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Adaptive synchronization and anti-synchronization of fractional order chaotic optical systems with uncertain parameters

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Abstract

This paper proposes an adaptive control algorithm to study the synchronization and anti-synchronization of fractional order chaotic optical systems. The Lyapunov stability theory verifies the convergence behavior and guarantees the robust asymptotic stability of the equilibrium point at the origin. In the sense of Lyapunov function, this paper also provides parameters adaptation laws that confirm the convergence of uncertain parameters to some constant values. The computer simulation results endorse the theoretical findings. The results of this study could be beneficial in the area of optics chaotic systems.

Keywords: Optics, synchronization, anti-synchronization, Lyapunov stability theory, fractional order.

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1. Introduction

The essence of studying of the fractional order chaotic systems is to understand their structure and behavior. These systems are deemed important as they reflect and fabricate the act of nature. One way to get a perspective glimpse of its complex dynamics is through synchronization or anti-synchronization, but synchronization and anti-synchronization of the fractional order chaotic systems is always almost an impossible task. This fact is due to the system's unpredictable behavior and sensitivity towards initial conditions. Much assumption has to be made artificially or unnecessarily to make practical engineering problems [3, 13, 20, 21, 25]. There is a need to synchronize and anti-synchronize the fractional order chaotic systems for the many applications found for chaotic systems. Different effective synchronization and anti-synchronization methodologies have been proposed to synchronize and anti-synchronize the fractional order chaotic systems. These include adaptive control, sliding mode control, linear active control technique, projective synchronization, and nonlinear active control [2, 4–7, 10, 12, 14, 17, 18, 22, 26]. However, to our best knowledge, the aforementioned methods and many other existing synchronization and anti-synchronization schemes of integer order can be improves to synchronize and anti-synchronize fractional order chaotic systems using some mathematical rigorous tools. Moreover, most of the aforementioned methods are stable only for the chaotic systems whose parameters are probably certain in

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prior. But in a practical engineering situation, some systems parameters are probably uncertain in prior, this effect will broke the synchronization. Therefore, there is a great need to effectively synchronize two fractional order chaotic systems with uncertain parameters. The prominence of the optics systems in the world has received more and more comment in recent years as its play an important role in the use of conventional calculus. Optics is a truly interdisciplinary topic in that specialists in many subjects study it. Many optics systems have been studied by researchers. For example the new fractional-order optical beams introduced by [11] using fractional calculus. In [19, 23, 24] investigate the Fourier transform and many researchers has been already studied its applications. A new fractional variational optical flow model, which combines the fractional derivative with the variational optical flow method, for motion estimation was introduced by [9]. A fractional-order version of the modified hybrid optical system introduced by [1]. The remaining structure of the paper is divided into seven sections. Sections 2 and 3 present and illustrate some concepts related to fractional derivative and adaptive synchronization and anti-synchronization control strategy. In Section 4, a brief description of the fractional order chaotic optical systems is presented. In Sections 5 and 6 a novel adaptive control with update laws of parameters are designed based on Lyapunov stability theory. The conclusion is drawn in Section 7.

2. Some concepts related to fractional derivative

The concept of an integer-order integro-differential operator can be extending by the fractional-order integro-differential operator using a generalisable formulation, that is

$$ {}_aD_t^p = \begin{cases} \frac{d^p}{dt^p}, & p > 0, \\ 1, & p = 0, \\ \int\limits_a^t (d\tau)^{-p}, & p < 0, \end{cases} $$

where p is the fractional order which could be a complex number, and a , t symbolize the limits of the operation. There are many definitions of the fractional integral and derivative which have been used in the recent literature, precisely, the following three definitions (Grünwald-Letnikov, Riemann-Liouville, and Caputo). The current study is dealing with the Riemann-Liouvile definition ([2, 25]), which is given by

$$ {}_aD_t^p f(t) = \frac{d^m}{dt^m} J_t^{m-p} f(t), \quad p > 0, $$

where $m = \lceil p \rceil$, J is the fractional Riemann-Liouville integral and

$$ J_t^{\varphi} \phi(t) = \frac{1}{\Gamma(\varphi)} \int\limits_0^t \frac{\phi(\upsilon)}{(t-\upsilon)^{1-\varphi}}\, d\upsilon, $$

with $0 < \varphi \leqslant 1$ and $\Gamma(.)$ is the gamma function. For $r > n \geqslant 0$, p and q are integers such that $0 \leqslant p-1 \leqslant r < p$, and $0 \leqslant q-1 \leqslant n < q$. Then,

$$ {}_aD_t^r \left({}_aD_t^{-m} f(t) \right) = {}_aD_t^{r-m} f(t). \tag{2.1} $$

For $r, m \geqslant 0$, there exist integers p and q such that $0 \leqslant p-1 \leqslant r < p$, and $0 \leqslant q-1 \leqslant m < q$. Then,

$$ {}_aD_t^r \left({}_aD_t^{m} f(t) \right) = {}_aD_t^{r+m} f(t) - \sum_{j=1}^{m} \left[{}_aD_t^{m-j} f(t) \right]_{t=a} \frac{(t-a)^{-r-j}}{\Gamma(1-r-j)}. \tag{2.2} $$

Suppose also that $f(t)$ has a continuous nth derivative in $[0, t] (n \in N, t > 0)$ and let $r, m > 0$. Then, there exists some $k \in N$ with $k \leqslant n$ and $r, r+m \in [k-1, k]$ such that

$$ {}_aD_t^r \, {}_aD_t^m f(t) = {}_aD_t^{r+m} f(t). $$

3. Adaptive synchronization and anti-synchronization control strategy

Consider a chaotic continuous drive system described by

$$D_t^s x = f(x) + F(x)\alpha, \tag{3.1}$$

where $x \in R^n$ is an n-dimensional state vector of the system (3.1), $\alpha \in R^m$ is the unknown parameter vector of the system, $f(x)$ is a continuous vector function, and $F(x)$ is is a matrix function. On the other hand, the controlled response system is assumed by

$$D_t^s y = g(y) + G(y)\beta + u, \tag{3.2}$$

where $y \in R^n$ is the state vector, $\beta \in R^p$ is the unknown parameter vector of the system, $f(y)$ is a continuous vector function, $F(y)$ is a matrix function, $u \in R^p$ is a controller. Let $e(t) = y(t) - x(t)$ be the error vector. Our aim is to find a suitable control function u which can able to achieve the synchronization such that,

$$\lim_{t\to\infty} \|e\| = \lim_{t\to\infty} \|y(t, y_0) - x(t, x_0)\| = 0.$$

3.1. *Adaptive synchronization*

Theorem 3.1. *If the adaptive control function u is defined as*

$$u = f(x) + F(x)\alpha - g(y) - G(y)\beta + D_t^{s-1}\left[- F(x)(\alpha - \tilde{\alpha}) + G(y)(\beta - \tilde{\beta}) - \left(D_t^{s-1} e\,(t)\right)\frac{(t)^{-(s-1)-1}}{\Gamma\,(-\,(s-1))} - e\right] \tag{3.3}$$

and the uncertain parameters update rule are taken as

$$\dot{\hat{\alpha}} = -[F(x)]^T e, \quad \dot{\hat{\beta}} = [G(y)]^T e, \tag{3.4}$$

where $\hat{\alpha} = \alpha - \tilde{\alpha}$, $\hat{\beta} = \beta - \tilde{\beta}$, $q \in [0, 1]$ is the order of the derivative, and $\tilde{\alpha}, \tilde{\beta}$ are the estimated parameters of α and β, respectively.

Proof. From (3.2) and (3.1) we get the error dynamical system as follows:

$$D_t^s e\,(t) = g\,(y) + G(y)\beta - f(x) - F(x)\alpha + u. \tag{3.5}$$

Inserting (3.3) into (3.5) yields the following:

$$D_t^s e(t) = D_t^{s-1}\left[- F(x)(\alpha - \tilde{\alpha}) + G(y)(\beta - \tilde{\beta}) - \left(D_t^{s-1} e(t)\right)\frac{(t)^{-(s-1)-1}}{\Gamma(-(s-1))} - e\right].$$

If a Lyapunov function candidate is chosen as

$$V = \frac{1}{2}\left[e^T e + (\alpha - \tilde{\alpha})^T(\alpha - \tilde{\alpha}) + (\beta - \tilde{\beta})^T(\beta - \tilde{\beta})\right], \tag{3.6}$$

differentiating (3.6) using (2.2) we get

$$\dot{V} = \left[D_t^{s-1}\left(D_t^s e(t)\right) + \left(D_t^{s-1} e(t)\right)\frac{(t)^{-(s-1)-1}}{\Gamma\,(-\,(s-1))}\right] + (\alpha - \tilde{\alpha})^T\,\dot{\hat{\alpha}} + (\beta - \tilde{\beta})^T \dot{\hat{\beta}}. \tag{3.7}$$

From (3.4) and (3.7), we get

$$\begin{aligned}\dot{V} = \Bigg[&D_t^{s-1}\Bigg(D_t^{s-1}\Big[- F(x)(\tilde{\alpha} - \alpha) + G(y)(\tilde{\beta} - \beta) - \left(D_t^{s-1} e(t)\right)\frac{(t)^{-(s-1)-1}}{\Gamma(-(s-1))} - e\Big]\Bigg)\\ &+ \left(D_t^{s-1} e(t)\right)\frac{(t)^{-(s-1)-1}}{\Gamma(-(s-1))}\Bigg]^T + (\alpha - \tilde{\alpha})^T\dot{\hat{\alpha}} + (\beta - \tilde{\beta})^T\dot{\hat{\beta}},\end{aligned} \tag{3.8}$$

since $\forall s \in [0,1], (1-s) > 0$ and $(s-1) < 0$. Now, using (2.1) and (3.4), (3.8) reduces to

$$\begin{aligned}\dot{V} &= \Bigg[-\left(F(x)(\alpha-\tilde{\alpha}) + G(y)(\beta-\tilde{\beta}) - \left(D_t^{s-1}e(t)\right)\frac{(t)^{-(s-1)-1}}{\Gamma(-(s-1))} - e\right) \\ &\quad + \left(D_t^{s-1}e(t)\right)\frac{(t)^{-(s-1)-1}}{\Gamma(-(s-1))}\Bigg]^T e - (\alpha-\tilde{\alpha})^T[F(x)]^T e - (\beta-\tilde{\beta})^T[G(y)]^T e \\ &= \Big[-(\alpha-\tilde{\alpha})^T F(x)^T + (\beta-\tilde{\beta})^T G(y)^T - e^T\Big] e + (\alpha-\tilde{\alpha})^T \Big[F(x)\Big]^T e - (\beta-\tilde{\beta})^T\Big[G(y)\Big]^T e \\ &= -e^T e \leqslant 0.\end{aligned}$$

Since V is positive definite and $\dot{V}$ is negative definite in the neighborhood of zero solution of system. Therefore, response system (3.2) can synchronize the drive system (3.1) asymptotically. This completes the proof. □

3.2. Adaptive anti-synchronization

Theorem 3.2. *If the nonlinear control function* u *is selected as*

$$\begin{aligned} u = -f(x) - F(x)\alpha - g(y) - G(y)\beta + D_t^{s-1}\Bigg[& F(x)(\alpha-\tilde{\alpha}) + G(y)(\beta-\tilde{\beta}) \\ & - \left(D_t^{s-1}e\,(t)\right)\frac{(t)^{-(s-1)-1}}{\Gamma\,(-\,(s-1))} - e\Bigg] \end{aligned} \tag{3.9}$$

and the adaptive laws of the uncertain are taken as

$$\dot{\hat{\alpha}} = [F(x)]^T e, \quad \dot{\hat{\beta}} = [G(y)]^T e, \tag{3.10}$$

where $\hat{\alpha} = \alpha - \tilde{\alpha}$, $\hat{\beta} = \beta - \tilde{\beta}$, $s \in [0,1]$ *is the order of the derivative, and* $\tilde{\alpha}, \tilde{\beta}$ *are the estimated parameters of* α *and* β, *respectively.*

Proof. From (3.2) and (3.1) we get the error dynamical system as follows:

$$D_t^s e\,(t) = g\,(y) + G(y)\beta + f(x) + F(x)\alpha + u. \tag{3.11}$$

Inserting (3.9) into (3.11) yields the following:

$$D_t^s e(t) = D_t^{s-1}\left[F(x)(\alpha-\tilde{\alpha}) + G(y)(\beta-\tilde{\beta}) - \left(D_t^{s-1}e(t)\right)\frac{(t)^{-(s-1)-1}}{\Gamma(-(s-1))} - e\right].$$

If a Lyapunov function candidate is chosen as

$$V = \frac{1}{2}\left[e^T e + (\alpha-\tilde{\alpha})^T(\alpha-\tilde{\alpha}) + (\beta-\tilde{\beta})^T(\beta-\tilde{\beta})\right], \tag{3.12}$$

differentiating (3.12) using (2.2) we get

$$\dot{V} = \left[D_t^{s-1}\left(D_t^s e(t)\right) + \left(D_t^{s-1}e(t)\right)\frac{(t)^{-(s-1)-1}}{\Gamma\,(-\,(s-1))}\right] + (\alpha-\tilde{\alpha})^T\,\dot{\hat{\alpha}} + (\beta-\tilde{\beta})^T\dot{\hat{\beta}}). \tag{3.13}$$

From (3.10) and (3.13), we get

$$\begin{aligned}\dot{V} = &\left[D_t^{s-1}\left(D_t^{s-1}\left[F(x)(\tilde{\alpha}-\alpha)+G(y)(\tilde{\beta}-\beta)-\left(D_t^{s-1}e(t)\right)\frac{(t)^{-(s-1)-1}}{\Gamma(-(s-1))}-e\right]\right)\right.\\ &\left.+\left(D_t^{s-1}e(t)\right)\frac{(t)^{-(s-1)-1}}{\Gamma(-(s-1))}\right]^T+(\alpha-\tilde{\alpha})^T\dot{\tilde{\alpha}}+(\beta-\tilde{\beta})^T\dot{\tilde{\beta}},\end{aligned} \tag{3.14}$$

since $\forall s \in [0,1], (1-s) > 0$ and $(s-1) < 0$. Now, using (2.1) and (3.4), (3.14) reduces to

$$\begin{aligned}\dot{V} &= \left[(F(x)(\alpha-\tilde{\alpha})+G(y)(\beta-\tilde{\beta})-\left(D_t^{s-1}e(t)\right)\frac{(t)^{-(s-1)-1}}{\Gamma(-(s-1))}-k)+\left(D_t^{s-1}e(t)\right)\right.\\ &\quad\left.\times\frac{(t)^{-(s-1)-1}}{\Gamma(-(s-1))}\right]^T e-(\alpha-\tilde{\alpha})^T[F(x)]^T e-(\beta-\tilde{\beta})^T\left[G(y)\right]^T e\\ &=\left[(\alpha-\tilde{\alpha})^T F(x)^T+(\beta-\tilde{\beta})^T G(y)^T-e^T\right]e-(\alpha-\tilde{\alpha})^T[F(x)]^T e-(\beta-\tilde{\beta})^T[G(y)]^T e\\ &=-e^T e\leqslant 0,\end{aligned}$$

since V is positive definite and $\dot{V}$ is negative definite in the neighborhood of zero solution of system. Therefore, response system (3.2) can synchronize the drive system (3.1) asymptotically. This completes the proof. □

4. Description of the systems

The nonlinear dynamic of the fractional-order single mode laser Lorenz [8, 16] is described by:

$$D_t^{s_1}x = a(y-x),\quad D_t^{s_2}y = (c-z)x-y,\quad D_t^{s_3}z = xy-bz, \tag{4.1}$$

where a, b and c are positive parameters, $s = (s_1, s_2, s_3)$ is the fractional-order, when $s = 0.99$, system (4.1) exhibits chaotic behaviors. The mathematical model of the fractional-order modified hybrid optical system [8] is described by:

$$D_t^{s_1}x = y,\quad D_t^{s_2}y = z,\quad D_t^{s_3}z = -az-y+bx(1-x^2), \tag{4.2}$$

where a and b are positive parameters, $s = (s_1, s_2, s_3)$ is the fractional-order, when $s = 0.99$, system (4.2) exhibits chaotic behaviors.

5. Adaptive synchronization of the fractional-order hybrid optical and single mode laser Lorenz chaotic systems

In order to achieve the adaptive synchronization with unknown parameters of the fractional-order hybrid optical and single mode laser Lorenz chaotic system (4.1) and the fractional-order modified hybrid optical system (4.2), system (4.1) is assumed to be the transmitter (drive) system and system (4.2) is assumed to be the receiver (response). The drive and the response systems are expressed as:

$$D_t^{s_1}x_1 = a_1(y_1-x_1),\quad D_t^{s_2}y_1 = (c_1-z_1)x_1-y_1,\quad D_t^{s_3}z_1 = x_1y_1-b_1z_1, \tag{5.1}$$

and

$$D_t^{s_1}x_2 = y_2+u_1,\quad D_t^{s_2}y = z_2+u_2,\quad D_t^{s_3}z = -a_2z_2-y+b_2x_2(1-x_2^2)+u_3, \tag{5.2}$$

where $U = (u_1, u_2, u_3)^T$ is the adaptive controller function to be determined for the purpose of adaptive synchronizing with unknown parameters in spite of the differences in initial conditions. The error system can be obtained by subtracting system (5.1) from system (5.2),

$$\begin{aligned} D_t^{s_1} e_1 &= y_2 - a_1(y_1 - x_1) + u_1, \\ D_t^{s_2} e_2 &= z_2 - (c_1 - z_1)x_1 + y_1 + u_2, \\ D_t^{s_3} e_3 &= -a_2 z_2 - y + b_2 x_2(1 - x_2^2) - x_1 y_1 + b_1 z_1 + u_3, \end{aligned} \tag{5.3}$$

where $e_1 = x_2 - x_1, e_2 = y_2 - y_1$ and $e_3 = z_2 - z_1$. So our aim is to design an effective adaptive control function U to achieve the adaptive synchronization between the fractional-order modified hybrid optical system (4.1) and the fractional-order modified hybrid optical system (4.2) with fully uncertain parameters, such that the states of response system (4.2) and the states of drive system (4.1) are globally synchronized asymptotically, i.e., $\lim_{t\to\infty} \|e_i(t)\| = 0$, $i = 1, 2, 3$, where $\|.\|$ represents the Euclidean norm.

Theorem 5.1. *If the adaptive control function* $U = (u_1, u_2, u_3)^T$ *is defined as*

$$\begin{aligned} u_1 &= -y_2 + a_1(y_1 - x_1) + D_t^{s_1 - 1}\left[\hat{a}_1(y_1 - x_1)\left(D_t^{s_1-1} e_1(t)\right)\frac{(t)^{-(s_1-1)-1}}{\Gamma(-(s_1-1))} - e_1\right], \\ u_2 &= -z_2 + (c_1 - z_1)x_1 - y_1 + D_t^{s_2-1}\left[\hat{c}_1 x_1 - \left(D_t^{s_2-1} e_2(t)\right)\frac{(t)^{-(s_2-1)-1}}{\Gamma(-(s_2-1))} - e_2\right], \\ u_3 &= a_2 z_2 + y - b_2 x_2(1 - x_2^2) + x_1 y_1 - b_1 z_1 + D_t^{s_3-1}\Bigg[- \hat{a}_2 z_2 + \hat{b}_2 x_2 - \hat{b}_2 x_2^3 - \hat{b}_1 z_1 \\ &\quad - \left(D_t^{s_3-1} e_3(t)\right)\frac{(t)^{-(s_3-1)-1}}{\Gamma(-(s_3-1))} - e_3\Bigg], \end{aligned} \tag{5.4}$$

and the unknown parameters update rule are taken as

$$\dot{\hat{a}}_1 = -(y_1 - x_1)e_1, \quad \dot{\hat{b}}_1 = z_1 e_3, \quad \dot{\hat{c}}_1 = -x_1 e_2, \quad \dot{\hat{a}}_2 = -z_2 e_3, \quad \dot{\hat{b}}_2 = (x_2 - x_2^3)e_3, \tag{5.5}$$

where, $\hat{a}_1, \hat{b}_1, \hat{c}_1, \hat{a}_2, \hat{b}_2$ *are estimates of* a_1, b_1, c_1, a_2, b_2, *respectively.*

Proof. Insert the adaptive control function (5.4) into (5.3), we get the following new error system which described by

$$\begin{aligned} D_t^{s_1} e_1 &= D_t^{s_1-1}\left[- \tilde{a}_1(y_1 - x_1) - \left(D_t^{s_1-1} e_1(t)\right)\frac{(t)^{-(s_1-1)-1}}{\Gamma(-(s_1-1))} - e_1\right], \\ D_t^{s_2} e_2 &= D_t^{s_2-1}\left[- \tilde{c}_1 x_1 - \left(D_t^{s_2-1} e_2(t)\right)\frac{(t)^{-(s_2-1)-1}}{\Gamma(-(s_2-1))} - e_2\right], \\ D_t^{s_3} e_3 &= D_t^{s_3-1}\left[- \tilde{a}_2 z_2 + \tilde{b}_2 x_2 - \tilde{b}_2 x_2^3 - \tilde{b}_1 z_1 - \left(D_t^{s_3-1} e_3(t)\right)\frac{(t)^{-(s_3-1)-1}}{\Gamma(-(s_3-1))} - e_3\right], \end{aligned} \tag{5.6}$$

where $\tilde{a}_1 = a_1 - \hat{a}_1, \tilde{b}_1 = b_1 - \hat{b}_1, \tilde{c}_1 = c_1 - \hat{c}_1, \tilde{a}_2 = a_2 - \hat{a}_2, \tilde{b}_2 = b_2 - \hat{b}_2$. Consider the following Lyapunov function candidate as:

$$V = \frac{1}{2}\left(e^T e + \tilde{a}_1^2 + \tilde{b}_1^2 + \tilde{c}_1^2 + \tilde{a}_2^2 + \tilde{b}_2^2\right), \tag{5.7}$$

Differentiating (5.7) with the time using (2.2) we get

$$
\begin{aligned}
\dot{V} = &\Big(e^T\dot{e} + \tilde{a}_1\dot{\tilde{a}}_1 + \tilde{b}_1\dot{\tilde{b}}_1 + \tilde{c}_1\dot{\tilde{c}}_1 + \tilde{a}_2\dot{\tilde{a}}_2 + \tilde{b}_2\dot{\tilde{b}}_2\Big),\\
= &e_1\left[D_t^{1-s_1}\Big(D_t^{s_1}e_1(t)\Big) + \Big(D_t^{s_1}e_1(t)\Big)\frac{(t)^{-(s_1-1)-1}}{\Gamma(-(s_1-1))}\right] + e_2\left[D_t^{1-s_2}\Big(D_t^{s_2}e_2(t)\Big)\right.\\
&\left. + \Big(D_t^{s_2}e_2(t)\Big)\frac{(t)^{-(s_2-1)-1}}{\Gamma(-(s_2-1))}\right] + e_3\left[D_t^{1-s_3}\Big(D_t^{s_3}e_3(t)\Big) + \Big(D_t^{s_3}e_3(t)\Big)\frac{(t)^{-(s_3-1)-1}}{\Gamma(-(s_3-1))}\right]\\
&+ \tilde{a}_1\dot{\tilde{a}}_1 + \tilde{b}_1\dot{\tilde{b}}_1 + \tilde{c}_1\dot{\tilde{c}}_1 + \tilde{a}_2\dot{\tilde{a}}_2 + \tilde{b}_2\dot{\tilde{b}}_2,\\
= &e_1\left[D_t^{1-s_1}\Big(D_t^{s_1-1}\Big[-\tilde{a}_1(y_1-x_1) - \Big(D_t^{s_1-1}e_1(t)\Big)\frac{(t)^{-(s_1-1)-1}}{\Gamma(-(s_1-1))} - e_1\Big]\Big) + \Big(D_t^{s_1}e_1(t)\Big)\right.\\
&\left.\times\frac{(t)^{-(s_1-1)-1}}{\Gamma(-(s_1-1))}\right] + e_2\left[D_t^{1-s_2}\Big(D_t^{s_2-1}\Big[-\tilde{c}_1x_1 - \Big(D_t^{s_2-1}e_2(t)\Big)\frac{(t)^{-(s_2-1)-1}}{\Gamma(-(s_2-1))} - e_2\Big]\Big)\right.\\
&\left. + \Big(D_t^{s_2}e_2(t)\Big)\frac{(t)^{-(s_2-1)-1}}{\Gamma(-(s_2-1))}\right] + e_3\left[D_t^{1-s_3}\Big(D_t^{1-s_3}\Big[-\tilde{a}_2z_2 + \tilde{b}_2x_2 - \tilde{b}_2x_2^3 - \tilde{b}_1z_1\right.\\
&\left.-(D_t^{1-s_3}e_3(t))\frac{(t)^{-(s_3-1)-1}}{\Gamma(-(s_3-1))} - e_3\Big]\Big) + \Big(D_t^{s_3}e_3(t)\Big)\frac{(t)^{-(s_3-1)-1}}{\Gamma(-(s_3-1))}\right] + \tilde{a}_1\dot{\tilde{a}}_1 + \tilde{b}_1\dot{\tilde{b}}_1 + \tilde{c}_1\dot{\tilde{c}}_1\\
&+ \tilde{a}_2\dot{\tilde{a}}_2 + \tilde{b}_2\dot{\tilde{b}}_2,
\end{aligned}
\tag{5.8}
$$

since $\forall s \in [0,1], (1-s) > 0$ and $(s-1) < 0$. Now using (2.1), (5.8) reduces to

$$
\begin{aligned}
\dot{V} = &e_1\Big[-\tilde{a}_1(y_1-x_1) - e_1\Big] + e_2\Big[-\tilde{c}_1x_1 - e_2\Big] + e_3\Big[-\tilde{a}_2z_2 + \tilde{b}_2x_2 - \tilde{b}_2x_2^3 - \tilde{b}_1z_1 - e_3\Big]\\
&+ \tilde{a}_1\Big((y_1-x_1)e_1\Big) + \tilde{b}_1\Big(-z_1e_3\Big) + \tilde{c}_1\Big(x_1e_2\Big) + \tilde{a}_2\Big(z_2e_3\Big) + \tilde{b}_2\Big(-(x_2-x_2^3)e_3\Big),
\end{aligned}
$$

then we get the from

$$\dot{V} = -e^Te \leqslant 0.$$

Since V is positive definite and $\dot{V}$ is negative definite in the neighborhood of zero solution of system (5.3), it follows that $e_1, e_2, e_3 \in L_\infty$ and $\hat{a}_1, \hat{b}_1, \hat{c}_1, \hat{a}_2, \hat{b}_2, \in L_\infty$. From (5.6), we have $\dot{e}_1, \dot{e}_2, \dot{e}_3 \in L_\infty$. Since $\dot{V} = -e^Te$ then we obtain

$$\int_0^t \|e\|^2 dt \leqslant \int_0^t e^Te\, dt = \int_0^t -\dot{V}dt = V(0) - V(t) \leqslant V(0).$$

Thus, $\dot{e}_1, \dot{e}_2, \dot{e}_3 \in L_2$ and by Barbalats lemma [15], we have $\lim_{t\to\infty} \|e(t)\| = 0$. Therefore, response system (5.2) can synchronize the drive system (5.1) asymptotically. This completes the proof. □

5.1. Numerical simulations

In the numerical results of the proposed adaptive synchronization method, Adams-Bashforth-Moulton method is used to solve the systems for the fractional order $s_i = 0.99, i = 1,2,3$, and the uncertain parameters are chosen as $a_1 = 10, b_1 = 8/3, c_1 = 28$ and $a_2 = 0.5, b_2 = 0.64$. The initial values of the fractional-order drive systems (4.1)-(4.2) and the estimated parameters are arbitrarily chosen in simulations as $x_1(0) = -15.8, y_1(0) = -17.48, z_1(0) = 35.64, x_2(0) = 1.5, y_2(0) = 0.01, z_2(0) = 0.02$ and $\tilde{a}_1(0) = 1, \tilde{b}_1(0) = 1, \tilde{c}_1(0) = 1, \tilde{a}_2(0) = 1, \tilde{b}_2(0) = 1$, respectively. Adaptive synchronization of the systems (4.1)-(4.2) via adaptive control law (5.4) and (5.5) are shown in Figs. (1)-(3). Fig. (2) (a)-(c) displays the adaptive synchronization of the fractional order chaotic (4.1)-(4.2). Fig. (2) (a)-(b) displays the time response of estimated values of parameters $\tilde{a}_1, \tilde{b}_1, \tilde{c}_1, \tilde{a}_2, \tilde{b}_2$ of drive and response system. Fig. (2) (c) displays the adaptive synchronization errors, e_1, e_2, e_3 with time t. Fig. (3) (a)-(c) displays the steady-state plane trajectories of drive and response system.

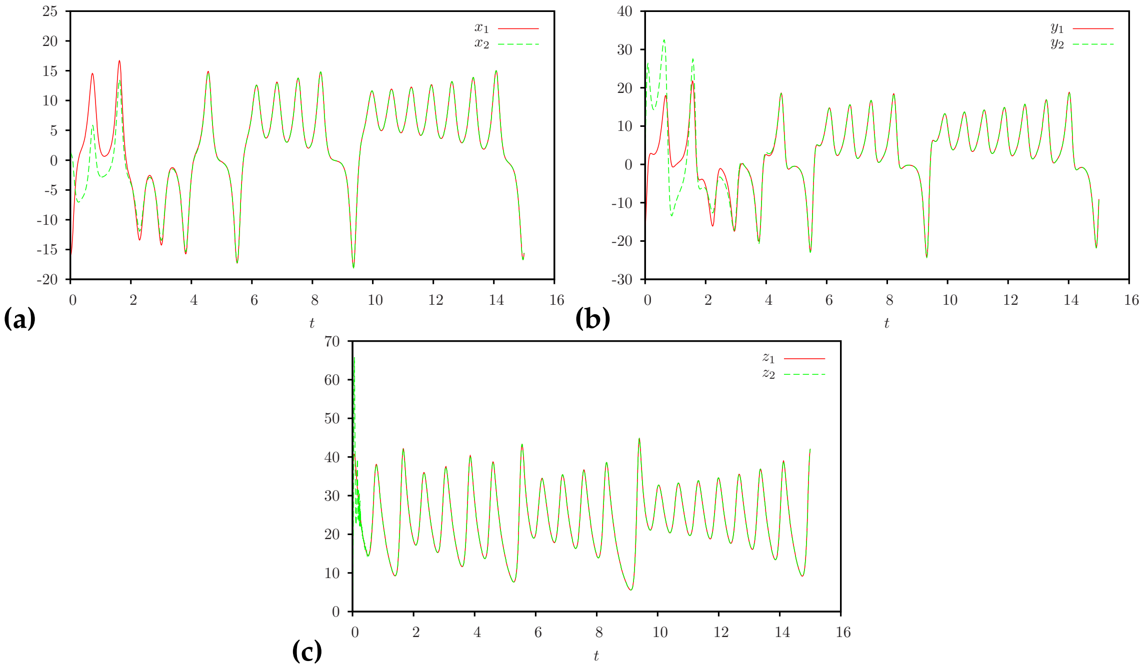

Figure 1: The adaptive synchronization of the fractional order chaotic (4.1) and (4.2): **(a):** Signals x_2 and x_1; **(b):** signals y_2 and y_1; **(c):** signals z_2 and z_1.

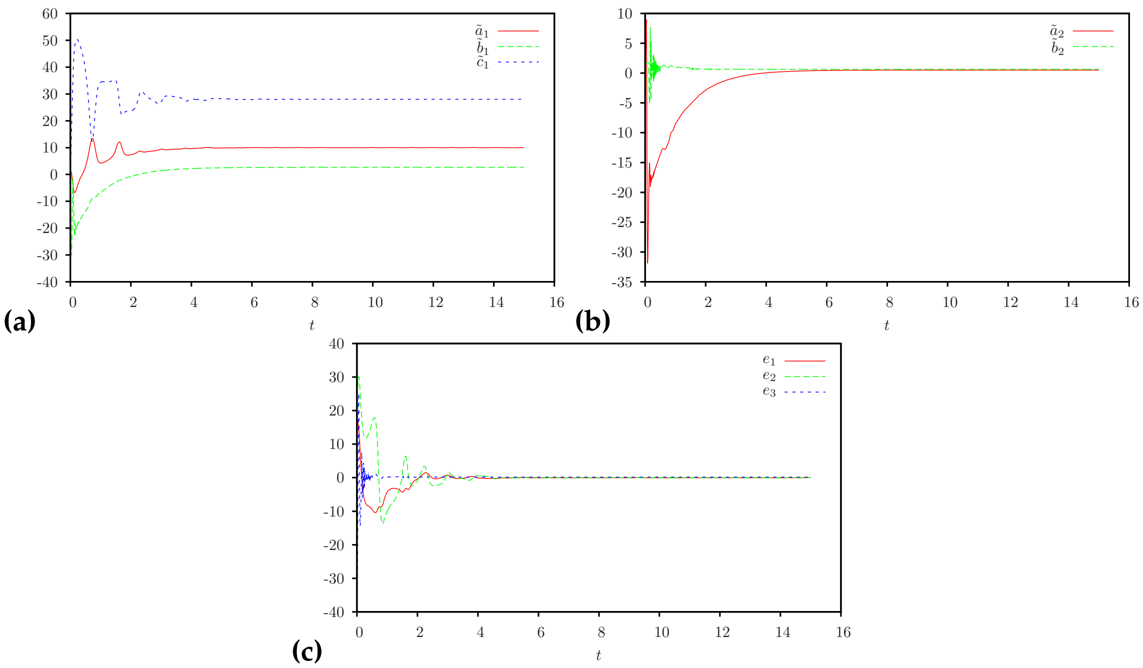

Figure 2: **(a)-(b):** The time response of estimated values of parameters $\tilde{a}_1, \tilde{b}_1, \tilde{c}_1, \tilde{a}_2, \tilde{b}_2$ of drive systems (4.1) and (4.2); **(c):** Adaptive synchronization errors, e_1, e_2, e_3 with time t .

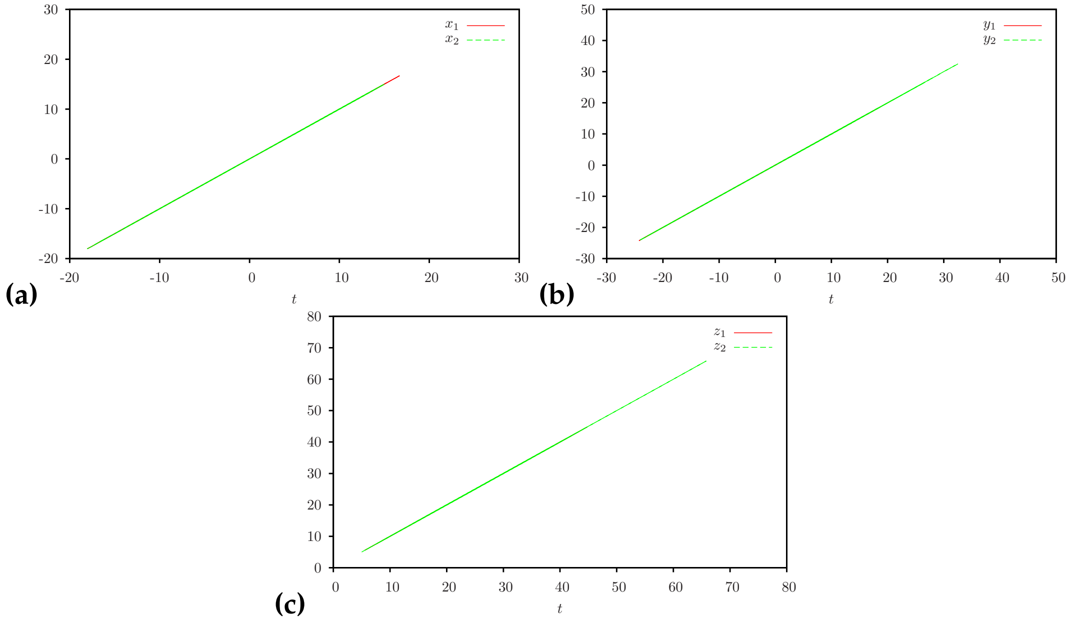

Figure 3: The steady-state plane trajectories of systems (4.1) and (4.1), **(a):** signals x_1 and x_2; **(b):** signals y_1 and y_2; **(c):** signals z_1 and z_2;

6. Adaptive anti-synchronization of the fractional-order hybrid optical and single mode laser Lorenz chaotic systems

In order to achieve the adaptive anti-synchronization with uncertain parameters of the fractional-order hybrid optical and single mode laser Lorenz chaotic system (4.1) and the fractional-order modified hybrid optical system (4.2), system (4.1) is assumed to be the transmitter (drive) system and system (4.2) is assumed to be the receiver (response). In this case the error system can be obtained by adding system (5.1) to system (5.2),

$$\begin{aligned}
D_t^{s_1} e_1 &= y_2 + a_1(y_1 - x_1) + u_1,\\
D_t^{s_2} e_2 &= z_2 + (c_1 - z_1)x_1 - y_1 + u_2,\\
D_t^{s_3} e_3 &= -a_2 z_2 - y + b_2 x_2(1 - x_2^2) + x_1 y_1 - b_1 z_1 + u_3,
\end{aligned} \tag{6.1}$$

where $e_1 = x_2 + x_1, e_2 = y_2 + y_1$ and $e_3 = z_2 + z_1$. So our aim is to design an effective adaptive control function U to achieve the adaptive anti-synchronization between the fractional-order modified hybrid optical system (4.1) and the fractional-order modified hybrid optical system (4.2) system with fully uncertain parameters, such that the states of response system (4.2) and the states of drive system (4.1) are globally, anti-synchronized asymptotically i.e., $\lim\limits_{t\to\infty} \|e_i(t)\| = 0$, $i = 1, 2, 3$, where $\|.\|$ represents the Euclidean norm.

Theorem 6.1. *If the adaptive control function* $U = (u_1, u_2, u_3)^T$ *is defined as*

$$\begin{aligned}
u_1 &= -y_2 - a_1(y_1 - x_1) + D_t^{s_1-1}\left[-\hat{a}_1(y_1 - x_1) - \left(D_t^{s_1-1} e_1(t)\right)\frac{(t)^{-(s_1-1)-1}}{\Gamma(-(s_1-1))} - e_1 \right],\\
u_2 &= -z_2 - (c_1 - z_1)x_1 + y_1 + D_t^{s_2-1}\left[\hat{c}_1 x_1 - \left(D_t^{s_2-1} e_2(t)\right)\frac{(t)^{-(s_2-1)-1}}{\Gamma(-(s_2-1))} - e_2 \right],\\
u_3 &= a_2 z_2 + y - b_2 x_2(1 - x_2^2) - x_1 y_1 + b_1 z_1 + D_t^{s_3-1}\Bigg[-\hat{a}_2 z_2 + \hat{b}_2 x_2 - \hat{b}_2 x_2^3 - \hat{b}_1 z_1\\
&\qquad - \left(D_t^{s_3-1} e_3(t)\right)\frac{(t)^{-(s_3-1)-1}}{\Gamma(-(s_3-1))} - e_3 \Bigg],
\end{aligned} \tag{6.2}$$

and the uncertain parameters update rule are taken as

$$\dot{\hat{a}}_1 = (y_1 - x_1)e_1, \quad \dot{\hat{b}}_1 = -z_1 e_3, \quad \dot{\hat{c}}_1 = x_1 e_2, \quad \dot{\hat{a}}_2 = -z_2 e_3, \quad \dot{\hat{b}}_2 = (x_2 - x_2^3)e_3,$$

where, $\hat{a}_1, \hat{b}_1, \hat{c}_1, \hat{a}_2, \hat{b}_2$ *are estimates of* a_1, b_1, c_1, a_2, b_2*, respectively.*

Proof. Inserting the adaptive control function (6.2) into (6.1), we get the following new error system which is described by

$$D_t^{s_1} e_1 = D_t^{s_1 - 1}\left[\tilde{a}_1(y_1 - x_1) - \left(D_t^{s_1-1} e_1(t)\right)\frac{(t)^{-(s_1-1)-1}}{\Gamma(-(s_1-1))} - e_1\right],$$
$$D_t^{s_2} e_2 = D_t^{s_2 - 1}\left[\tilde{c}_1 x_1 - \left(D_t^{s_2-1} e_2(t)\right)\frac{(t)^{-(s_2-1)-1}}{\Gamma(-(s_2-1))} - e_2\right],$$
$$D_t^{s_3} e_3 = D_t^{s_3 - 1}\left[-\tilde{a}_2 z_2 + \tilde{b}_2 x_2 - \tilde{b}_2 x_2^3 - \tilde{b}_1 z_1 - \left(D_t^{s_3-1} e_3(t)\right)\frac{(t)^{-(s_3-1)-1}}{\Gamma(-(s_3-1))} - e_3\right],$$

where $\tilde{a}_1 = a_1 - \hat{a}_1, \tilde{b}_1 = b_1 - \hat{b}_1, \tilde{c}_1 = c_1 - \hat{c}_1, \tilde{a}_2 = a_2 - \hat{a}_2, \tilde{b}_2 = b_2 - \hat{b}_2$. Consider the following Lyapunov function candidate as:

$$V = \frac{1}{2}\left(e^T e + \tilde{a}_1^2 + \tilde{b}_1^2 + \tilde{c}_1^2 + \tilde{a}_2^2 + \tilde{b}_2^2\right). \tag{6.3}$$

Differentiating (6.3) with the time using (2.2) we get

$$\begin{aligned}
\dot{V} &= \left(e^T \dot{e} + \tilde{a}_1\dot{\tilde{a}}_1 + \tilde{b}_1\dot{\tilde{b}}_1 + \tilde{c}_1\dot{\tilde{c}}_1 + \tilde{a}_2\dot{\tilde{a}}_2 + \tilde{b}_2\dot{\tilde{b}}_2\right),\\
&= e_1\left[D_t^{1-s_1}(D_t^{s_1} e_1(t)) + (D_t^{s_1} e_1(t))\frac{(t)^{-(s_1-1)-1}}{\Gamma(-(s_1-1))}\right] + e_2\left[D_t^{1-s_2}\left(D_t^{s_2} e_2(t)\right) + \left(D_t^{s_2} e_2(t)\right)\right.\\
&\quad \left.\times\frac{(t)^{-(s_2-1)-1}}{\Gamma(-(s_2-1))}\right] + e_3\left[D_t^{1-s_3}\left(D_t^{s_3} e_3(t)\right) + \left(D_t^{s_3} e_3(t)\right)\frac{(t)^{-(s_3-1)-1}}{\Gamma(-(s_3-1))}\right]\\
&\quad + \tilde{a}_1\dot{\tilde{a}}_1 + \tilde{b}_1\dot{\tilde{b}}_1 + \tilde{c}_1\dot{\tilde{c}}_1 + \tilde{a}_2\dot{\tilde{a}}_2 + \tilde{b}_2\dot{\tilde{b}}_2,\\
&= e_1\left[D_t^{1-s_1}\left(D_t^{s_1-1}\left[\tilde{a}_1(y_1-x_1) - \left(D_t^{s_1-1}e_1(t)\right)\frac{(t)^{-(s_1-1)-1}}{\Gamma(-(s_1-1))} - e_1\right]\right) + \left(D_t^{s_1}e_1(t)\right)\frac{(t)^{-(s_1-1)-1}}{\Gamma(-(s_1-1))}\right]\\
&\quad + e_2\left[D_t^{1-s_2}\left(D_t^{s_2-1}\left[\tilde{c}_1 x_1 - \left(D_t^{s_2-1}e_2(t)\right)\frac{(t)^{-(s_2-1)-1}}{\Gamma(-(s_2-1))} - e_2\right]\right) + \left(D_t^{s_2}e_2(t)\right)\frac{(t)^{-(s_2-1)-1}}{\Gamma(-(s_2-1))}\right]\\
&\quad + e_3\left[D_t^{1-s_3}\left(D_t^{1-s_3}\left[-\tilde{a}_2 z_2 + \tilde{b}_2 x_2 - \tilde{b}_2 x_2^3 - \tilde{b}_1 z_1 - \left(D_t^{1-s_3}e_3(t)\right)\frac{(t)^{-(s_3-1)-1}}{\Gamma(-(s_3-1))} - e_3\right]\right)\right.\\
&\quad \left. + \left(D_t^{s_3}e_3(t)\right)\frac{(t)^{-(s_3-1)-1}}{\Gamma(-(s_3-1))}\right] + \tilde{a}_1\dot{\tilde{a}}_1 + \tilde{b}_1\dot{\tilde{b}}_1 + \tilde{c}_1\dot{\tilde{c}}_1 + \tilde{a}_2\dot{\tilde{a}}_2 + \tilde{b}_2\dot{\tilde{b}}_2,
\end{aligned} \tag{6.4}$$

since $\forall s \in [0,1], (1-s) > 0$ and $(s-1) < 0$. Now using (2.1), (6.4) reduces to

$$\begin{aligned}
\dot{V} &= e_1\left[\tilde{a}_1(y_1-x_1) - e_1\right] + e_2\left[\tilde{c}_1 x_1 - e_2\right] + e_3\left[-\tilde{a}_2 z_2 + \tilde{b}_2 x_2 - \tilde{b}_2 x_2^3 - \tilde{b}_1 z_1 - e_3\right]\\
&\quad + \tilde{a}_1\left(-(y_1-x_1)e_1\right) + \tilde{b}_1\left(z_1 e_3\right) + \tilde{c}_1\left(-x_1 e_2\right) + \tilde{a}_2\left(z_2 e_3\right) + \tilde{b}_2\left(-(x_2-x_2^3)e_3\right),
\end{aligned}$$

then we get the from

$$\dot{V} = -e^T e \leqslant 0.$$

Since V is positive definite and $\dot{V}$ is negative definite in the neighborhood of zero solution of system (5.3), it follows that $e_1, e_2, e_3 \in L_\infty$ and $\hat{a}_1, \hat{b}_1, \hat{c}_1, \hat{a}_2, \hat{b}_2, \in L_\infty$. From (5.6), we have $\dot{e}_1, \dot{e}_2, \dot{e}_3 \in L_\infty$. Since $\dot{V} = -e^T e$, then we obtain

$$\int_0^t \|e\|^2 dt \leqslant \int_0^t e^T e \; dt = \int_0^t -\dot{V} dt = V(0) - V(t) \leqslant V(0).$$

Thus, $\dot{e}_1, \dot{e}_2, \dot{e}_3 \in L_2$ and by Barbalats lemma [15], we have $\lim_{t\to\infty} \|e(t)\| = 0$. Therefore, response system (5.2) can anti-synchronize the drive system (5.1) asymptotically. This completes the proof. □

6.1. Numerical simulations

In the numerical results of the proposed adaptive synchronization method, Adams-Bashforth-Moulton method is used to solve the systems for the fractional order $s_i = 0.99, i = 1, 2, 3$, and the uncertain parameters are chosen as $a_1 = 10, b_1 = 8/3, c_1 = 28$ and $a_2 = 0.5_2, b_2 = 0.64$. The initial values of the fractional-order drive systems (4.1)-(4.2) and the estimated parameters are arbitrarily chosen in simulations as $x_1(0) = -15.8, y_1(0) = -17.48, z_1(0) = 35.64, x_2(0) = 1.5, y_2(0) = 0.01, z_2(0) = 0.02$ and $\tilde{a}_1(0) = 1, \tilde{b}_1(0) = 1, \tilde{c}_1(0) = 1, \tilde{a}_2(0) = 1, \tilde{b}_2(0) = 1$, respectively. Adaptive anti-synchronization of the systems (4.1)-(4.2) via adaptive control law (5.4) and (5.5) are shown in Figs. (4)-(6). Fig. (4) (a)-(c) displays the adaptive anti-synchronization of the fractional order chaotic (4.1)-(4.2). Fig. (5) (a)-(b) displays the time response of estimated values of parameters $\tilde{a}_1, \tilde{b}_1, \tilde{c}_1, \tilde{a}_2, \tilde{b}_2$ of drive and response system. Fig. (5) (c) displays the adaptive synchronization errors, e_1, e_2, e_3 with time t. Fig. (6) (a)-(c) displays the steady-state plane trajectories of drive and response system.

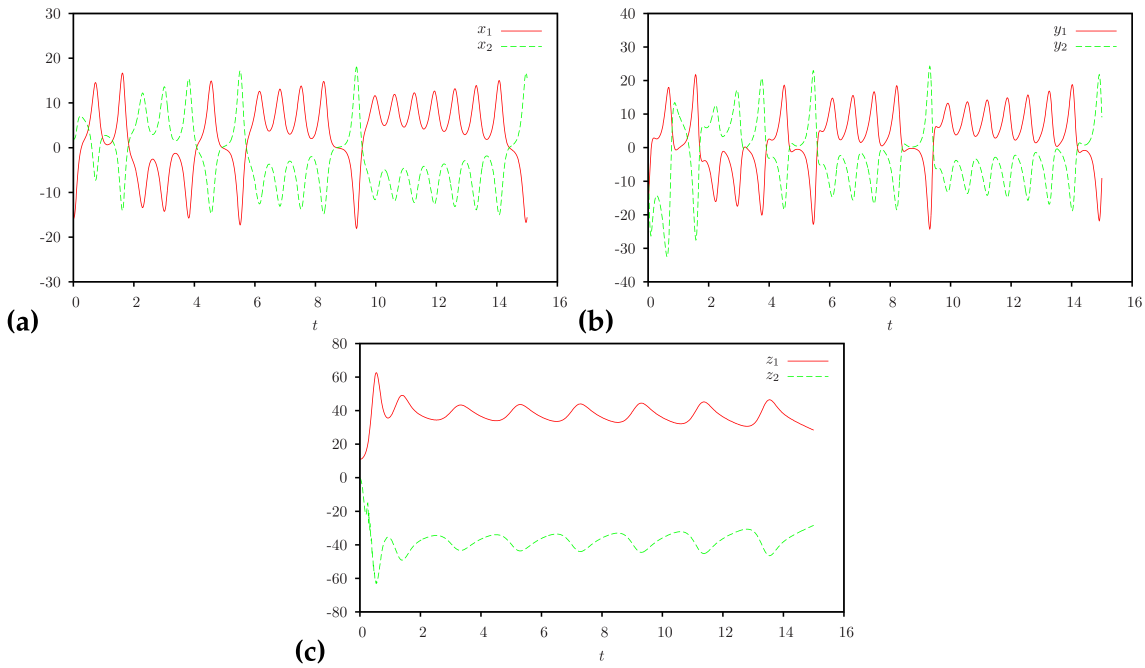

Figure 4: The adaptive anti-synchronization of the fractional order chaotic (4.1) and (4.2): **(a):** Signals x_2 and x_1; **(b):** signals y_2 and y_1; **(c):** signals z_2 and z_1.

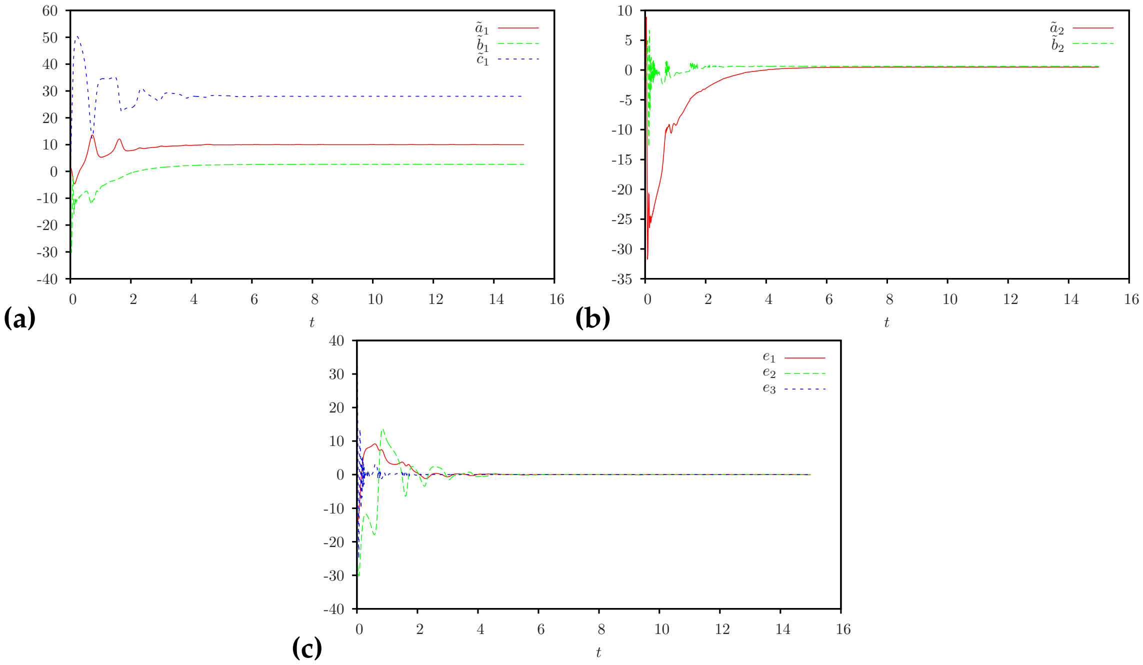

Figure 5: **(a)-(b):** The time response of estimated values of parameters $\tilde{a}_1, \tilde{b}_1, \tilde{c}_1, \tilde{a}_2, \tilde{b}_2$ of drive systems (4.1) and (4.2); **(c):** Adaptive anti-synchronization errors, e_1, e_2, e_3 with time t .

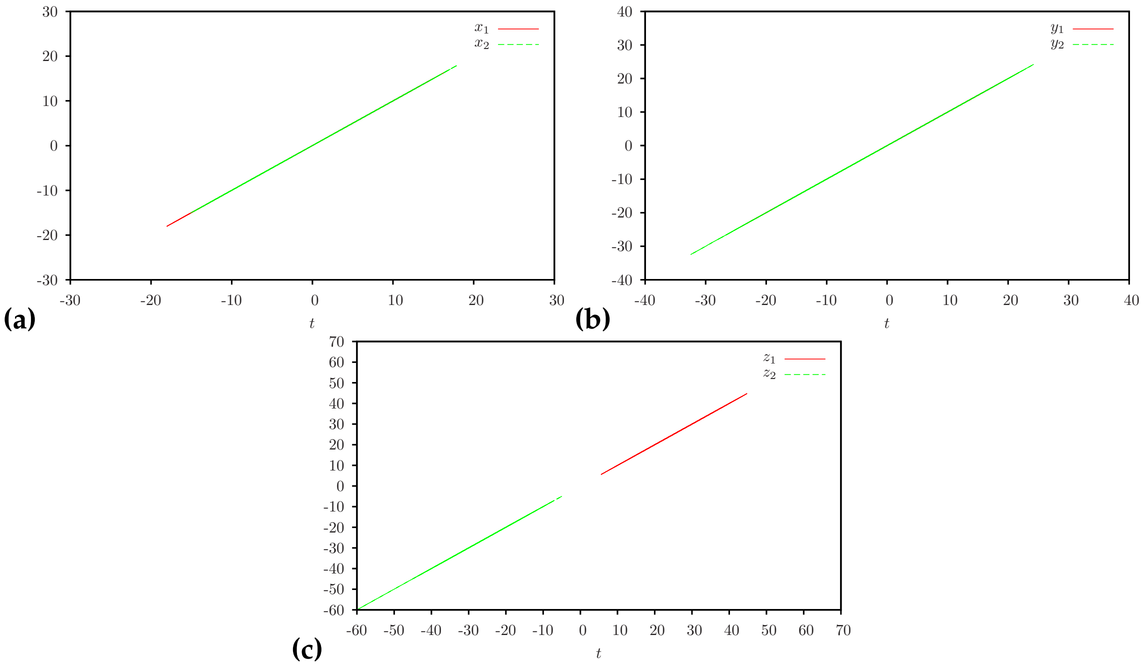

Figure 6: The steady-state plane trajectories of systems (4.1) and (4.1),**(a):** signals x_1 and x_2; **(b):** signals y_1 and y_2; **(c):** signals z_1 and z_2;

7. Conclusion

This paper proposes a robust adaptive control technique and studies the synchronization and anti-synchronization of fractional order chaotic optical systems in the presence of uncertain parameters. The

Lyapunov stability technique proves the asymptotic stability of the closed-loop system. The proposed control scheme is fast convergence of the state trajectories to the origin. Comparative examples are provided to show the performance and efficiency of the proposed control technique. Numerical simulation further validates the theoretical findings. The results of this study could be beneficial in the area of other fractional order chaotic optical systems, both theoretically as well as practically.

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