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Sequences of fuzzy star-shaped numbers

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Abstract

The concept of fuzzy star-shaped numbers was initially introduced by Diamond [P. Diamond, Fuzzy Sets and Systems, **37** (1990), 193–199]. In this paper, we define the concepts of convergent, Cauchy, and bounded sequences of fuzzy star-shaped numbers in $\mathbb{R}^n$ with respect to L_p-metric and study some properties of these new notions.

Keywords: Fuzzy star-shaped number, sequences of fuzzy star-shaped number, L_p-metric.

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1. Introduction

Let $\mathbb{N}$ be the set of all natural numbers, $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^n$ and $\mathbb{K}^n$ denotes the class of all nonempty compact sets in $\mathbb{R}^n$. If $A, B \in \mathbb{K}^n$, then the Hausdorff distance between A and B is given by

$$d_H(A, B) = \max\{\sup_{a\in A} \inf_{b\in B} \|a-b\|, \sup_{b\in B} \inf_{a\in A} \|a-b\|\}.$$

It is well known that $(\mathbb{K}^n, d_H)$ is a complete and separable space (see [5]).

Recall from [1] that a fuzzy star-shaped number is a fuzzy set $u : \mathbb{R}^n \longrightarrow [0, 1]$, satisfying the following conditions:

(1) u is normal, that is, there exists $t_0 \in \mathbb{R}^n$ such that $u(t_0) = 1$;
(2) u is upper semicontinuous;
(3) $\operatorname{supp} u = \operatorname{cl}\{t \in \mathbb{R}^n : u(t) > 0\}$, is compact;
(4) u is fuzzy star-shaped with respect to t, i.e., if there exists $t \in \mathbb{R}^n$ such that for any $s \in \mathbb{R}^n$ and $\lambda \in [0, 1]$,

$$u(\lambda s + (1-\lambda)t) \geqslant u(s).$$

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Let S^n denotes the set of all fuzzy star-shaped numbers in $\mathbb{R}^n$. For $u \in S^n$ and each $\alpha \in [0,1]$, one defines α-level set of u as follows:

$$[u]^\alpha = \begin{cases} \{t \in \mathbb{R}^n : u(t) \geqslant \alpha\}, & \text{if } \alpha \in (0,1], \\ \text{supp } u, & \text{if } \alpha = 0. \end{cases}$$

$\mathbb{R}^n$ can be embedded in S^n, as any $m \in \mathbb{R}^n$ can be viewed as the fuzzy star-shaped number

$$\hat{m}(t) = \begin{cases} 1, & \text{if } t = m, \\ 0, & \text{if } t \neq m. \end{cases}$$

Then it is obvious that $\hat{m} \in S^n$ for each $m \in \mathbb{R}^n$; for more details on fuzzy star-shaped numbers and other related concepts we refer to [2, 6–8]. The linear structure of the set of all fuzzy star-shaped numbers S^n induces an addition $u + v$ and a scalar multiplication λu, $\lambda \in \mathbb{R}$ in terms of α-level sets by

$$[u+v]^\alpha = [u]^\alpha + [v]^\alpha, \quad [\lambda u]^\alpha = \lambda [u]^\alpha$$

for each $\alpha \in [0,1]$. It directly follows that $u + v, \lambda u \in S^n$. Recall in [10] that for each $1 \leqslant p < \infty$, one defines

$$d_p(u,v) = \left(\int_0^1 d_H([u]^\alpha, [v]^\alpha)^p \, d\alpha \right)^{\frac{1}{p}}$$

and $d_\infty(u,v) = \sup\limits_{0 \leqslant \alpha \leqslant 1} d_H([u]^\alpha, [v]^\alpha)$. Clearly $d_\infty(u,v) = \lim_{p\to\infty} d_p(u,v)$ with $d_p \leqslant d_q$ if $p \leqslant q$, for all $u, v \in S^n$, then d_p is called the L_p-metric on S^n; some related works of L_p-metric can be found in [3, 4, 9].

2. Main results

In this section, we consider the sequence of fuzzy star-shaped numbers and define the notions of Cauchy, convergence and boundedness for these sequences and study some properties of these new resulting. A sequence of fuzzy star-shaped numbers $u = (u_k)$, is a function u from the set $\mathbb{N}$ into S^n. The fuzzy star-shaped number u_k denotes the value of the function at a point $k \in \mathbb{N}$ and is called the k-th term of the sequence. We denote by $\omega^*(S^n)$ the set of all sequences $u = (u_k)$ of fuzzy star-shaped numbers in $\mathbb{R}^n$. We construct the following examples of sequences of fuzzy star-shaped numbers with respect to origin.

Remark 2.1. It is obvious that every fuzzy number is a fuzzy star-shaped number and in the set of real numbers $\mathbb{R}$ the converse is also true. The example given below is a counter example of the remark in $\mathbb{R}^n$ for $n \neq 1$.

Example 2.1.

$$u : \mathbb{R}^2 \to [0,1] \text{ such that } u_k(t,s) = \begin{cases} 1 - kt, & \text{if } t \in [0, \frac{1}{k}], \\ 1 - ks, & \text{if } s \in [0, \frac{1}{k}], \\ 0, & \text{otherwise.} \end{cases}$$

Example 2.2.

$$u : \mathbb{R} \to [0,1] \text{ such that } u_k(t) = \begin{cases} \left(1 - |t|^{\frac{k+1}{k}}\right)^{\frac{k}{k+1}}, & \text{if } t \in [-1,1], \\ 0, & \text{otherwise.} \end{cases}$$

Before proceeding with our main results, we rewrite a formal definition of convergence of a sequence of fuzzy star-shaped numbers defined in paper of Zhao and Wu [10, Lemma 12 (4), Page 3].

Definition 2.2. A sequence $u = (u_k) \in \omega^*(S^n)$ is convergent to $u_0 \in \omega^*(S^n)$ if there exists a positive integer $N = N(\epsilon)$ such that

$$d_p(u_k, u_0) < \epsilon \text{ for all } k \geqslant N.$$

In this case, we write $\lim_{k\to\infty} d_p(u_k, u_0) = 0$ and u_0 is called the limit of the sequence (u_k). If $u_0 = \bar{0}$ then the sequence (u_k) is called null sequence. We define

$$c(S^n) = \text{The space of all convergent sequences of fuzzy star-shaped numbers in } \mathbb{R}^n,$$
$$c_0(S^n) = \text{The space of all null sequences of fuzzy star-shaped numbers in } \mathbb{R}^n.$$

Definition 2.3. Consider the metric space (S^n, d_p). Let $u \in S^n$ and $r > 0$ then the open ball centered at u with radius r is defined as $B_{d_p}(u, r) = \{v \in S^n : d_p(v, u) < r\}$.

Definition 2.4. A subset U of a metric space (S^n, d_p) is said to be bounded if there exists a positive real number $r > 0$ such that $U \subseteq B_{d_p}(u, r)$ for some $u \in S^n$.

Definition 2.5. A nonempty subset U of S^n is said to be open if, for all $u \in U$ there exists $r > 0$ such that $B_{d_p}(u, r) \subseteq U$, i.e., $\{v \in S^n : d_p(v, u) < r\} \subseteq U$.

Definition 2.6. A sequence $u = (u_k) \in \omega^*(S^n)$ is said to be bounded if, there exists a positive integer M such that

$$d_p(u_k, \bar{0}) \leqslant M \text{ for all } k \in \mathbb{N},$$

or if the set of its terms is bounded. We define

$$l_\infty(S^n) = \text{The space of all bounded sequences of fuzzy star-shaped numbers in } \mathbb{R}^n.$$

Definition 2.7. A sequence $u = (u_k) \in \omega^*(S^n)$ is said to be Cauchy sequence if, for every $\epsilon > 0$, there exists a positive integer $N = N(\epsilon)$ such that

$$d_p(u_k, u_m) < \epsilon \text{ for all } k, m \geqslant N.$$

Definition 2.8. Let $u = (u_k) \in \omega^*(S^n)$ and (k_i), $i = 1, 2, 3, \ldots$ is an increasing sequence of positive integers, then the sequence (u_{k_i}) is called a subsequence of (u_k).

Theorem 2.9. *The space of all fuzzy star-shaped numbers S^n with L_p-metric is a Hausdorff metric space.*

Proof. Consider the metric space (S^n, d_p) and let $u, v \in S^n$ such that $u \neq v$. Let $r = d_p(u, v)$ and $U = B_{d_p}(u, \frac{r}{2})$ and $V = B_{d_p}(v, \frac{r}{2})$ such that $u \in U$ and $v \in V$. We need to show that $U \cap V = \emptyset$. Let on contrary $z \in U \cap V$, this implies that $d_p(u, z) < \frac{r}{2}$ and $d_p(v, z) < \frac{r}{2}$. Now, by triangle inequality

$$r = d_p(u, v) \leqslant d_p(u, z) + d_p(z, v) < \frac{r}{2} + \frac{r}{2} = r,$$

which is a contradiction. Thus, $U \cap V = \phi$. □

Corollary 2.10. *A convergent sequence $(u_k) \in \omega^*(S^n)$ has a unique limit.*

Theorem 2.11. *Every open ball in the metric space (S^n, d_p) is an open set.*

Proof. Consider $B_{d_p}(u, r)$ to be an open ball. Let $r_1 = r - d_p(u, v)$. To show that $B_{d_p}(v, r_1) \subseteq B_{d_p}(u, r)$, let $w \in B_{d_p}(v, r_1)$ this implies that

$$\begin{aligned} d_p(w, v) < r_1 = r - d_p(u, v) &\implies d_p(u, v) + d_p(v, w) < r \\ &\implies d_p(u, w) < r \implies w \in B_{d_p}(u, r). \end{aligned}$$

Thus, $B_{d_p}(u, r)$ is an open set. □

Theorem 2.12. *A convergent sequence* (u_k) *of fuzzy star-shaped numbers is bounded but the converse is not true.*

Proof. Let $\lim_{k\to\infty} d_p(u_k, u_0) = 0$ and $\Delta = \{(u_k) : k = 1, 2, 3, \dots\}$. Let $\epsilon = 1$, thus by definition there exists a number $N \in \mathbb{N}$ such that $d_p(u_k, u_0) < 1$ for all $k \geqslant N$. Let $\delta = \max\{d_p(u_1, u_0), d_p(u_2, u_0), \dots, d_p(u_k, u_0), 1\}$. Then $d_p(u_k, u_0) \leqslant \delta$ for any $k \in \mathbb{N}$. Since d_p is a metric on S^n therefore, by triangle inequality we have

$$d_p(u_k, u_l) \leqslant d_p(u_k, u_0) + (d_p(u_0, u_l) \leqslant 2\delta.$$

This implies $\delta(\Delta) \leqslant 2\delta$, i.e., the sequence (u_k) is bounded. We construct the following counter example to verify our claim.

Example 2.3. Consider the sequence of fuzzy star-shaped numbers $u_k(t) : \mathbb{R} \to [0, 1]$ defined by

$$u_k(t) = \begin{cases} \frac{3t}{2}, & \text{if } 0 \leqslant t \leqslant \frac{2}{3}, \\ 1, & \text{if } \frac{2}{3} \leqslant t \leqslant \frac{4}{3}, \\ \frac{-3}{2}(t-2), & \text{if } \frac{4}{3} \leqslant t \leqslant 2, \\ 0, & \text{otherwise.} \end{cases}$$

Thus $(u_k) \in c(S^n)$. Let A be the collection of all finite subsets of $\mathbb{N}$, then there exists a subset K of $\mathbb{N}$ such that $K \notin A$ and $\mathbb{N} - K \notin A$, define $\nu = (\nu_k)$ by

$$\nu_k = \begin{cases} u_k, & k \in K, \\ 0, & \text{otherwise.} \end{cases}$$

Then, the sequence $(\nu_k) \in l_\infty(S^n)$ but $(\nu_k) \notin c(S^n)$. □

Theorem 2.13. *If* $(u_k) \in \omega^*(S^n)$ *is convergent to* u_0 *then any subsequence of* (u_k) *also converges to the same limit.*

Proof. Let $\lim_{k\to\infty} d_p(u_k, u_0) = 0$, then we show that for any subsequence (u_{k_l}) of (u_k),

$$\lim_{k\to\infty} d_p(u_{k_l}, u_0) = 0,$$

that is, for every $\epsilon > 0$, there exists a number $N_0 \in \mathbb{N}$ such that $(u_{k_l}) \in B_{d_p}(u_0, \epsilon)$ for all $l \geqslant N_0$. Since $\lim_{k\to\infty} d_p(u_k, u_0) = 0$, there exists a number $N \in \mathbb{N}$ such that $(u_k) \in B_{d_p}(u_0, \epsilon)$ for all $k \geqslant N$. Also, since (k_l) is an increasing sequence of positive integers there exists a number $N_0 \in \mathbb{N}$ such that $k_l \geqslant N$ whenever $l \geqslant N_0$. Hence, $(u_{k_l}) \in B_{d_p}(u_0, \epsilon)$. □

Theorem 2.14. *If* d_H *is a translation invariant metric, then* $l_\infty(S^n)$, $c(S^n)$, *and* $c_0(S^n)$ *are linear spaces over the field* $\mathbb{R}$.

Proof. We prove the result for the sequence space $c_0(S^n)$. The result for the other mentioned spaces can be proved correspondingly. Let $u = (u_k)$ and $\nu = (\nu_k)$ in $c_0(S^n)$, then, for every $\epsilon > 0$ and $\lambda, \mu \in \mathbb{R}$ are scalars, we have

$$d_p(u, \bar{0}) \leqslant \frac{\epsilon}{2|\lambda|} \text{ and } d_p(\nu, \bar{0}) \leqslant \frac{\epsilon}{2|\mu|}.$$

Therefore,

$$\begin{aligned} d_p(\lambda u + \mu\nu, \bar{0}) &= \left(\int_0^1 d_H([\lambda u_k + \mu\nu_k]^\alpha, \bar{0})^p \, d\alpha \right)^{\frac{1}{p}} \\ &= \left(\int_0^1 d_H([\lambda u_k]^\alpha + [\mu\nu_k]^\alpha, \bar{0})^p \, d\alpha \right)^{\frac{1}{p}} \end{aligned}$$

$$\leqslant \left(\int_0^1 d_H([\lambda u_k]^\alpha,\bar{0})^p d\alpha\right)^{\frac{1}{p}} + \left(\int_0^1 d_H([\mu v_k]^\alpha,\bar{0})^p d\alpha\right)^{\frac{1}{p}}$$
$$\leqslant |\lambda|\left(\int_0^1 d_H([u_k]^\alpha,\bar{0})^p d\alpha\right)^{\frac{1}{p}} + |\mu|\left(\int_0^1 d_H([v_k]^\alpha,\bar{0})^p d\alpha\right)^{\frac{1}{p}}$$
$$= |\lambda| d_p(u,\bar{0}) + |\mu| d_p(v,\bar{0}) \leqslant |\lambda|\frac{\epsilon}{2|\lambda|} + |\mu|\frac{\epsilon}{2|\mu|} = \epsilon.$$

Thus, $\lambda u + \mu v \in c_0(S^n)$. Hence, $c_0(S^n)$ is a linear space. □

Theorem 2.15. *The space $l_\infty(S^n)$ is a complete metric space with the metric ρ defined by*

$$\rho(u,v) = \sup_k d_p(u_k,v_k).$$

Proof. Let $\left(u_k^{(i)}\right)$ for $i = 1,2,\dots$ be a Cauchy sequence in $l_\infty(S^n)$. Then for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\rho\left(u_k^{(i)},u_k^{(j)}\right) < \epsilon \text{ for all } i,j \geqslant N$$

so that

$$d_p\left(u_k^{(i)},u_k^{(j)}\right) < \epsilon^k. \tag{2.1}$$

Hence, for each $k \in \mathbb{N}$, we have

$$d_p\left(u_k^{(i)},u_k\right) < \epsilon,$$

i.e., $\left(u_k^{(i)}\right)$ is convergent to (u_k). But then by equation (2.1) we infer that

$$\rho\left(u_k^{(i)},u_k\right) < \epsilon \text{ as } j \to \infty.$$

We have to show that $(u_k) \in l_\infty(S^n)$. For this

$$d_p(u_k,\bar{0}) \leqslant d_p\left(u_k,u_k^{(i)}\right) + d_p\left(u_k^{(i)},\bar{0}\right) \leqslant d_p\left(u_k^{(i)},\bar{0}\right) + \epsilon.$$

Since $\left(u_k^{(i)}\right)$ is a Cauchy sequence in $l_\infty(S^n)$, therefore $d_p(u_k^i,\bar{0}) \leqslant M$. This implies that

$$d_p(u_k,\bar{0}) \leqslant M + \epsilon = M' \text{ (say)}.$$

□

Theorem 2.16. *The space $c(S^n) \cap l_\infty(S^n)$ is a closed subset of the complete metric space $l_\infty(S^n)$.*

Proof. Let $\left(u_k^{(i)}\right)$ be a Cauchy sequence in $c(S^n) \cap l_\infty(S^n)$ and $u_k^{(i)} \to u_k$ in $l_\infty(S^n)$. Since $\left(u_k^{(i)}\right) \in c(S^n) \cap l_\infty(S^n)$, therefore, there exists a fuzzy star-shaped number (v^i) such that

$$\lim u_k^{(i)} = v^{(i)}.$$

Since $\left(u_k^{(i)}\right)$ is a Cauchy sequence it implies that there exists a positive integer N such that

$$\rho\left(u_k^{(j)},u_k^{(i)}\right) < \frac{\epsilon}{3} \text{ for } i,j \geqslant N.$$

Also,

$$\lim u_k^{(j)} = v^{(j)}.$$

Then, we have

$$\rho\left(v^{(j)}, v^{(i)}\right) \leqslant \rho\left(u_k^{(j)}, v^{(j)}\right) + \rho\left(u_k^{(j)}, u_k^{(i)}\right) + \rho\left(u_k^{(i)}, v^{(i)}\right) \leqslant \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

This shows that $(v^{(i)})$ is a Cauchy sequence in $l_\infty(S^n)$ and, therefore convergent, say to v, i.e.,

$$\lim v^{(i)} = v. \tag{2.2}$$

Now, it remains to show that (u_k) is convergent to v. Since $\left(u_k^{(i)}\right) \to u_k$, so for each $\epsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that

$$\rho(u_k^{(i)}, u_k) < \frac{\epsilon}{3} \text{ for all } i \geqslant N_0.$$

Also from equation (2.2) we have for every $\epsilon > 0$ there exists $N_1 \in N$ such that

$$\rho(v^{(i)}, v) < \frac{\epsilon}{3} \text{ for all } i \geqslant N_1.$$

Furthermore, since $\left(u_k^{(i)}\right) \to v^{(i)}$ for every $\epsilon > 0$ there exists $N_2 \in \mathbb{N}$ such that

$$\rho\left(u_k^{(i)}, v^{(i)}\right) < \frac{\epsilon}{3} \text{ for all } i \geqslant N_2.$$

Let $N_3 = \max\{N_0, N_1, N_2\}$. Now for $\epsilon > 0$ we have

$$\rho\left(u_k, v\right) \leqslant \rho\left(u_k, u_k^{(i)}\right) + \rho\left(u_k^{(i)}, v^{(i)}\right) + \rho\left(v^{(i)}, v\right) \leqslant \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

This shows that (u_k) is convergent to v that is, $(u_k) \in c(S^n) \cap l_\infty(S^n)$. This shows that the intersection is closed subset of the complete metric space $l_\infty(S^n)$. □

In view of the above theorem and Example 2.3 we deduce the following result.

Theorem 2.17. *The space $c(S^n) \cap l_\infty(S^n)$ is nowhere dense in the complete metric space $l_\infty(S^n)$.*

Proof. Since $c(S^n) \cap l_\infty(S^n)$ is a closed subset of $l_\infty(S^n)$ therefore $\overline{c(S^n) \cap l_\infty(S^n)} = c(S^n) \cap l_\infty(S^n)$. It suffices to show that
$(c(S^n) \cap l_\infty(S^n))^o = \phi$. Consider an arbitrary point $(u_k) \in c(S^n) \cap l_\infty(S^n)$. Then for any $r > 0$ there exists a point (v_k) in the neighborhood $B_{d_p}(u_k, r)$ such that $(v_k) \notin c(S^n)$ and $(v_k) \notin l_\infty(S^n)$. Thus $B_{d_p}(u_k, r) \nsubseteq c(S^n) \cap l_\infty(S^n)$. Thus an arbitrary (u_k) is not an interior point and hence $(c(S^n) \cap l_\infty(S^n))^o = \phi$. □

3. Conclusion

In this paper, we have formally defined the notions of convergent, Cauchy and bounded sequences of fuzzy star-shaped numbers in $\mathbb{R}^n$ with respect to L_p-metric and introduced new convergent and bounded sequence spaces, namely $c(S^n)$ and $l_\infty(S^n)$, respectively. Furthermore, we proved that $l_\infty(S^n)$ with L_p-metric is complete. Also, we proved that the inclusion $c(S^n) \subset l_\infty(S^n)$ holds and is strict in compliance with Example 2.3. These new results will further help the researchers expand their work in the area of sequence spaces in view of fuzzy theory.

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