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Ideal theory of BCK/BCI-algebras based on hybrid structures

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Abstract

In this paper, the concept of hybrid ideals in BCK/BCI-algebras is introduced, and related properties are investigated. Further, it is proved that every hybrid ideal is hybrid sub-algebra in BCK/BCI-algebras, but the converse is not valid in general, and an example is provided in this regard. Some characterizations of hybrid ideals in BCK/BCI-algebras are given. Moreover, the notion of hybrid closed ideals in BCI-algebras is introduced, and some associated properties are studied. Furthermore, the hybrid intersection and hybrid union are also discussed.

Keywords: BCK/BCI-algebras, hybrid sub-algebra, hybrid ideal, hybrid closed ideal.

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1. Introduction and Preliminaries

BCK and BCI-algebras are two classes of non-classical logic algebras which were introduced by Imai and Iseki in 1966 [16, 17]. They are algebraic formulation of BCK-system and BCI-system in combinatory logic. Since then a great deal of literature has been produced on the theory of BCK/BCI-algebras, in particular, emphasis seems to have been put on the ideal theory of BCK/BCI-algebras.

Fuzzy sets, which were introduced by Zadeh [34], deal with possibilistic uncertainty, connected with imprecision of states, perceptions and preferences. After the introduction of fuzzy sets by Zadeh, fuzzy set theory has become an active area of research in various fields. These are widely scattered over many disciplines such as artificial intelligence, computer science, control engineering, expert systems, management science, operations research, pattern recognition, robotics, and others. Molodtsov [28] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets. Molodtsov successfully applied the soft set theory in several directions, such as smoothness of functions, game theory, operations research, Riemann integration, Perron integration, probability, theory of measurement and so on (see [28–31]). Soft set theory is applied to algebraic structures such as BCK/BCI-algebra ([18, 21, 23]), d-algebras ([20]), ring ([1]), semiring ([9, 14]), group (see [7]), ordered semigroup ([19]), decision making ([10, 11, 26]) and BL-algebra ([35]).

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By combining the concepts of fuzzy sets and soft sets, Jun et al. [24] introduced the notion of hybrid structure in a set of parameters over an initial universe set, and investigated several properties. Using this notion, they introduce the concepts of a hybrid sub-algebra, a hybrid field and a hybrid linear space. Recently, the notion of hybrid structure and its applications to BCI-algebras and semigroups have been studied in (see [8, 12, 13, 25] and references there in). Also, more concepts related to our study in different aspect have been studied in [2–6, 15, 22, 32, 33].

Motivated by a lot of work on ideal theory in BCK/BCI-algebras, in this paper, we studied the ideal theory of BCK/BCI-algebras based on hybrid structures. In fact, the notions of hybrid ideals and hybrid closed ideals in BCK/BCI-algebras are introduced and related properties are investigated. Furthermore, the relations among hybrid closed ideals, hybrid sub-algebras and hybrid ideals are studied. Also, the hybrid intersection and hybrid union are discussed.

An algebra $(P;*,0)$ of type $(2,0)$ is said to be a *BCI-algebra* if the following conditions are satisfied:

(K_1) $((p*l)*(p*q))*(q*l)=0$;
(K_2) $(p*(p*l))*l=0$;
(K_3) $p*p=0$;
(K_4) $p*l=0$ and $l*p=0 \Rightarrow p=l$; for all $p,q,l \in P$.

If a BCI-algebra P satisfies the condition:

(K_5) $0*p=0$,

then P is a BCI-algebra, for all $p \in P$.

The following is true in any BCK/BCI-algebra P:

(P_1) $p*0=p$;
(P_2) $(p*l)*q=(p*q)*l$;
(P_3) $p \leqslant l \Rightarrow p*q \leqslant l*q$ and $q*l \leqslant q*p$;
(P_4) $0*(p*l)=(0*p)*(0*l)$;
(P_5) $0*(0*(p*l))=0*(l*p)$;
(P_6) $(p*q)*(l*q) \leqslant (p*l)$;
(P_7) $p*(p*(p*l))=p*l$;
(P_8) $0*(0*((p*q)*(l*q)))=(0*l)*(0*p)$;
(P_9) $0*(0*(p*l)=(0*l)*(0*p)$,

where $p \leqslant l$ if and only if $p*l=0$ for all $p,q,l \in P$. Note that $(P,\leqslant)$ is a partially ordered set.

A nonempty subset S of P is said to be a sub-algebra of P if $p*l \in S$ for all $p,l \in S$.

A nonempty subset I of P is said to be an ideal of P if:

(I_1) $0 \in I$;
(I_2) $\forall p,l \in P, p*l \in I$ and $l \in I \Rightarrow p \in I$.

For more details we refer the readers to [27].

Further, we collect some basic notions and results on hybrid structures due to Jun et al. [24]. Let I, P and $\mathfrak{P}(U)$ be the unit interval, a set of parameters and the power set of an initial universe set U, respectively.

Definition 1.1 ([24])**.** A hybrid structure in P over U is a mapping

$$\tilde{f}_\lambda := (\tilde{f};\lambda) : P \to \mathfrak{P}(U) \times I; p \mapsto (\tilde{f}(p);\lambda(p)),$$

where $\tilde{f}: P \to \mathfrak{P}(U)$ and $\lambda : P \to I$ are mappings.

Definition 1.2 ([24])**.** For hybrid structures $\tilde{f}_\lambda$ and $\tilde{g}_\gamma$ in P over U, the hybrid intersection denoted by $\tilde{f}_\lambda \Cap \tilde{g}_\gamma$ is a hybrid structure

$$\tilde{f}_\lambda \Cap \tilde{g}_\gamma : P \to \mathfrak{P}(U) \times I, \quad p \mapsto ((\tilde{f} \tilde{\cap} \tilde{g})(p), (\lambda \vee \gamma)(p)),$$

where

$$\tilde{f} \tilde{\cap} \tilde{g} : P \to \mathfrak{P}(U), p \mapsto \tilde{f}(p) \cap \tilde{g}(p), \quad \lambda \vee \gamma : P \to I, p \mapsto \bigvee\{\lambda(p), \gamma(p)\}.$$

Definition 1.3 ([24]). For hybrid structures $\tilde{f}_\lambda$ and $\tilde{g}_\gamma$ in P over U, the hybrid union denoted by $\tilde{f}_\lambda \Cup \tilde{g}_\gamma$ is a hybrid structure

$$\tilde{f}_\lambda \Cup \tilde{g}_\gamma : P \to \mathfrak{P}(U) \times I, \quad p \mapsto ((\tilde{f} \tilde{\cup} \tilde{g})(p), (\lambda \wedge \gamma)(p)),$$

where

$$\tilde{f} \tilde{\cup} \tilde{g} : P \to \mathfrak{P}(U), p \mapsto \tilde{f}(p) \cup \tilde{g}(p), \quad \lambda \wedge \gamma : P \to I, p \mapsto \bigwedge\{\lambda(p), \gamma(p)\}.$$

Definition 1.4 ([24]). Let P be a BCK/BCI-algebra. For a hybrid structure $\tilde{f}_\lambda$ in P over U, $\tilde{f}_\lambda$ is said to be a hybrid sub-algebra of P if the following statements are valid:

$$(\forall p, l \in P) \left(\begin{array}{c} \tilde{f}(p * l) \supseteq \tilde{f}(p) \cap \tilde{f}(l), \\ \lambda(p * l) \leqslant \bigvee\{\lambda(p), \lambda(l)\} \end{array} \right).$$

Lemma 1.5 ([24]). *Every hybrid sub-algebra $\tilde{f}_\lambda$ of a BCK/BCI-algebra P over U satisfies:*

$$(\forall p \in P)\ (\tilde{f}(0) \supseteq \tilde{f}(p), \lambda(0) \leqslant \lambda(p)).$$

2. Hybrid ideals in BCK/BCI-algebras

Definition 2.1. Let P be a BCK/BCI-algebra. For a hybrid structure $\tilde{f}_\lambda$ in P over U, $\tilde{f}_\lambda$ is said to be a hybrid ideal of P if the following statements are valid:

(H_1) $(\forall p \in P) \left(\begin{array}{c} \tilde{f}(0) \supseteq \tilde{f}(p), \\ \lambda(0) \leqslant \lambda(p) \end{array} \right)$;

(H_2) $(\forall p, l \in P) \left(\begin{array}{c} \tilde{f}(p) \supseteq \tilde{f}(p * l) \cap \tilde{f}(l), \\ \lambda(p) \leqslant \bigvee\{\lambda(p * l), \lambda(l)\} \end{array} \right)$.

Example 2.2. Let U be the initial universe set given by the doctors as follows:

$$U = \{d_1, d_2, d_3, d_4, d_5\}.$$

Where, d_1 stands for "cardiologist", d_2 stands for "pulmonologist", d_3 stands for "orthopedics", d_4 stands for "gastroenterologist", and d_5 stands for "gynecologist".

As a set of parameters, we consider $P = \{h, l, b, s, p\}$ be a status of doctors in which h stands for the parameter "heart", l stands for the parameter "lungs", b stands for the parameter "bone", s stands for the parameter "stomach", and p stands for the parameter "pregnancy".

We define a binary operation on P by Table 1.

Table 1

$*$	h	l	b	s	p
h	h	h	h	s	s
l	l	h	l	p	s
b	b	b	h	s	s
s	s	s	s	h	h
p	p	s	p	l	h

Then $(P, *, h)$ is a BCI-algebra. Let $\tilde{f}_\lambda$ be a hybrid structure represented by Table 2.

It is routine to verify that $\tilde{f}_\lambda$ is a hybrid ideal of P over U.

Table 2

P	$\tilde{f}$	λ
h	U	0.2
l	$\{d_1, d_2, d_3, d_4\}$	0.2
b	$\{d_2, d_4\}$	0.5
s	$\{d_2, d_4\}$	0.5
p	$\{d_2, d_4\}$	0.5

Proposition 2.3. *Let $\tilde{f}_\lambda = (\tilde{f}, \lambda)$ be a hybrid ideal of P. If for all $p, q, l \in P$, $p * l \leqslant q$ holds in P, then*

$$\tilde{f}(p) \supseteq \tilde{f}(l) \cap \tilde{f}(q) \ \ \textit{and} \ \ \lambda(p) \leqslant \bigvee\{\lambda(l), \lambda(q)\}.$$

Proof. Assume that for all $p, q, l \in P$ the inequality $p * l \leqslant q$ holds in P. Then

$$\tilde{f}(p * l) \supseteq \tilde{f}((p * l) * q) \cap \tilde{f}(q) = \tilde{f}(0) \cap \tilde{f}(q) = \tilde{f}(q)$$

and

$$\lambda(p * l) \leqslant \bigvee\{\lambda((p * l) * q), \lambda(q)\} = \bigvee\{\lambda(0), \lambda(q)\} = \lambda(q).$$

It follows from (H_1) and (H_2) that

$$\tilde{f}(p) \supseteq \tilde{f}(p * l) \cap \tilde{f}(l) \supseteq \tilde{f}(q) \cap \tilde{f}(l) \ \ \text{and} \ \ \lambda(p) \leqslant \bigvee\{\lambda(p * l), \lambda(l)\} \leqslant \bigvee\{\lambda(q), \lambda(l)\}.$$

This completes the prove. □

Proposition 2.4. *For a hybrid ideal $\tilde{f}_\lambda$ of P over U, if the inequality $p \leqslant l$ holds in P, then the following are equivalent:*

(1) $(\forall p, l \in P)\ (\tilde{f}(p) \supseteq \tilde{f}(l), \lambda(p) \leqslant \lambda(l))$;
(2) $(\forall p \in P)\ (\tilde{f}(0) = \tilde{f}(p), \lambda(0) = \lambda(p))$.

Proof. If we take $l = 0$ in (1), then $(\tilde{f}(p) \supseteq \tilde{f}(0)$ and $\lambda(p) \leqslant \lambda(0))$ for all $p \in P$. Combining this and (H_1), we have $(\tilde{f}(0) = \tilde{f}(p), \lambda(0) = \lambda(p))$ for all $p \in P$.

Conversely, assume that (2) is valid. Then, for all $p, l \in P$

$$\tilde{f}(l) = \tilde{f}(0) \cap \tilde{f}(l) = \tilde{f}(p * l) \cap \tilde{f}(l) \subseteq \tilde{f}(p), \quad \lambda(l) = \bigvee\{\lambda(0), \lambda(l)\} = \bigvee\{\lambda(p * l), \lambda(l)\} \geqslant \lambda(p).$$

□

Proposition 2.5. *Let $\tilde{f}_\lambda = (\tilde{f}, \lambda)$ be a hybrid ideal of P. If for all $p, l \in P$, $p \leqslant l$ holds in P, then*

$$\tilde{f}(p) \supseteq \tilde{f}(l) \ \ \textit{and} \ \ \lambda(p) \leqslant \lambda(l).$$

Proof. Let $\tilde{f}_\lambda = (\tilde{f}_\lambda, \lambda)$ be a hybrid ideal of P. Then for all $p, l \in P$, where $p \leqslant l$, we have

$$\tilde{f}(p) \supseteq \tilde{f}(0) \cap \tilde{f}(l) = \tilde{f}(l) \quad \text{and} \quad \lambda(p) \leqslant \bigvee\{\lambda(0), \lambda(l)\} = \lambda(l).$$

□

Theorem 2.6. *If $\tilde{f}_\lambda = (\tilde{f}, \lambda)$ be a hybrid ideal of P, then for any $p, a_1, a_2, \ldots, a_n \in P$, $(\cdots((p * a_1) * a_2) * \cdots) * a_n = 0$, we have,*

$$\tilde{f}(p) \supseteq \tilde{f}(a_1) \cap \tilde{f}(a_2) \cap \cdots \cap \tilde{f}(a_n) \quad \textit{and} \quad \lambda(p) \leqslant \bigvee\{\lambda(a_1), \lambda(a_2), \ldots, \lambda(a_n)\}.$$

Proof. The proof is straightforward by using induction on n and also by using Propositions 2.4 and 2.5. □

Theorem 2.7. *Every hybrid ideal of* P *is a hybrid sub-algebra of* P.

Proof. Assume that $\tilde{f}_\lambda = (\tilde{f}, \lambda)$ is a hybrid ideal of P. Since $p * l \leqslant p$ for all $p, l \in P$, it follows from Proposition 2.5 that

$$\tilde{f}(p * l) \supseteq \tilde{f}(p) \text{ and } \lambda(p * l) \leqslant \lambda(p)$$

so by (H_2)

$$\tilde{f}(p * l) \supseteq \tilde{f}(p) \supseteq \tilde{f}(p * l) \cap \tilde{f}(l) \supseteq \tilde{f}(p) \cap \tilde{f}(l) \quad \text{and} \quad \lambda(p * l) \leqslant \lambda(p) \leqslant \bigvee\{\lambda(p * l), \lambda(l)\} \leqslant \bigvee\{\lambda(p), \lambda(l)\}.$$

This shows that $\tilde{f}_\lambda$ is a hybrid sub-algebra of P. □

The following example shows that the converse of Theorem 2.7 may not be true.

Example 2.8. Consider the initial universe U and let $(P; *, h)$ be a BCK-algebra where $P = \{h, b, s, p\}$ is the set of parameters and the operation $*$ is given by Table 3.

Table 3

$*$	h	b	s	p
h	h	h	h	h
b	b	h	h	b
s	s	b	h	s
p	p	p	p	h

Let $\tilde{f}_\lambda = (\tilde{f}, \lambda)$ be a hybrid structure in P over U given by Table 4.

Table 4

P	$\tilde{f}$	λ
h	$\{d_1, d_2, d_3, d_4, d_5\}$	0.3
b	$\{d_1, d_4, d_5\}$	0.6
s	$\{d_1, d_3, d_4\}$	0.8
p	$\{d_2, d_3, d_5\}$	0.4

Then by routine calculations, $\tilde{f}_\lambda = (\tilde{f}, \lambda)$ is a hybrid sub-algebra of a BCK-algebra over U but not a hybrid ideal of a BCK-algebra as

$$\tilde{f}(s) = \{d_1, d_3, d_4\} \not\supseteq \{d_1, d_4, d_5\} = \tilde{f}(s * b) \cap \tilde{f}(b).$$

The following theorem gives a condition for a hybrid sub-algebra to be a hybrid ideal.

Theorem 2.9. *Let* $\tilde{f}_\lambda = (\tilde{f}, \lambda)$ *be a hybrid sub-algebra of* P. *If* $\tilde{f}_\lambda = (\tilde{f}, \lambda)$ *satisfies the following conditions:*

$$\tilde{f}(p) \supseteq \tilde{f}(l) \cap \tilde{f}(q) \text{ and } \lambda(p) \leqslant \bigvee\{\lambda(l), \lambda(q)\} \tag{2.1}$$

for all $p, l, q \in P$ *with* $p * l \leqslant q$, *then* $\tilde{f}_\lambda = (\tilde{f}, \lambda)$ *is a hybrid ideal of* P.

Proof. Let $\tilde{f}_\lambda = (\tilde{f}, \lambda)$ be a hybrid sub-algebra of P. By Lemma 1.5, we have $(\tilde{f}(0) \supseteq \tilde{f}(p)$ and $\lambda(0) \leqslant \lambda(p))$ for all $p \in P$. Since $p * (p * l) \leqslant l$ for all $p, l \in P$, using (2.1) we have,

$$\tilde{f}(p) \supseteq \tilde{f}(p * l) \cap \tilde{f}(l) \text{ and } \lambda(p) \leqslant \bigvee\{\lambda(p * l), \lambda(l)\}.$$

Thus, $\tilde{f}_\lambda = (\tilde{f}, \lambda)$ is a hybrid ideal of P. □

Theorem 2.10. *Let* P *be a BCK/BCI-algebra. For a hybrid structure* $\tilde{f}_\lambda$ *in* P *over* U, *the following statements are equivalent:*

(1) $\tilde{f}_\lambda$ *is a hybrid ideal of* P;

(2) *for any* $\alpha \in \mathfrak{P}(U)$ *and* $t \in I$, *the nonempty sets* $\tilde{f}_\lambda(\alpha) := \{p \in P \mid \alpha \subseteq \tilde{f}(p)\}$ *and* $\tilde{f}_\lambda(t) := \{p \in P \mid \lambda(p) \leqslant t\}$ *are ideals of* P.

Proof. Let $\tilde{f}_\lambda$ be a hybrid ideal of P. Let $\alpha \in \mathfrak{P}(U)$ and $t \in I$ such that $\tilde{f}_\lambda(\alpha) \neq \emptyset$ and $\tilde{f}_\lambda(t) \neq \emptyset$. If $p * l, l \in \tilde{f}_\lambda(\alpha) \cap \tilde{f}_\lambda(t)$, then $\alpha \subseteq \tilde{f}(p * l), \alpha \subseteq \tilde{f}(l)$ and $\lambda(p * l) \leqslant t, \lambda(l) \leqslant t$. It follows from (H_2) that

$$\tilde{f}(p) \supseteq \tilde{f}(p * l) \cap \tilde{f}(l) \supseteq \alpha \quad \text{and} \quad \lambda(p) \leqslant \bigvee\{\lambda(p * l), \lambda(l)\} \leqslant t.$$

Hence $p \in \tilde{f}_\lambda(\alpha) \cap \tilde{f}_\lambda(t)$, and so $\tilde{f}_\lambda(\alpha)$ and $\tilde{f}_\lambda(t)$ are ideal of P. Now suppose that the second assertion is valid. Let $p, l \in P$ be such that $\tilde{f}(p * l) = \alpha_p$ and $\tilde{f}(l) = \alpha_l$. Taking $\alpha = \alpha_p \cap \alpha_l$ implies that $p * l, l \in \tilde{f}_\lambda(\alpha)$, and so $p \in \tilde{f}_\lambda(\alpha)$. Hence,

$$\tilde{f}(p) \supseteq \alpha = \alpha_p \cap \alpha_l = \tilde{f}(p * l) \cap \tilde{f}(l).$$

For any $p, l \in P$, let $t := \bigvee\{\lambda(p * l), \lambda(l)\}$. Then $p * l, l \in \tilde{f}_\lambda(t)$, and so $p \in \tilde{f}_\lambda(t)$. It follows that

$$\lambda(p) \leqslant t = \bigvee\{\lambda(p * l), \lambda(l)\}.$$

Hence, $\tilde{f}_\lambda$ is a hybrid ideal of P over U. □

Theorem 2.11. *Let* $\tilde{f}_\lambda$ *be a hybrid ideal of* P *over* U. *The set*

$$\Omega := \{p \in P \mid \tilde{f}(p) \cap \alpha \neq \emptyset, \lambda(p) \leqslant t\}$$

is an ideal of P *for all* $(\alpha, t) \in \mathfrak{P}(U) \times I$ *with* $\alpha \neq \emptyset$ *whenever it is nonempty.*

Proof. Let $(\alpha, t) \in \mathfrak{P}(U) \times I$ be such that $\Omega \neq \emptyset \neq \alpha$. Let $p * l, l \in \Omega$. Then $\tilde{f}(p * l) \cap \alpha \neq \emptyset \neq \tilde{f}(l) \cap \alpha$, $\lambda(p * l) \leqslant t$ and $\lambda(l) \leqslant t$. It follows from (H_2) that

$$\tilde{f}(p) \cap \alpha \supseteq (\tilde{f}(p * l) \cap \tilde{f}(l)) \cap \alpha = (\tilde{f}(p * l) \cap \alpha) \cap (\tilde{f}(l) \cap \alpha) \neq \emptyset$$

and

$$\lambda(p) \leqslant \bigvee\{\lambda(p * l), \lambda(l)\} \leqslant t.$$

Hence $p \in \Omega$, and so Ω is an ideal of P. □

Proposition 2.12. *In a BCI-algebra* P, *every hybrid ideal* $\tilde{f}_\lambda$ *of* P *over* U *satisfies the following condition:*

$$(\forall p \in P)(\tilde{f}(0 * (0 * p)) \supseteq \tilde{f}(p) \text{ and } \lambda(0 * (0 * p)) \leqslant \lambda(p)).$$

Proof. Let $\tilde{f}_\lambda$ hybrid ideal of P over U. Then

$$\tilde{f}(0 * (0 * p)) \supseteq \tilde{f}((0 * (0 * p)) * p) \cap \tilde{f}(p) = \tilde{f}(0) \cap \tilde{f}(p) = \tilde{f}(p)$$

and

$$\lambda(0 * (0 * p)) \leqslant \bigvee\{\lambda((0 * (0 * p)) * p), \lambda(p)\} = \bigvee\{\lambda(0), \lambda(p)\} = \lambda(p)$$

for all $p \in P$. □

Definition 2.13. Let P be a BCI-algebra. A hybrid structure $\tilde{f}_\lambda$ in P over U is said to be a hybrid closed ideal of P if (H_1), (H_2), and following condition are satisfied:

(H_3) $(\forall p \in P)\left(\begin{array}{l}\tilde{f}(0 * p) \supseteq \tilde{f}(p), \\ \lambda(0 * p) \leqslant \lambda(p)\end{array}\right).$

Theorem 2.14. *In a BCI-algebra* P*, every hybrid closed ideal is a hybrid sub-algebra of* P *over* U.

Proof. Let $\tilde{f}_\lambda$ be a hybrid closed ideal of P. Then $\tilde{f}(0*p) \supseteq \tilde{f}(p)$ and $\lambda(0*p) \leqslant \lambda(p)$. It follows that

$$\tilde{f}(p*l) \supseteq \tilde{f}((p*l)*p) \cap \tilde{f}(p) = \tilde{f}(0*l) \cap \tilde{f}(p) \supseteq \tilde{f}(p) \cap \tilde{f}(l)$$

and

$$\lambda(p*l) \leqslant \bigvee\{\lambda((p*l)*p), \lambda(p)\} = \bigvee\{\lambda(0*l), \lambda(p)\} \leqslant \bigvee\{\lambda(p), \lambda(l)\}$$

for all $p, l \in P$. Hence, $\tilde{f}_\lambda$ is a hybrid sub-algebra of P. □

Theorem 2.15. *Let* $\tilde{f}_\lambda$ *be a hybrid ideal of a BCI-algebra* P *over* U*. Then it is closed if and only if it satisfies:*

$$(\forall p, l \in P)(\tilde{f}(p*l) \supseteq \tilde{f}(p) \cap \tilde{f}(l) \text{ and } \lambda(p*l) \leqslant \bigvee\{\lambda(p), \lambda(l)\}). \tag{2.2}$$

Proof. Let $\tilde{f}_\lambda$ be closed. Since $((p*l)*p)*(0*l) = 0$, it follows from Proposition 2.3 that

$$\tilde{f}(p*l) \supseteq \tilde{f}(p) \cap \tilde{f}(0*l) \supseteq \tilde{f}(p) \cap \tilde{f}(l) \quad \text{and} \quad \lambda(p*l) \leqslant \bigvee\{\lambda(p), \lambda(0*l)\} \leqslant \bigvee\{\lambda(p), \lambda(l)\}.$$

For the converse, assume that $\tilde{f}_\lambda$ satisfies (2.2). Since $\tilde{f}(0) \supseteq \tilde{f}(p), \lambda(0) \leqslant \lambda(p)$ for all $p \in P$, the following is true

$$\tilde{f}(0*p) \supseteq \tilde{f}(0) \cap \tilde{f}(p) \supseteq \tilde{f}(p) \cap \tilde{f}(p) = \tilde{f}(p) \quad \text{and} \quad \lambda(0*p) \leqslant \bigvee\{\lambda(0), \lambda(p)\} \leqslant \bigvee\{\lambda(p), \lambda(p)\} = \lambda(p).$$

Hence, $\tilde{f}_\lambda$ is a hybrid closed ideal of P. □

Theorem 2.16. *Let* P *be a BCK/BCI-algebra. If* $\tilde{f}_\lambda$ *and* $\tilde{g}_\gamma$ *are hybrid ideals of* P *over* U*, then the hybrid intersection of them is also a hybrid ideal.*

Proof. Let $p, l \in P$, then

$$\begin{aligned}(\tilde{f} \tilde{\cap} \tilde{g})(p) = \tilde{f}(p) \cap \tilde{g}(p) &\supseteq (\tilde{f}(p*l) \cap \tilde{f}(l)) \cap (\tilde{g}(p*l) \cap \tilde{g}(l)) \\ &= (\tilde{f}(p*l) \cap \tilde{g}(p*l)) \cap (\tilde{f}(l) \cap \tilde{g}(l)) \\ &= (\tilde{f} \tilde{\cap} \tilde{g})(p*l) \cap (\tilde{f} \tilde{\cap} \tilde{g})(l)\end{aligned}$$

and

$$\begin{aligned}(\lambda \vee \gamma)(p) = \bigvee\{\lambda(p), \gamma(p)\} &\leqslant \bigvee\left\{\bigvee\{\lambda(p*l), \lambda(l)\}, \bigvee\{\gamma(p*l), \gamma(l)\}\right\} \\ &= \bigvee\left\{\bigvee\{\lambda(p*l), \gamma(p*l)\}, \bigvee\{\lambda(l), \gamma(l)\}\right\} \\ &= \bigvee\{(\lambda \vee \gamma)(p*l), (\lambda \vee \gamma)(l)\}.\end{aligned}$$

Hence, $\tilde{f}_\lambda \Cap \tilde{g}_\gamma$ is a hybrid ideal of P over U. □

Remark 2.17. The hybrid union of two hybrid ideals is not necessarily a hybrid ideal. For example, it can be shown that the hybrid structures $\tilde{f}_\lambda$ and $\tilde{g}_\gamma$ given in [24, Example 4.3] are hybrid ideals of P but the hybrid union of them is not an ideal since

$$(\tilde{f} \tilde{\cup} \tilde{g})(t) = \{p_2, p_4\} \not\supseteq \{p_2, p_3, p_4\} = (\tilde{f} \tilde{\cup} \tilde{g})(e) \cap (\tilde{f} \tilde{\cup} \tilde{g})(h)$$

and/or

$$(\lambda \wedge \gamma)(t) = 0.6 \nleqslant 0.5 = \bigvee\{(\lambda \wedge \gamma)(e), (\lambda \wedge \gamma)(h)\}.$$

For any hybrid structure $\tilde{f}_\lambda$ in P over U, let $\tilde{f}^*_\lambda := (\tilde{f}^*, \lambda^*)$ be a hybrid structure in P over U defined by

$$\tilde{f}^* : P \to \mathfrak{P}(U), p \mapsto \begin{cases} \tilde{f}(p), & \text{if } p \in \tilde{f}_\lambda(\alpha), \\ \beta, & \text{otherwise,} \end{cases} \quad \text{and} \quad \lambda^* : P \to I, p \mapsto \begin{cases} \lambda(p), & \text{if } p \in \tilde{f}_\lambda(t), \\ s, & \text{otherwise,} \end{cases}$$

where $\alpha, \beta \in \mathfrak{P}(U)$ and $s, t \in I$ with $\beta \subsetneq \tilde{f}(p)$ and $s > \lambda(p)$.

Theorem 2.18. *Let* P *be a BCK/BCI-algebra. Then* $\tilde{f}^*_\lambda$ *is a hybrid ideal of* P *over* U*, if* $\tilde{f}_\lambda$ *is.*

Proof. Let $\tilde{f}_\lambda$ be a hybrid ideal of a BCK/BCI-algebra P. Then $\tilde{f}_\lambda(\alpha)$ and $\tilde{f}_\lambda(t)$ are ideals of P for all $\alpha \in \mathfrak{P}(U)$ and $t \in I$ provided that they are nonempty by Theorem 2.10. Therefore, if $p \in \tilde{f}_\lambda(\alpha)$ then $\tilde{f}^*(0) = \tilde{f}(0) \supseteq \tilde{f}(p) = \tilde{f}^*(p)$ and $\lambda^*(0) = \lambda(0) \leqslant \lambda(p) = \lambda^*(p)$. If $p \notin \tilde{f}_\lambda(\alpha)$ then $\tilde{f}^*(p) = \beta$ and $\lambda^*(p) = s$. As $\beta \subsetneq \tilde{f}(p)$ and $s > \lambda(p)$, it follows that $\tilde{f}^*(0) = \tilde{f}(0) \supseteq \tilde{f}(p) \supset \beta = \tilde{f}^*(p)$ and $\lambda^*(0) = \lambda(0) \leqslant \lambda(p) < s = \lambda^*(p)$. That is, $\tilde{f}^*_\lambda$ satisfies (H_1) in both cases.

Also, for $p, l \in P$, if $p * l, l \in \tilde{f}_\lambda(\alpha)$, then $p \in \tilde{f}_\lambda(\alpha)$. Thus,

$$\tilde{f}^*(p) = \tilde{f}(p) \supseteq \tilde{f}(p * l) \cap \tilde{f}(l) = \tilde{f}^*(p * l) \cap \tilde{f}^*(l).$$

If $p * l \notin \tilde{f}_\lambda(\alpha)$ or $l \notin \tilde{f}_\lambda(\alpha)$, then $\tilde{f}^*(p * l) = \beta$ or $\tilde{f}^*(l) = \beta$. Hence,

$$\tilde{f}^*(p) \supseteq \beta = \tilde{f}^*(p * l) \cap \tilde{f}^*(l).$$

Now, if $p * l, l \in \tilde{f}_\lambda(t)$, then $p \in \tilde{f}_\lambda(t)$. Thus,

$$\lambda^*(p) = \lambda(p) \leqslant \bigvee\{\lambda(p * l), \lambda(l)\} = \bigvee\{\lambda^*(p * l), \lambda^*(l)\}.$$

If $p * l \notin \tilde{f}_\lambda(t)$ or $l \notin \tilde{f}_\lambda(t)$, then $\lambda^*(p * l) = s$ or $\lambda^*(l) = s$. Hence,

$$\lambda^*(p) \leqslant s = \bigvee\{\lambda^*(p * l), \lambda^*(l)\}.$$

Therefore, $\tilde{f}^*_\lambda$ is a hybrid ideal of P over U. □

3. Conclusion

The main goal of the present paper is to apply the hybrid structure to the ideal theory in BCK/BCI-algebras. In fact, the ideal theory in BCK/BCI-algebras developed by introducing the concepts of hybrid ideals and hybrid closed ideals. Further, the relations among hybrid closed ideals, hybrid sub-algebras and hybrid ideals are studied.

In our furure study, we intend to apply the notions of the present paper to different algebras such as BL-algebras, MTL-algebras, R0-algebras, MV-algebras, EQ-algebras, and lattice implication algebras, etc.

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