On some properties of β-Laplace integral transform

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Abstract

This paper aims to discuss and resolve some shortcomings of the classical Laplace integral transform of a particular class of functions such as poles (if exist) lie on fixed points, fixed s-domain, and analyticity region for any function of that particular class of functions. We mainly try to solve these shortcomings by employing a new generalized Laplace transform named the β-Laplace integral transform.

1. Introduction

An integral transform is any transform T of the following form [4]

$$T(f)_{(s)} = \int_{t_1}^{t_2} K(t,s)f(t)dt.$$

Here, $f(t)$ is input, $T(f)_{(s)}$ is output function with a new parameter s independent of t and $K(t,s)$ is called the kernel of integral transform. Mainly integral transform is a function from its original domain into another more suitable domain where it might be solved much more easily than its previous domain. Thereafter, the solution is transformed back to the original domain using the inverse of that integral transform [22].

The Laplace integral transform [18] is an old, famous, and widely used integral transform than other integral transforms. The Laplace transform plays a vital role in solving engineering sciences problems [2, 5, 6, 11, 20, 23].

In recent past, many new integral transforms have been introduced, for instance, Sumudu, Kamal, Mohand, Natural, Polynomial, Sawi, ZZ-transform, Sadik, Tarig, Aboodh, Mahgoub, AF Transform. These

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all-new transforms are commonly using the exponential type kernel, and they are a generalized form of the Laplace transform [1, 3, 7, 8, 12–16, 21, 24, 26].

But, recently Gaur et. al. [10] has introduced a very new form of generalized Laplace integral transform with a new parameter $\beta > 1$ and defined as

$$\mathcal{L}_\beta\{f(t)\}_{(s)} = \int_0^\infty \beta^{-st} f(t) dt, \quad \beta > 1.$$

Every function $f(t), t \geqslant 0$ of exponential order and sectionally continuous on $[0, \infty)$ is called a function of class A. It is a sufficient condition for existence of the Laplace and the β-Laplace integral transform that a function f is of class A.

On the function of class A, the Laplace transform has many disadvantage as poles of a class A function (if exists) lie at a fixed points, constant s-domain $(\mathrm{Re}(s) > \alpha)$, therefore these are drawback of the Laplace transform which restrict to its full applicability in many fields of basic science and engineering such as signal processing, digital communication, electrical engineering, control theory, medical science [9, 17, 19, 25].

This is the motivation of our paper to find the solution of above-mentioned problems. On this class A, The β-Laplace integral transform has many advantages over the Laplace integral transform such as variable s-domain, duality with the Laplace transform, shifting of poles, variable domain of analyticity, necessary condition for existence of the β-Laplace integral transform, which is also can be applied on the Laplace integral transform.

2. Main results

Theorem 2.1 (Expansion and compression of s-domain). *For a function* $f(t), t \geqslant 0$ *which is a function of class A,*

(i) *if* $\beta > e$, *then* $(\mathrm{SD})_L \subset (\mathrm{SD})_{L_\beta}$;

(ii) *if* $\beta > e$, *then* $(\mathrm{SD})_L \subset (\mathrm{SD})_{L_\beta}$,

where $(\mathrm{SD})_L$ *and* $(\mathrm{SD})_{L_\beta}$ *are s-domain of the Laplace and the* β-*Laplace integral transform of function* $f \in A$, *respectively.*

Proof. Firstly f is of exponential order therefore there exist $M_1 > 0, \alpha > 0$ and $t_0 \geqslant 0$ such that

$$|f(t)| \leqslant M_1 e^{\alpha t}, \qquad t \geqslant t_0.$$

Also, $f(t)$ is sectionally continuous on $[0, t_0]$ and hence bounded, so there exists M_2, say

$$|f(t)| \leqslant M_2, \qquad 0 < t \leqslant t_0.$$

Since $e^{\alpha t}$ has a positive minimum on $[0, t_0]$, a constant M can be chosen sufficiently large so that

$$|f(t)| \leqslant M e^{\alpha t}, \qquad t \geqslant 0.$$

Therefore,

$$\int_0^\tau |\beta^{-st} f(t)| dt \leqslant M \int_0^\tau e^{-(\xi \log \beta - \alpha)t} dt = \left\{ \frac{M e^{-(\xi \log \beta - \alpha)t}}{-(\xi \log \beta - \alpha)} \right\}_0^\tau = \frac{M}{(\xi \log \beta - \alpha)} - \frac{M e^{-(\xi \log \beta - \alpha)t}}{-(\xi \log \beta - \alpha)}.$$

Letting limit $\tau \to \infty$ and noting that $\mathrm{Re}(s) = \xi$, yields

$$\int_0^\infty |\beta^{-st} f(t)| dt \leqslant \frac{M}{(\xi \log \beta - \alpha)}.$$

Thus, the Laplace transform converges absolutely for

$$\xi = Re(s) > \frac{\alpha}{\log \beta}.$$

Therefore, s-domain of the Laplace transform for function f is

$$(SD)_L = \{s \in \mathbb{C} : Re(s) > \alpha\}.$$

Similarly, we get s-domain of the β-Laplace transform for function f

$$(SD)_{L_\beta} = \left\{s \in \mathbb{C} : Re(s) > \frac{\alpha}{\log \beta}\right\}.$$

Obviously, if $\beta > e$, then

$$\log \beta > 1 \Rightarrow \alpha > \frac{\alpha}{\log \beta} \Rightarrow (SD)_{L_\beta} \supset (SD)_L.$$

And also, if $1 < \beta \leqslant e$, then

$$\log \beta \leqslant 1 \Rightarrow \alpha \leqslant \frac{\alpha}{\log \beta} \Rightarrow (SD)_{L_\beta} \subseteq (SD)_L.$$

$\square$

Theorem 2.2 (Duality of the Laplace and β-Laplace integral transform). *A function* $f(t), t \geqslant 0$ *has the Laplace integral transform if and only if* $f(t)$ *has the β-Laplace integral transform.*

Proof. Let the function $f(t)$ has the Laplace integral transform $\mathcal{L}\{f(t)\}_{(s)}$ then by the definition of the β-Laplace integral transform

$$\mathcal{L}\{f(t)\}_{(s)} = \int_0^\infty e^{-st} f(t) dt.$$

If $\beta > 1$ be any parameter then we can re-write the above equation as

$$\mathcal{L}_\beta\{f(t)\}_{(s)} = \int_0^\infty \beta^{\log_\beta e^{-st}} f(t) dt = \mathcal{L}_\beta\{f(t)\}_{(s)} = \int_0^\infty \beta^{(\log_\beta e)t} f(t) dt = \mathcal{L}_\beta\{f(t)\}_{(s \log_\beta e)}.$$

Conversely let the function $f(t)$ has the β-Laplace integral transform $\mathcal{L}_\beta\{f(t)\}_{(s)}$ then by the definition of the β-Laplace integral transform

$$\mathcal{L}_\beta\{f(t)\}_{(s)} = \int_0^\infty \beta^{-st} f(t) dt, \quad \beta > 1.$$

Therefore, if whenever the Laplace transform of a function exists, then β-Laplace integral transform also exists and converges. $\square$

Theorem 2.3 (Shifting of poles). *The function* $f(t), t \geqslant 0$ *is any function of class A and if the Laplace transform has a pole at point a of order* m *then the β-Laplace integral transform also has pole at point* $\frac{a}{\log \beta}$ *of order* m, *i.e., pole at point a can be shifted to other point by choosing appropriate value of β.*

Proof. Let the Laplace transform of the function $\mathcal{L}\{f(t)\}_{(s)}$ has pole at point SaS of order m, then by the definition of pole

$$\lim_{s \to a} |\mathcal{L}\{f(t)\}_{(s)}| \to \infty.$$

And we can also write as

$$\mathcal{L}\{f(t)\}_{(s)} = \frac{\phi(s)}{(s-a)^m}, \quad \text{where } \phi(a) \neq 0.$$

By Theorem 2.2 the β-Laplace integral transform of the function f(t) will be

$$\mathcal{L}_\beta\{f(t)\}_{(s)} = \mathcal{L}\{f(t)\}_{(s\log_e \beta)} = \frac{\phi(s\log\beta)}{(s\log\beta - a)^m}.$$

In this case, we get pole at point $\frac{a}{\log\beta}$, which shows that it depends on the value of the β therefore by choosing value of β the pole at a can be shifted to any other point.   □

Theorem 2.4 (Region of analyticity). *Let* $f \in A$, *then* $\mathcal{L}_\beta\{f(t)\}_{(s)}$ *is an analytic function in the domain* $\mathrm{Re}(s) > \frac{\alpha}{\log\beta}$ *and also*

   (i) *if* $\beta > e$, *then* $(DA)_L \subset (DA)_{L_\beta}$;

   (ii) *if* $1 < \beta \leqslant e$, *then* $(DA)_L \supseteq (DA)_{L_\beta}$.

Proof. Let $f \in A$ therefore $\exists M > 0, \alpha > 0$ for some $t_0 \geqslant 0$ such that

$$|f(t)| \leqslant Me^{\alpha t}, \quad t > t_0,$$

for $s = x + iy$,

$$\mathcal{L}_\beta\{f(t)\}_{(s)} = \int_0^\infty \beta^{-st} dt = \int_0^\infty \beta^{-(x+iy)t} dt = \int_0^\infty \beta^{-xt}(\cos(yt\log\beta) - \sin(yt\log\beta))f(t)dt$$

$$= \int_0^\infty \beta^{-xt}\cos(yt\log\beta)dt \int_0^\infty \beta^{-st}\sin(yt\log\beta)f(t)dt$$

$$= u(x,y) + iv(x,y).$$

Now consider

$$\left| \int_{t_0}^\infty \frac{\partial}{\partial x}\left(\beta^{-xt}\cos(yt\log\beta)\right)f(t)dt \right| = \left| \int_{t_0}^\infty -t\log\beta\beta^{-xt}\cos(yt\log\beta)f(t)dt \right|$$

$$\leqslant \int_{t_0}^\infty (t\log\beta)\beta^{-xt}|f(t)|dt \leqslant \frac{M}{(x\log\beta - \alpha)}\beta^{-(x-\alpha)t_0}.$$

Then for $x\log\beta > \alpha$, implying that the integral $\int_0^\infty \left(\frac{\partial}{\partial x}\right)\beta^{-xt}\cos(yt\log\beta)f(t)dt$ converge uniformly in the region $\mathrm{Re}(s) > \frac{\alpha}{\log\beta}$.

   Likewise, the integral $\int_0^\infty \left(\frac{\partial}{\partial x}\right)(-\beta^{-xt}\cos(yt\log\beta))f(t)dt$ converges uniformly in $\mathrm{Re}(s) > \frac{\alpha}{\log\beta}$. Because of this uniform convergence, and the absolute convergence of $\mathcal{L}_\beta\{f\}_{(s)}$, we can differentiate under the integral sign, that is to say,

$$u_x = \int_0^\infty \left(\frac{\partial}{\partial x}\right)\left(\beta^{-xt}\cos(yt\log\beta)f(t)\right)dt = \int_0^\infty (-t\log\beta)\beta^{-xt}\cos(yt\log\beta)f(t)dt,$$

$$v_y = \int_0^\infty \left(\frac{\partial}{\partial y}\right)\left(-\beta^{-xt}\sin(yt\log\beta)f(t)\right)dt = \int_0^\infty (-t\log\beta)\beta^{-xt}\cos(yt\log\beta)f(t)dt,$$

and so $u_x = v_y$. In a similar fashion we can show that $u_y = -v_x$. These partial derivatives are continuous. Thus, the Cauchy-Riemann conditions [12] are satisfied and $\mathcal{L}_\beta\{f\}_{(s)} = u(x,y) + iv(x,y)$ is an analytic function in the domain $\mathrm{Re}(s) > \frac{\alpha}{\log\beta}$. In similar fashion as above applied on the Laplace transform, we get $\mathcal{L}_\beta\{f\}_{(s)}$ is an analytic function in the domain $\mathrm{Re}(s) > \alpha$. Now domain of analyticity of the Laplace transform of function f, $(DA)_L$, can be defined as

$$(DA)_L = \{s \in \mathbb{C} : \mathrm{Re}(s) > \alpha\}.$$

And, domain of analyticity of the β-Laplace transform of function f, $(DA)_{L_\beta}$, can be defined as

$$(DA)_{L_\beta} = \left\{ s \in \mathbb{C} : Re(s) > \frac{\alpha}{\log \beta} \right\}.$$

Obviously, if $\beta > e$, then $(DA)_L \subset (DA)_{L_\beta}$, and also, if $1 < \beta \leqslant e$, then $(DA)_L \supseteq (DA)_{L_\beta}$. $\qquad \square$

Theorem 2.5 (Necessary conditions of existence of β-Laplace integral transform of class A function). *Let* $f \in A$ *and* $\mathcal{L}_\beta\{f(t)\}_{(s)}$ *be the β-Laplace integral transform of function f, then it satisfies*

$$\left\{ s\frac{\partial}{\partial s} - \beta \log \beta \frac{\partial}{\partial \beta} \right\} \mathcal{L}\{f(t)\}_{(s)} = 0,$$

or,

$$\mathcal{T}\mathcal{L}\{f(t)\}_{(s)} = 0, \quad where \quad \mathcal{T} \equiv \left\{ s\frac{\partial}{\partial s} - \beta \log \beta \frac{\partial}{\partial \beta} \right\}.$$

By the definition of the β-Laplace integral transform

$$\mathcal{L}_\beta\{f(t)\}_{(s)} = \int_0^\infty \beta^{-st} f(t) dt, \quad \beta > 1.$$

By Theorem 2.4, if $f \in A$, *then* $\mathcal{L}_\beta\{f(t)\}_{(s)}$ *is analytic in the domain* $Re(s) > \frac{\alpha}{\log \beta}$. *Partial differentiating with respect to s, we obtain*

$$\frac{\partial \mathcal{L}_\beta\{f(t)\}_{(s)}}{\partial s} = \int_0^\infty (-t \log \beta) \beta^{-st} f(t) dt = -\log \beta \mathcal{L}_\beta\{tf(t)\}_{(s)}. \tag{2.1}$$

Taking the partial derivative of equation with respect to β, we get

$$\frac{\partial \mathcal{L}_\beta\{f(t)\}_{(s)}}{\partial \beta} = \int_0^\infty (-\frac{st}{\beta}) \beta^{-st} f(t) dt = -\frac{s}{\beta} \mathcal{L}_\beta\{tf(t)\}_{(s)}. \tag{2.2}$$

By equations (2.1) and (2.2)

$$\frac{\partial \mathcal{L}_\beta\{f(t)\}_{(s)}}{\partial s} = \frac{\beta \log \beta}{s} \frac{\partial \mathcal{L}_\beta\{f(t)\}_{(s)}}{\partial \beta}$$

$$\Rightarrow \left\{ s\frac{\partial}{\partial s} - \beta \log \beta \frac{\partial}{\partial \beta} \right\} \mathcal{L}\{f(t)\}_{(s)} = 0,$$

$$\Rightarrow \mathcal{T}\mathcal{L}\{f(t)\}_{(s)} = 0, \quad where \quad \mathcal{T} \equiv \left\{ s\frac{\partial}{\partial s} - \beta \log \beta \frac{\partial}{\partial \beta} \right\}.$$

We can get necessary condition for existence of the Laplace transform for a function of class A by taking limit $\beta \to e$ *and we obtain*

$$\lim_{\beta \to e} \mathcal{T}\mathcal{L}\{f(t)\}_{(s)} = 0.$$

3. Conclusion

We have discussed and proved that the β-Laplace transform has some advantage (for having some special properties) over the classical Laplace transform for the functions of class A. This new generalised Laplace transform possesses more powerful tools (special properties) than the Laplace transform, such as shifting of poles, variable s-domain, variable domain of analyticity, and a necessary condition for the existence.

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