t)f(t)\left(\Theta^\sigma(t)\right)^{p-1}}{\Upsilon^{\gamma-\alpha}(t)}\Delta_\alpha t. \tag{3.8}$$

From (3.1), and (3.8), we see that

$$\begin{aligned}\int_a^\infty \frac{\lambda(t)}{(\Upsilon^\sigma(t))^{\gamma-\alpha+1}}(\Theta^\sigma(t))^p\Delta_\alpha t &\leqslant K\int_a^\infty \frac{\lambda(t)f(t)(\Theta^\sigma(t))^{p-1}}{\Upsilon^{\gamma-\alpha}(t)}\Delta_\alpha t\\ &= K\int_a^\infty\left(\frac{\lambda^{\frac{p-1}{p}}(t)(\Theta^\sigma(t))^{p-1}}{(\Upsilon^\sigma(t))^{\frac{(\gamma-\alpha+1)(p-1)}{p}}}\frac{\lambda^{\frac{1}{p}}(t)(\Upsilon^\sigma(t))^{\frac{(\gamma-\alpha+1)(p-1)}{p}}f(t)}{\Upsilon^{\gamma-\alpha}(t)}\right)\Delta_\alpha t,\end{aligned}$$

by applying Hölder's inequality (2.4) with indices p and $p/(p-1)$, we have

$$\begin{aligned}\int_a^\infty \frac{\lambda(t)}{(\Upsilon^\sigma(t))^{\gamma-\alpha+1}}(\Theta^\sigma(t))^p\Delta_\alpha t &\leqslant K\int_a^\infty\left(\frac{\lambda^{\frac{p-1}{p}}(t)(\Theta^\sigma(t))^{p-1}}{(\Upsilon^\sigma(t))^{\frac{(\gamma-\alpha+1)(p-1)}{p}}}\times\frac{\lambda^{\frac{1}{p}}(t)(\Upsilon^\sigma(t))^{\frac{(\gamma-\alpha+1)(p-1)}{p}}f(t)}{\Upsilon^{\gamma-\alpha}(t)}\right)\Delta_\alpha t\\ &\leqslant K\left(\int_a^\infty \frac{\lambda(t)}{(\Upsilon^\sigma(t))^{\gamma-\alpha+1}}(\Theta^\sigma(t))^p\Delta_\alpha t\right)^{\frac{p-1}{p}}\\ &\quad\times\left(\int_a^\infty \frac{\lambda(t)(\Upsilon^\sigma(t))^{(\gamma-\alpha+1)(p-1)}f^p(t)}{\Upsilon^{p(\gamma-\alpha)}(t)}\Delta_\alpha t\right)^{\frac{1}{p}},\end{aligned}$$

then

$$\left(\int_a^\infty \frac{\lambda(t)}{(\Upsilon^\sigma(t))^{\gamma-\alpha+1}}(\Theta^\sigma(t))^p\Delta_\alpha t\right)^{\frac{1}{p}} \leqslant K\left(\int_a^\infty \frac{\lambda(t)(\Upsilon^\sigma(t))^{(\gamma-\alpha+1)(p-1)}f^p(t)}{\Upsilon^{p(\gamma-\alpha)}(t)}\Delta_\alpha t\right)^{\frac{1}{p}}.$$

This leads to

$$\int_a^\infty \frac{\lambda(t)}{(\Upsilon^\sigma(t))^{\gamma-\alpha+1}}(\Theta^\sigma(t))^p\Delta_\alpha t \leqslant K^p\int_a^\infty \frac{\lambda(t)(\Upsilon^\sigma(t))^{(\gamma-\alpha+1)(p-1)}f^p(t)}{\Upsilon^{p(\gamma-\alpha)}(t)}\Delta_\alpha t,$$

which is the desired inequality (3.2). The proof is complete. □

Remark 3.2. In Theorem 3.1 at $\alpha = 1$, we have the inequality

$$\int_a^\infty \frac{\lambda(t)}{(\Upsilon^\sigma(t))^{\gamma}}(\Theta^\sigma(t))^p\Delta t \leqslant K^p\int_a^\infty \frac{(\Upsilon^\sigma(t))^{\gamma(p-1)}}{\Upsilon^{p(\gamma-1)}(t)}\lambda(t)f^p(t)\Delta t,$$

where

$$\gamma-1+\frac{q^\Delta(t)\Upsilon^\sigma(t)\Theta^p(t)}{\lambda(t)(q^\sigma(t))^2(\Theta^\sigma(t))^p}\geqslant\frac{p}{K},\quad \text{for } t\in[a,\infty)_{\mathbb{T}},$$

that is the time scales version of inequality (2.7) in [34].

Remark 3.3. In Theorem 3.1 if $q(t) = 1$, we have the inequality

$$\int_a^\infty \frac{\lambda(t)}{(\Upsilon^\sigma(t))^{\gamma-\alpha+1}}(\Theta^\sigma(t))^p\Delta_\alpha t \leqslant \left(\frac{p}{\gamma-\alpha}\right)^p\int_a^\infty \frac{(\Upsilon^\sigma(t))^{(\gamma-\alpha+1)(p-1)}}{\Upsilon^{p(\gamma-\alpha)}(t)}\lambda(t)f^p(t)\Delta_\alpha t,$$

that is the inequality (22) in [33].

Remark 3.4. In Theorem 3.1 at $\alpha = q(t) = 1$, we have the inequality

$$\int_a^\infty \frac{\lambda(t)}{(\Upsilon^\sigma(t))^{\gamma}}(\Theta^\sigma(t))^p\Delta t \leqslant \left(\frac{p}{\gamma-1}\right)^p\int_a^\infty \frac{(\Upsilon^\sigma(t))^{\gamma(p-1)}}{\Upsilon^{p(\gamma-1)}(t)}\lambda(t)f^p(t)\Delta t,$$

which is the time scales version of inequality (2.8) in [35].

If $\mathbb{T}=\mathbb{R}$ ($\Upsilon^{\sigma}(t)=\Upsilon(t)$), we have the following Copson integral inequality

$$\int_a^{\infty}\frac{\lambda(t)}{\Upsilon^{\gamma}(t)}\left(\int_a^t\lambda(s)f(s)ds\right)^p dt\leqslant\left(\frac{p}{\gamma-1}\right)^p\int_a^{\infty}\Upsilon^{p-\gamma}(t)\lambda(t)f^p(t)dt,$$

when $\lambda(t)=1$ and $a=0$ $\left(\Upsilon(t)=\int_0^t\lambda(s)ds=t\right)$, then we have Hardy integral inequality (1.3),

$$\int_0^{\infty}\frac{1}{t^{\gamma}}\left(\int_0^t f(s)ds\right)^p dt\leqslant\left(\frac{p}{\gamma-1}\right)^p\int_0^{\infty}\frac{1}{t^{\gamma-p}}f^p(t)dt.$$

Also if $\gamma=p$, we have the inequality

$$\int_0^{\infty}\frac{1}{t^{p}}\left(\int_0^t f(s)ds\right)^p dt\leqslant\left(\frac{p}{p-1}\right)^p\int_0^{\infty}f^p(t)dt,$$

which is the classical Hardy inequality (1.2).

When $\mathbb{T}=\mathbb{R}$, then $\sigma(t)=t$ and Theorem 3.1 gives us the following results.

Corollary 3.5. *Let $1<\gamma\leqslant p$, and $\alpha\in(0,1]$, and $q(t)$ be an increasing function on $[a,\infty)$. Furthermore, assume that there exists a constant $K>0$ such that*

$$\gamma-\alpha+\frac{q'(t)t^{1-\alpha}\Upsilon(t)}{\lambda(t)q^2(t)}\geqslant\frac{p}{K},\quad \textit{for } t\in[a,\infty),$$

then

$$\int_a^{\infty}\frac{\lambda(t)}{\Upsilon^{\gamma-\alpha+1}(t)}(\Theta(t))^p d_{\alpha}t\leqslant K^p\int_a^{\infty}\Upsilon^{p-\gamma+\alpha-1}(t)\lambda(t)f^p(t)d_{\alpha}t, \tag{3.9}$$

where

$$\Upsilon(t):=\int_a^t\lambda(s)q(s)d_{\alpha}s,\qquad \textit{and}\quad \Theta(t):=\int_a^t\lambda(s)q(s)f(s)d_{\alpha}s,$$

if $\alpha=1$, then the inequality (3.9) becomes

$$\int_a^{\infty}\frac{\lambda(t)}{\Upsilon^{\gamma}(t)}(\Theta(t))^p dt\leqslant K^p\int_a^{\infty}\Upsilon^{p-\gamma}(t)\lambda(t)f^p(t)dt,$$

where

$$\gamma-1+\frac{q'(t)\Upsilon(t)}{\lambda(t)q^2(t)}\geqslant\frac{p}{K},\quad \textit{for } t\in[a,\infty),$$

and if $\gamma=p$, then we have the Yang and Hwang's inequality (1.8).

In the following, we prove a new inequality with different operators $\overline{\Theta}(t)$ and $\Upsilon(t)$ which are defined by

$$\Upsilon(t):=\int_a^t\lambda(s)q(s)\Delta_{\alpha}s,\qquad \text{and}\quad \overline{\Theta}(t):=\int_t^{\infty}\lambda(s)q(s)f(s)\Delta_{\alpha}s,$$

where

$$\overline{\Theta}(a)<\infty,\qquad \text{and}\qquad \int_a^{\infty}\frac{\lambda(s)}{(\Upsilon^{\sigma}(s))^{\gamma-\alpha+1}}\Delta_{\alpha}s<\infty.$$

Theorem 3.6. *Let $\mathbb{T}$ be a time scale with $a\in[0,\infty)_{\mathbb{T}}$, $p>1$, and $0\leqslant\gamma<\alpha\leqslant1$, and $q(t)$ be an increasing function on $[a,\infty)_{\mathbb{T}}$. Furthermore, assume that there exists a constant $K>0$ such that*

$$\alpha-\gamma-\frac{T_{\alpha}^{\Delta}q(t)\Upsilon^{\sigma}(t)}{\lambda(t)q^2(t)}\geqslant\frac{p}{K},\quad \textit{for } t\in[a,\infty)_{\mathbb{T}}. \tag{3.10}$$

Then

$$\int_a^{\infty}\frac{\lambda(t)}{(\Upsilon^{\sigma}(t))^{\gamma-\alpha+1}}\left(\overline{\Theta}(t)\right)^p\Delta_{\alpha}t\leqslant K^p\int_a^{\infty}(\Upsilon^{\sigma}(t))^{p-\gamma+\alpha-1}\lambda(t)f^p(t)\Delta_{\alpha}t. \tag{3.11}$$

Proof. By employing the formula of integration by parts (2.3) on the term

$$\int_a^\infty \frac{\lambda(t)}{(\Upsilon^\sigma(t))^{\gamma-\alpha+1}} \left(\overline{\Theta}(t)\right)^p \Delta_\alpha t,$$

with $u(t) = \left(\overline{\Theta}(t)\right)^p / q(t)$ and $T_\alpha^\Delta v(t) = \lambda(t) q(t) \left(\Upsilon^\sigma(t)\right)^{\alpha-\gamma-1}$, we have that

$$\int_a^\infty \frac{\lambda(t)}{(\Upsilon^\sigma(t))^{\gamma-\alpha+1}} \left(\overline{\Theta}(t)\right)^p \Delta_\alpha t = v(t) \frac{\overline{\Theta}^p(t)}{q(t)}\bigg|_a^\infty + \int_a^\infty v^\sigma(t) \left(-T_\alpha^\Delta \left(\frac{\overline{\Theta}^p(t)}{q(t)}\right)\right) \Delta_\alpha t, \tag{3.12}$$

where

$$v(t) = \int_a^t \lambda(s) q(s) \left(\Upsilon^\sigma(s)\right)^{\alpha-\gamma-1} \Delta_\alpha s = \int_a^t T_\alpha^\Delta \Upsilon(s) \left(\Upsilon^\sigma(s)\right)^{\alpha-\gamma-1} \Delta_\alpha s.$$

Since $T_\alpha^\Delta \Upsilon(t) = \lambda(t) q(t) \geqslant 0$, and by using chain rule (2.2), we obtain that

$$T_\alpha^\Delta \left(\Upsilon^{\alpha-\gamma}(t)\right) = (\alpha-\gamma)\Upsilon^{\alpha-\gamma-1}(d) T_\alpha^\Delta \Upsilon(t), \text{ where } d \in [t, \sigma(t)] = \frac{(\alpha-\gamma) T_\alpha^\Delta \Upsilon(t)}{\Upsilon^{\gamma-\alpha+1}(d)} \geqslant \frac{(\alpha-\gamma) T_\alpha^\Delta \Upsilon(t)}{(\Upsilon^\sigma(t))^{\gamma-\alpha+1}}.$$

Then we have

$$T_\alpha^\Delta \Upsilon(t) \left(\Upsilon^\sigma(t)\right)^{\alpha-\gamma-1} \leqslant \frac{1}{\alpha-\gamma} T_\alpha^\Delta \left(\Upsilon^{\alpha-\gamma}(t)\right),$$

and thus

$$v^\sigma(t) = \int_a^{\sigma(t)} T_\alpha^\Delta \Upsilon(s) \left(\Upsilon^\sigma(s)\right)^{\alpha-\gamma-1} \Delta_\alpha s \leqslant \int_a^{\sigma(t)} \frac{1}{\alpha-\gamma} T_\alpha^\Delta \left(\Upsilon^{\alpha-\gamma}(s)\right) \Delta_\alpha s \leqslant \frac{1}{\alpha-\gamma} \left(\Upsilon^\sigma(t)\right)^{\alpha-\gamma}. \tag{3.13}$$

By using the quotient rule (2.1), we see that

$$-T_\alpha^\Delta \left(\frac{\overline{\Theta}^p(t)}{q(t)}\right) = \frac{-q(t) T_\alpha^\Delta \overline{\Theta}^p(t) + \overline{\Theta}^p(t) T_\alpha^\Delta q(t)}{q(t) q^\sigma(t)} = \frac{-T_\alpha^\Delta \overline{\Theta}^p(t)}{q^\sigma(t)} + \frac{\overline{\Theta}^p(t) T_\alpha^\Delta q(t)}{q(t) q^\sigma(t)}, \tag{3.14}$$

and by using chain rule (2.2), we have that

$$-T_\alpha^\Delta \left(\overline{\Theta}^p(t)\right) = -p\overline{\Theta}^{p-1}(d) T_\alpha^\Delta \overline{\Theta}(t), \text{ where } d \in [t,\, \sigma(t)],$$

since $T_\alpha^\Delta \overline{\Theta}(t) = -\lambda(t) q(t) f(t) \leqslant 0$ and $t \leqslant d$ we have

$$-T_\alpha^\Delta \left(\overline{\Theta}^p(t)\right) = -p\overline{\Theta}^{p-1}(d) T_\alpha^\Delta \overline{\Theta}(t) \leqslant p\lambda(t) q(t) f(t) \left(\overline{\Theta}(t)\right)^{p-1}. \tag{3.15}$$

From (3.14) and (3.15), we have that

$$-T_\alpha^\Delta \left(\frac{\overline{\Theta}^p(t)}{q(t)}\right) = \frac{-T_\alpha^\Delta \overline{\Theta}^p(t)}{q^\sigma(t)} + \frac{\overline{\Theta}^p(t) T_\alpha^\Delta q(t)}{q(t) q^\sigma(t)} \leqslant \frac{p\lambda(t) q(t) f(t) \left(\overline{\Theta}(t)\right)^{p-1}}{q^\sigma(t)} + \frac{\overline{\Theta}^p(t) T_\alpha^\Delta q(t)}{q(t) q^\sigma(t)}. \tag{3.16}$$

Since $\overline{\Theta}(\infty) = 0$, $v(a) = 0$, and from (3.13), (3.16), and (3.12) we have

$$\begin{aligned}
\int_a^\infty \frac{\lambda(t)}{(\Upsilon^\sigma(t))^{\gamma-\alpha+1}} \left(\overline{\Theta}(t)\right)^p \Delta_\alpha t &\leqslant \int_a^\infty v^\sigma(t) \left(-T_\alpha^\Delta \left(\frac{\overline{\Theta}^p(t)}{q(t)}\right)\right) \Delta_\alpha t \\
&\leqslant \int_a^\infty \frac{(\Upsilon^\sigma(t))^{\alpha-\gamma}}{\alpha-\gamma} \left(\frac{p\lambda(t) q(t) f(t) \left(\overline{\Theta}(t)\right)^{p-1}}{q^\sigma(t)} \frac{\overline{\Theta}^p(t) T_\alpha^\Delta q(t)}{q(t) q^\sigma(t)}\right) \Delta_\alpha t,
\end{aligned}$$

since $T_\alpha^\Delta\Upsilon(t)=\lambda(t)q(t)\geqslant 0$, and $d\leqslant\sigma(t)$, then we have (note that $q(t)$ be an increasing function on $[a,\infty)_{\mathbb{T}}$)

$$\int_a^\infty \frac{\lambda(t)}{(\Upsilon^\sigma(t))^{\gamma-\alpha+1}}\left(\overline{\Theta}(t)\right)^p\Delta_\alpha t\leqslant \frac{p}{\alpha-\gamma}\int_a^\infty \lambda(t)f(t)\left(\Upsilon^\sigma(t)\right)^{\alpha-\gamma}\left(\overline{\Theta}(t)\right)^{p-1}\Delta_\alpha t + \frac{1}{\alpha-\gamma}\int_a^\infty \frac{\overline{\Theta}^p(t)T_\alpha^\Delta q(t)}{(\Upsilon^\sigma(t))^{\gamma-\alpha}q^2(t)}\Delta_\alpha t,$$

hence

$$\int_a^\infty \frac{\lambda(t)}{(\Upsilon^\sigma(t))^{\gamma-\alpha+1}}\left(\overline{\Theta}(t)\right)^p\Delta_\alpha t\left[\alpha-\gamma-\frac{T_\alpha^\Delta q(t)\Upsilon^\sigma(t)}{\lambda(t)q^2(t)}\right]\Delta_\alpha t\leqslant p\int_a^\infty \frac{\lambda(t)f(t)\left(\overline{\Theta}(t)\right)^{p-1}}{(\Upsilon^\sigma(t))^{\gamma-\alpha}}\Delta_\alpha t. \tag{3.17}$$

From (3.10), and (3.17), we see that

$$\begin{aligned}\int_a^\infty \frac{\lambda(t)}{(\Upsilon^\sigma(t))^{\gamma-\alpha+1}}\left(\overline{\Theta}(t)\right)^p\Delta_\alpha t&\leqslant K\int_a^\infty \frac{\lambda(t)f(t)\left(\overline{\Theta}(t)\right)^{p-1}}{(\Upsilon^\sigma(t))^{\gamma-\alpha}}\Delta_\alpha t\\ &=K\int_a^\infty\left(\frac{\lambda^{\frac{p-1}{p}}(t)\left(\overline{\Theta}(t)\right)^{p-1}}{(\Upsilon^\sigma(t))^{\frac{(\gamma-\alpha+1)(p-1)}{p}}}\times\frac{\lambda^{\frac{1}{p}}(t)\left(\Upsilon^\sigma(t)\right)^{\frac{(\gamma-\alpha+1)(p-1)}{p}}f(t)}{(\Upsilon^\sigma(t))^{\gamma-\alpha}}\right)\Delta_\alpha t,\end{aligned}$$

by applying Hölder's inequality (2.4) with indices p and $p/(p-1)$, we have

$$\begin{aligned}\int_a^\infty \frac{\lambda(t)}{(\Upsilon^\sigma(t))^{\gamma-\alpha+1}}\left(\overline{\Theta}(t)\right)^p\Delta_\alpha t&\leqslant K\int_a^\infty\left(\frac{\lambda^{\frac{p-1}{p}}(t)\left(\overline{\Theta}(t)\right)^{p-1}}{(\Upsilon^\sigma(t))^{\frac{(\gamma-\alpha+1)(p-1)}{p}}}\times\frac{\lambda^{\frac{1}{p}}(t)\left(\Upsilon^\sigma(t)\right)^{\frac{(\gamma-\alpha+1)(p-1)}{p}}f(t)}{(\Upsilon^\sigma(t))^{\gamma-\alpha}}\right)\Delta_\alpha t\\ &\leqslant K\left(\int_a^\infty\frac{\lambda(t)\left(\overline{\Theta}(t)\right)^p}{(\Upsilon^\sigma(t))^{\gamma-\alpha+1}}\Delta_\alpha t\right)^{\frac{p-1}{p}}\left(\int_a^\infty\frac{\lambda(t)\left(\Upsilon^\sigma(t)\right)^{(\gamma-\alpha+1)(p-1)}f^p(t)}{(\Upsilon^\sigma(t))^{p(\gamma-\alpha)}(t)}\Delta_\alpha t\right)^{\frac{1}{p}},\end{aligned}$$

then

$$\left(\int_a^\infty \frac{\lambda(t)}{(\Upsilon^\sigma(t))^{\gamma-\alpha+1}}\left(\overline{\Theta}(t)\right)^p\Delta_\alpha t\right)^{\frac{1}{p}}\leqslant K\left(\int_a^\infty\left(\Upsilon^\sigma(t)\right)^{p-\gamma+\alpha-1}\lambda(t)f^p(t)\Delta_\alpha t\right)^{\frac{1}{p}}.$$

This leads to

$$\int_a^\infty \frac{\lambda(t)}{(\Upsilon^\sigma(t))^{\gamma-\alpha+1}}\left(\overline{\Theta}(t)\right)^p\Delta_\alpha t\leqslant K^p\int_a^\infty\left(\Upsilon^\sigma(t)\right)^{p-\gamma+\alpha-1}\lambda(t)f^p(t)\Delta_\alpha t,$$

which is the desired inequality (3.11). The proof is complete. □

Remark 3.7. In Theorem 3.6 at $\alpha=1$, we have the inequality

$$\int_a^\infty \frac{\lambda(t)}{(\Upsilon^\sigma(t))^{\gamma}}\left(\overline{\Theta}(t)\right)^p\Delta t\leqslant K^p\int_a^\infty\left(\Upsilon^\sigma(t)\right)^{p-\gamma}\lambda(t)f^p(t)\Delta t,$$

where

$$1-\gamma-\frac{q^\Delta(t)\Upsilon^\sigma(t)}{\lambda(t)q^2(t)}\geqslant\frac{p}{K},\quad \text{for } t\in[a,\infty)_{\mathbb{T}},$$

that is the time scales version of inequality (2.27) in [34].

Remark 3.8. In Theorem 3.6, if $q(t)=1$, we have the inequality

$$\int_a^\infty \frac{\lambda(t)}{(\Upsilon^\sigma(t))^{\gamma-\alpha+1}}\left(\overline{\Theta}(t)\right)^p\Delta_\alpha t\leqslant\left(\frac{p}{\alpha-\gamma}\right)^p\int_a^\infty\left(\Upsilon^\sigma(t)\right)^{p-\gamma+\alpha-1}\lambda(t)f^p(t)\Delta_\alpha t,$$

that is the inequality (26) in [33].

Remark 3.9. In Theorem 3.6 at $\alpha = q(t) = 1$, we have the inequality

$$\int_a^\infty \frac{\lambda(t)}{(\Upsilon^\sigma(t))^\gamma}\left(\overline{\Theta}(t)\right)^p \Delta t \leqslant \left(\frac{p}{1-\gamma}\right)^p \int_a^\infty \left(\Upsilon^\sigma(t)\right)^{p-\gamma}\lambda(t)f^p(t)\Delta t,$$

which is the time scales version of inequality (2.22) in [35].

If $\mathbb{T} = \mathbb{R}$ (so that $\Upsilon^\sigma(t) = \Upsilon(t)$), we have the following Copson integral inequality

$$\int_a^\infty \frac{\lambda(t)}{\Upsilon^\gamma(t)}\left(\int_t^\infty \lambda(s)f(s)ds\right)^p dt \leqslant \left(\frac{p}{1-\gamma}\right)^p \int_a^\infty \Upsilon^{p-\gamma}(t)\lambda(t)f^p(t)dt,$$

when $\lambda(t) = 1$ and $a = 0$ $\left(\Upsilon(t) = \int_0^t \lambda(s)ds = t\right)$, then we have Hardy integral inequality (1.4),

$$\int_0^\infty \frac{1}{t^\gamma}\left(\int_t^\infty f(s)ds\right)^p dt \leqslant \left(\frac{p}{1-\gamma}\right)^p \int_0^\infty \frac{1}{t^{\gamma-p}}f^p(t)dt.$$

When $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = t$ and Theorem 3.6 gives us the following results.

Corollary 3.10. *Let $p > 1$, and $0 \leqslant \gamma < \alpha \leqslant 1$, and $q(t)$ be an increasing function on $[a,\infty)$. Furthermore, assume that there exists a constant $K > 0$ such that*

$$\alpha - \gamma - \frac{q'(t)t^{1-\alpha}\Upsilon(t)}{\lambda(t)q^2(t)} \geqslant \frac{p}{K}, \quad \textit{for } t \in [a,\infty).$$

Then

$$\int_a^\infty \frac{\lambda(t)}{\Upsilon^{\gamma-\alpha+1}(t)}\left(\overline{\Theta}(t)\right)^p d_\alpha t \leqslant K^p \int_a^\infty \Upsilon^{p-\gamma+\alpha-1}(t)\lambda(t)f^p(t)d_\alpha t, \tag{3.18}$$

where

$$\Upsilon(t) := \int_a^t \lambda(s)q(s)d_\alpha s, \quad \textit{and } \overline{\Theta}(t) := \int_t^\infty \lambda(s)q(s)f(s)d_\alpha s.$$

If $\alpha = 1$, then the inequality (3.18) becomes

$$\int_a^\infty \frac{\lambda(t)}{\Upsilon^\gamma(t)}\left(\overline{\Theta}(t)\right)^p dt \leqslant K^p \int_a^\infty \Upsilon^{p-\gamma}(t)\lambda(t)f^p(t)dt,$$

where

$$1 - \gamma - \frac{q'(t)\Upsilon(t)}{\lambda(t)q^2(t)} \geqslant \frac{p}{K}, \quad \textit{for } t \in [a,\infty).$$

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