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# On the stability of a sum form functional equation related to nonadditive entropies

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**Abstract**

In this paper we intend to discuss the stability of

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i q_j) = \sum_{i=1}^{n} M_1(p_i)\sum_{j=1}^{m} g(q_j) + \sum_{j=1}^{m} M_2(q_j)\sum_{i=1}^{n} h(p_i),$$

where $f : I \to \mathbb{R}$, $g : I \to \mathbb{R}$, $h : I \to \mathbb{R}$ are unknown mappings; $M_1 : I \to \mathbb{R}$, $M_2 : I \to \mathbb{R}$ are fixed multiplicative mappings both different from identity mapping; $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$ ; $n \geqslant 3$, $m \geqslant 3$ are fixed integers.

**Keywords:** Stability, bounded mapping, logarithmic mapping, multiplicative mapping.

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## 1. Introduction

Let $\mathbb{N}$ denote the set of natural numbers; $\mathbb{R}$ denote the set of real numbers; I denote the closed unit interval $[0,1]$, i.e., $I = [0,1] = \{x \in \mathbb{R} : 0 \leqslant x \leqslant 1\}$. For $n \in \mathbb{N}$; let $\Gamma_n = \{(p_1, \ldots, p_n);\ p_i \geqslant 0,\ i = 1, \ldots, n;\ \sum_{i=1}^{n} p_i = 1\}$ denote the set of all $n$-component discrete probability distributions.

A mapping $a : I \to \mathbb{R}$ is said to be additive on I or on the unit triangle $\Delta = \{(x,y) : 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1, 0 \leqslant x+y \leqslant 1\}$ if it satisfies $a(x+y) = a(x) + a(y)$ for all $(x,y) \in \Delta$. Similarly, a mapping $A : \mathbb{R} \to \mathbb{R}$ is said to be additive on $\mathbb{R}$ if it satisfies $A(x+y) = A(x) + A(y)$ for all $x \in \mathbb{R}$, $y \in \mathbb{R}$. An interesting relation between these fore mentioned additive mappings was given by Daróczy and Losonczi [4]. They proved that there exists a unique additive extension of the additive mapping $a : I \to \mathbb{R}$ to the set of real numbers.

A mapping $\ell : I \to \mathbb{R}$ is said to be logarithmic on I if it satisfies $\ell(0) = 0$ and $\ell(xy) = \ell(x) + \ell(y)$ for all $x \in ]0,1]$, $y \in ]0,1]$.

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A mapping $m : I \to \mathbb{R}$ is said to be multiplicative on I if it satisfies $m(0) = 0$, $m(1) = 1$ and $m(xy) = m(x)m(y)$ for all $x \in ]0,1[$, $y \in ]0,1[$.

Over the years, the exploration of an approximate solution which is in close proximity of the exact solution of an arbitrary functional equation has gained a lot of momentum. This problem was initially raised in reference to group homomorphisms by Ulam (see [24]) during his comprehensive presentation at the University of Wisconsin in 1940. In the following year Hyers [7] answered this problem for linear mappings $f : E \to E'$ on Banach spaces $E$ and $E'$. He considered a perturbation of the given transformation by considering the inequality $\|f(x+y)-f(x)-f(y)\| < \delta$ for all $x \in E$, $y \in E$. Then using this inequality he provided an affirmative answer to Ulam's problem. With this seminal idea proposed by Hyers [7], the stability problem entered the corpus of functional equations and laid the foundation for a new domain of research referred as "Hyers-Ulam stability problem" in the literature.

It needs to be highlighted that Hyers' methodology, also known as the direct method is widely used for discussing the stability of functional equations. Following Hyers' approach, numerous research papers in reference to the stability problem of functional equations have appeared. Some of their results and methods can be found in the book of Hyers, Isac and Rassias [9]. In brief, these papers reflected upon: generalizations; new methods; significance and applications. Notwithstanding all these endeavours, researchers missed discussing the stability problem for sum form functional equations. In this direction, Maksa [15] presented an affirmative answer to Ulam's open problem by developing an interesting result similar to that of Hyers'. This was a significant breakthrough that opened a new area of research known as "Stability problem for sum form functional equations".

Recently few significant contributions that have appeared in this field are as follows. Dutta and Kumar [5] deployed Găvruta's perspective to establish the stability of a functional equation; Najati and Sahoo [16] investigated the stability for functional equations by following Hyers' approach; Narasimann [17] briefly reflected upon generalizations of Hyers' result, followed by establishing the stability of a generalized k-additive functional equation. Further it is worth mentioning that stability problem for the sum form functional equations is a new area of research which is comparatively less explored. There are many sum form functional equations for which the stability problem needs to be addressed providing ample opportunities to the researches. Intrigued by one of these, the objective of this paper is to discuss the stability of functional equation

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i q_j) = \sum_{i=1}^{n} M_1(p_i) \sum_{j=1}^{m} g(q_j) + \sum_{j=1}^{m} M_2(q_j) \sum_{i=1}^{n} h(p_i), \tag{A}$$

where $f : I \to \mathbb{R}$, $g : I \to \mathbb{R}$, $h : I \to \mathbb{R}$ are unknown mappings; $M_1 : I \to \mathbb{R}$, $M_2 : I \to \mathbb{R}$ are fixed multiplicative mappings both different from identity mapping; $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$ ; $n \geqslant 3$, $m \geqslant 3$ be fixed integers. We would now like to reflect upon the significance of functional equation (A) which motivated us to study it.

If $M_1(p) = p$ and $M_2(q) = q$ for all $p \in I$, $q \in I$, then (A) reduces to the functional equation

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i q_j) = \sum_{j=1}^{m} g(q_j) + \sum_{i=1}^{n} h(p_i), \tag{1.1}$$

where $f : I \to \mathbb{R}$, $g : I \to \mathbb{R}$, $h : I \to \mathbb{R}$ are unknown mappings; $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$; $n \geqslant 3$, $m \geqslant 3$ be fixed integers. Nath and Singh [18] have obtained the general solutions of (1.1). Indeed it plays a significant role in characterizing the Shannon [20] entropies $H_n : \Gamma_n \to R$, $n \in \mathbb{N}$ defined as

$$H_n(p_1, \ldots, p_n) = -\sum_{i=1}^{n} p_i \log_2 p_i \quad ((p_1, \ldots, p_n) \in \Gamma_n;\ 0 \log_2 0 := 0). \tag{1.2}$$

The relation of these entropies (1.2) with the general solutions of (1.1) is clearly established by the first author of this paper in [18].

Further if $g = f$, $h = f$; $M_1(p) = p^\alpha$, $M_2(q) = q^\beta$ for fixed positive real powers $\alpha \neq 1$ and $\beta \neq 1$ satisfying the conventions

$$0^\alpha := 0,\ 0^\beta := 0,\ 1^\alpha := 1,\ 1^\beta := 1, \tag{1.3}$$

then (A) reduces to the functional equation given by Behara and Nath [2] that is

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i q_j) = \sum_{i=1}^{n} p_i^\alpha \sum_{j=1}^{m} f(q_j) + \sum_{j=1}^{m} q_j^\beta \sum_{i=1}^{n} f(p_i), \tag{1.4}$$

where $f : I \to \mathbb{R}$ is an unknown mapping; $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$; $n \geqslant 3$, $m \geqslant 3$ be fixed integers. Behara and Nath were first who came across this functional equation and found its continuous solutions which plays a key role in characterizing the entropies of type $(\alpha, \beta)$. For $n \in \mathbb{N}$, $(p_1, \ldots, p_n) \in \Gamma_n$, an entropy of type $(\alpha, \beta)$ is defined as

$$H_n^{(\alpha,\beta)}(p_1, \ldots, p_n) = \begin{cases} (2^{1-\alpha} - 2^{1-\beta})^{-1}\left(\sum\limits_{i=1}^{n} p_i^\alpha - \sum\limits_{i=1}^{n} p_i^\beta\right), & \text{if } \alpha \neq \beta, \\ -2^{\beta-1}\sum\limits_{i=1}^{n} p_i^\beta \log_2 p_i, & \text{if } \alpha = \beta, \end{cases}$$

where $H_n^{(\alpha,\beta)} : \Gamma_n \to \mathbb{R}$; $\alpha$ and $\beta$ are fixed positive real powers satisfying (1.3) and $0^\beta \log_2 0 := 0$.

Losonczi and Maksa [13] were first who obtained the general solutions of (1.4) without assuming any conditions on the mapping $f : I \to \mathbb{R}$ for $\alpha \neq 1$, $\beta \neq 1$ and $n \geqslant 3$, $m \geqslant 2$ being fixed integers. Then, Kocsis and Maksa [11] discussed its stability. Further, Kocsis [12] also considered a generalization of (1.4) by studying

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i q_j) = \sum_{i=1}^{n} M_1(p_i) \sum_{j=1}^{m} f(q_j) + \sum_{j=1}^{m} M_2(q_j) \sum_{i=1}^{n} f(p_i), \tag{1.5}$$

where $f : I \to \mathbb{R}$ is an unknown mapping; $M_1 : I \to \mathbb{R}$, $M_2 : I \to \mathbb{R}$ are fixed multiplicative mappings both different from identity mapping; $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$ ; $n \geqslant 3$, $m \geqslant 3$ be fixed integers. In this paper [12], he obtained its general solutions and also discussed its stability. A more inspiring and novel generalization of (1.4) was observed by Singh and Dass [21].

We observe that functional equations (1.4) and (1.5) were addressed a number of times and studied comprehensively resulting in emergence of new aspects but no one discussed its Pexiderized form, i.e., (A), which seems to have missed the attention. Further as (A) is arising from (1.4) and (1.5), it is useful in the characterization of entropies of type $(\alpha, \beta)$, so it seems interesting to study (A).

Singh and Grover [22] have obtained the general solutions of (A) when the multiplicative mappings $M_1 : I \to \mathbb{R}$ and $M_2 : I \to \mathbb{R}$ are power mappings. Indeed they have added more stimulating aspects to the functional equation by first demonstrating its relation with the expected value of a random variable and then connecting its general solutions not only with entropies of type $(\alpha, \beta)$ but also with diversity index. In what follows, we shed some light on the phenomenon of diversity index.

An index of diversity is a nonnegative real valued mapping with probability distribution as its domain which indicates the differences within a sample space. As a matter of fact, there is spectrum of definitions on index of diversity and its applications. We would prefer references [6, 10, 19, 23] for the readers to be familiar with the concept of diversity index and its various fields of research. Thus, it can easily be concluded that (A) is related to entropies of type $(\alpha, \beta)$ and diversity index which is a quantitative measure that has evolved with a multidisciplinary approach over the years. This evokes our interest to discuss functional equation (A) in detail. We notice that the general solutions are obtained but stability is yet to be established. So, in this paper we discuss the stability of (A).

The rest of the paper is structured as follows. In Section 2, we mention few preliminary results which will be used in the upcoming sections. In Section 3, we discuss the stability of the functional equation (A) for $n \geqslant 3$, $m \geqslant 3$ being fixed integers.

## 2. Preliminary results

In this section, we state some known previous results which will be used in the upcoming section.

**Result 2.1** ([14]). *Suppose a mapping* $\phi : I \to \mathbb{R}$ *satisfies the functional equation* $\sum_{i=1}^{n} \phi(p_i) = c_1$ *for all* $(p_1, \dots, p_n) \in \Gamma_n$, $n \geqslant 3$ *a fixed integer and* $c_1$ *a real constant. Then there exists an additive mapping* $a : \mathbb{R} \to \mathbb{R}$ *such that* $\phi(p) = a(p) - \frac{1}{n}a(1) + \frac{c_1}{n}$ *for all* $p \in I$.

**Result 2.2** ([15]). *Let* $0 \leqslant \varepsilon \in \mathbb{R}$, $n \geqslant 3$ *be fixed integer and* $\psi : I \to \mathbb{R}$ *be a mapping which satisfy the functional inequality* $\left|\sum_{i=1}^{n} \psi(p_i)\right| \leqslant \varepsilon$ *for all* $(p_1, \dots, p_n) \in \Gamma_n$. *Then there exist an additive mapping* $A_1 : \mathbb{R} \to \mathbb{R}$ *and a mapping* $B_1 : \mathbb{R} \to \mathbb{R}$ *such that* $|B_1(p)| \leqslant 18\varepsilon$ *for all* $p \in \mathbb{R}$, $B_1(0) = 0$ *and* $\psi(p) - \psi(0) = A_1(p) + B_1(p)$ *for all* $p \in I$.

**Result 2.3** ([12]). *Let* $A_2 : \mathbb{R} \to \mathbb{R}$ *be an additive mapping,* $M : I \to \mathbb{R}$ *a multiplicative mapping,* $B_2 : \mathbb{R} \to \mathbb{R}$ *a bounded mapping and* $c_2 \in \mathbb{R}$. *If* $A_2(p) = M(p) + c_2$ *for all* $p \in I$, *then* $A_2(p) = dp$, $p \in \mathbb{R}$ *for some* $d \in \mathbb{R}$ *and* $M(p) = 0$ *or* $M(p) = p$, $p \in I$. *Also if* $A_2(p) = M(p) + B_2(p)$ *for all* $p \in I$, *then* $A_2(p) = dp$, $p \in \mathbb{R}$ *for some* $d \in \mathbb{R}$ *and* $M(p) = 0$ *or* $M(p) = p^{\alpha}$, $p \in I$ *for some* $0 \leqslant \alpha \in \mathbb{R}$.

**Result 2.4** ([12]). *Let* $M_1, M_2 : I \to \mathbb{R}$ *be fixed multiplicative mappings,* $M_1 \neq M_2$, $A_3 : \mathbb{R} \to \mathbb{R}$ *be an additive mapping and* $c_3 \in \mathbb{R}$. *If* $M_1(p) - M_2(p) = A_3(p) + c_3$ *holds for all* $p \in I$, *then* $M_1$ *and* $M_2$ *are zero or identity mappings on* $I$.

**Result 2.5** ([25]). *If* $f$ *is a solution to the functional equation*

$$f(p + q) = f(p) + f(q),$$

*which is bounded over an interval* $[a, b]$, *then it is of the form* $f(p) = c_4 p$ *for some real number* $c_4$.

**Result 2.6** ([3, 11]). *Suppose that* $\beta, \varepsilon' \in [0, +\infty[$; $G : I \to \mathbb{R}$ *and*

$$|G(pq) - p^{\beta}G(q) - q^{\beta}G(p)| \leqslant \varepsilon', \qquad (p, q \in I).$$

*Then there exists a logarithmic mapping* $\ell : I \to \mathbb{R}$ *and a mapping* $B_3 : \mathbb{R} \to \mathbb{R}$ *such that* $|B_3(p)| \leqslant 4e\varepsilon'$ *(e, a natural base of the logarithmic mapping) and* $G(p) = p^{\beta}\ell(p) + B_3(p)$ *for all* $p \in I$.

**Result 2.7** ([22]). *Let* $n \geqslant 3$, $m \geqslant 3$ *be fixed integers;* $M_1 : I \to \mathbb{R}$, $M_2 : I \to \mathbb{R}$ *be fixed multiplicative mappings different from identity mapping and* $f : I \to \mathbb{R}$, $g : I \to \mathbb{R}$, $h : I \to \mathbb{R}$ *be real valued mappings satisfying the functional equation* (A) *for all* $(p_1, \dots, p_n) \in \Gamma_n$, $(q_1, \dots, q_m) \in \Gamma_m$. *Then for all* $p \in I$, *any general solution* $(f, g, h)$ *of functional equation* (A) *is of the form (for* $M_1 = M_2$*)*

$$\left.\begin{aligned}
&\text{(i)} \quad f(p) = M_2(p)\ell(p) + [g(1) + (m-1)g(0) + h(1) + (n-1)h(0)]M_2(p)\\
&\qquad\quad + a_1(p) + f(0),\ a_1(1) = -nmf(0),\\
&\text{(ii)} \quad g(p) = M_2(p)\ell(p) + [g(1) + (m-1)g(0)]M_2(p) + a_2(p) + g(0),\\
&\qquad a_2(1) = -mg(0),\\
&\text{(iii)} \quad h(p) = M_2(p)\ell(p) + [h(1) + (n-1)h(0)]M_2(p) + a_3(p) + h(0),\\
&\qquad a_3(1) = -nh(0),
\end{aligned}\right\} \qquad (\alpha_1)$$

*and (for* $M_1 \neq M_2$*)*

$$\left.\begin{aligned}
&\text{(i)} \quad f(p) = d(M_1(p) - M_2(p)) + [g(1) + (m-1)g(0) + h(1) + (n-1)h(0)]\\
&\qquad\qquad \times M_2(p) + a_4(p) + f(0),\ a_4(1) = -nmf(0),\\
&\text{(ii)} \quad g(p) = d(M_1(p) - M_2(p)) + [g(1) + (m-1)g(0)]M_2(p)\\
&\qquad\qquad + a_5(p) + g(0),\ a_5(1) = -mg(0),\\
&\text{(iii)} \quad h(p) = \{d - [g(1) + (m-1)g(0)]\}(M_1(p) - M_2(p))\\
&\qquad\qquad + [h(1) + (n-1)h(0)]M_2(p) + a_6(p) + h(0),\ a_6(1) = -nh(0),
\end{aligned}\right\} \qquad (\alpha_2)$$

*where* $a_i : \mathbb{R} \to \mathbb{R}$ $(i = 1,2,3,4,5,6)$ *are additive mappings;* $\ell : I \to \mathbb{R}$ *is a logarithmic mapping and* $d$ *is an arbitrary real constant.*

## 3. The stability of the functional equation (A)

In this section we discuss the stability of functional equation (A). For this we refer to survey paper of Hyers and Rassias [8] and Hyers, Isac and Rassias [9]. Indeed in the sense of [9] we consider a perturbation of (A) given by the functional inequality

$$\left|\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_iq_j) - \sum_{i=1}^{n} M_1(p_i)\sum_{j=1}^{m} g(q_j) - \sum_{j=1}^{m} M_2(q_j)\sum_{i=1}^{n} h(p_i)\right| \leqslant \varepsilon \tag{B}$$

for all $(p_1, \dots, p_n) \in \Gamma_n$, $(q_1, \dots, q_m) \in \Gamma_m$; $n \geqslant 3$, $m \geqslant 3$ be fixed integers and $\varepsilon$ be a positive real number.

Our aim is to find the solutions of inequality (B) and observe: *What is the difference between the solutions (given by Result 2.7) of equation (A) and inequality (B)?* If the difference between their solutions is only a bounded mapping, we would say that functional equation (A) is stable. Following this we establish the stability of (A) and prove the main result of this section.

**Theorem 3.1.** *Let* $n \geqslant 3$, $m \geqslant 3$ *be fixed integers;* $\varepsilon$ *be a positive real number and* $M_1 : I \to \mathbb{R}$, $M_2 : I \to \mathbb{R}$ *be fixed multiplicative mappings different from identity mapping. Suppose* $f : I \to \mathbb{R}$, $g : I \to \mathbb{R}$, $h : I \to \mathbb{R}$ *be mappings which satisfy the functional inequality* (B) *for all* $(p_1, \dots, p_n) \in \Gamma_n$, $(q_1, \dots, q_m) \in \Gamma_m$. *Then, (for* $M_1 = M_2$*)*

$$\left.\begin{array}{rl}
\text{(i)} & f(p) - f(0) = M_2(p)\ell(p) + [g(1) + (m-1)g(0) + h(1) + (n-1)h(0)]M_2(p) + a_1(p) + b_1(p),\\
\text{(ii)} & g(p) - g(0) = M_2(p)\ell(p) + [g(1) + (m-1)g(0)]M_2(p) + a_2(p) + b_2(p),\\
\text{(iii)} & h(p) - h(0) = M_2(p)\ell(p) + [h(1) + (n-1)h(0)]M_2(p) + a_3(p) + b_3(p),
\end{array}\right\} \tag{$\beta_1$}$$

*and (for* $M_1 \neq M_2$*)*

$$\left.\begin{array}{rl}
\text{(i)} & f(p) - f(0) = d(M_1(p) - M_2(p)) + [g(1) + (m-1)g(0) + h(1)\\
& \qquad +(n-1)h(0)]M_2(p) + a_4(p) + b_4(p),\\
\text{(ii)} & g(p) - g(0) = d(M_1(p) - M_2(p)) + [g(1) + (m-1)g(0)]M_2(p) + a_5(p) + b_5(p),\\
\text{(iii)} & h(p) - h(0) = \{d - [g(1) + (m-1)g(0)]\}(M_1(p) - M_2(p))\\
& \qquad +[h(1) + (n-1)h(0)]M_2(p) + a_6(p) + b_6(p),
\end{array}\right\} \tag{$\beta_2$}$$

*where* $a_i : \mathbb{R} \to \mathbb{R}$ $(i = 1,2,3,4,5,6)$ *are additive mappings;* $b_j : \mathbb{R} \to \mathbb{R}$ $(j = 1,2,3,4,5,6)$ *are bounded mappings;* $\ell : I \to \mathbb{R}$ *is a logarithmic mapping and* $d$ *is an arbitrary real constant.*

*Proof.* Let us put $p_1 = 1$, $p_2 = \cdots = p_n = 0$ in (B). We obtain

$$\left|\sum_{j=1}^{m}\left\{f(q_j) + (n-1)\,f(0) - g(q_j) - [h(1) + (n-1)\,h(0)]\,M_2(q_j)\right\}\right| \leqslant \varepsilon$$

for all $(q_1, \dots, q_m) \in \Gamma_m$. By Result 2.2, there exists an additive mapping $\overline{A}_1 : \mathbb{R} \to \mathbb{R}$ and a mapping $B_1^* : \mathbb{R} \to \mathbb{R}$ such that $|B_1^*(q)| \leqslant 18\varepsilon$, $B_1^*(0) = 0$ and

$$f(q) - g(q) - [h(1) + (n-1)h(0)]M_2(q) - f(0) + g(0) = \overline{A}_1(q) + B_1^*(q).$$

From this, one can easily obtain the expression

$$f(q) = g(q) + \overline{A}_1(q) + \overline{B}_1(q) + [h(1) + (n-1)h(0)]M_2(q), \tag{3.1}$$

where $\overline{B}_1 : \mathbb{R} \to \mathbb{R}$ is a bounded mapping defined as $\overline{B}_1(x) = B_1^*(x) + f(0) - g(0)$. Using (3.1), inequality (B) can be written as

$$\begin{aligned} &\left| \sum_{i=1}^{n} \sum_{j=1}^{m} g(p_i q_j) + \overline{A}_1(1) + \sum_{i=1}^{n} \sum_{j=1}^{m} \overline{B}_1(p_i q_j) - \sum_{i=1}^{n} M_1(p_i) \sum_{j=1}^{m} g(q_j) \right. \\ &\qquad \left. - \left\{ \sum_{i=1}^{n} h(p_i) - [h(1) + (n-1)h(0)] \sum_{i=1}^{n} M_2(p_i) \right\} \sum_{j=1}^{m} M_2(q_j) \right| \leqslant \varepsilon \end{aligned} \tag{3.2}$$

for all $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$. Now substituting $q_1 = 1, q_2 = \cdots = q_m = 0$ in (3.2), we have

$$\begin{aligned} &\left| \sum_{i=1}^{n} \left\{ g(p_i) + (m-1)g(0) + \overline{A}_1(1)p_i + \overline{B}_1(p_i) + (m-1)\overline{B}_1(0) \right. \right. \\ &\quad \left. \left. - [g(1) + (m-1)g(0)]M_1(p_i) - h(p_i) + [h(1) + (n-1)h(0)]M_2(p_i) \right\} \right| \leqslant \varepsilon \end{aligned}$$

for all $(p_1, \ldots, p_n) \in \Gamma_n$. By Result 2.2, there exists an additive mapping $\overline{A}_2 : \mathbb{R} \to \mathbb{R}$ and a mapping $B_2^* : \mathbb{R} \to \mathbb{R}$ such that $|B_2^*(p)| \leqslant 18\varepsilon$, $B_2^*(0) = 0$ and

$$\begin{aligned} &g(p) + \overline{A}_1(1)p + \overline{B}_1(p) - [g(1) + (m-1)g(0)]M_1(p) - h(p) \\ &+ [h(1) + (n-1)h(0)]M_2(p) - g(0) - \overline{B}_1(0) + h(0) = \overline{A}_2(p) + B_2^*(p). \end{aligned}$$

From this, we can easily obtain the expression

$$\begin{aligned} h(p) = g(p) - \overline{A}_2(p) - \overline{B}_2(p) + \overline{A}_1(1)p + \overline{B}_1(p) - [g(1) + (m-1)g(0)]M_1(p) \\ + [h(1) + (n-1)h(0)]M_2(p), \end{aligned} \tag{3.3}$$

where $\overline{B}_2 : \mathbb{R} \to \mathbb{R}$ is a bounded mapping defined as $\overline{B}_2(x) = B_2^*(x) + g(0) + \overline{B}_1(0) - h(0)$. Substituting (3.3) in inequality (3.2), we get

$$\begin{aligned} &\left| \sum_{i=1}^{n} \sum_{j=1}^{m} G(p_i q_j) + \sum_{i=1}^{n} \sum_{j=1}^{m} \overline{B}_1(p_i q_j) + \overline{A}_1(1) - \sum_{i=1}^{n} M_1(p_i) \sum_{j=1}^{m} G(q_j) - \sum_{j=1}^{m} M_2(q_j) \right. \\ &\quad \left. \times \left[ \sum_{i=1}^{n} G(p_i) - \sum_{i=1}^{n} \overline{B}_2(p_i) + \sum_{i=1}^{n} \overline{B}_1(p_i) - \overline{A}_2(1) + \overline{A}_1(1) \right] \right| \leqslant \varepsilon, \end{aligned} \tag{3.4}$$

where $G : I \to \mathbb{R}$ is a mapping defined as

$$G(x) = g(x) - [g(1) + (m-1)g(0)]M_2(x) \tag{3.5}$$

for all $x \in I$. By Result 2.2, there exists a mapping $\overline{A}_3 : \Gamma_n \times \mathbb{R} \to \mathbb{R}$, additive in the second variable and a mapping $\overline{B}_3 : \Gamma_n \times \mathbb{R} \to \mathbb{R}$, bounded in the second variable by $18\varepsilon$ with $\overline{B}_3(p_1, \ldots, p_n; 0) = 0$, such that

$$\begin{aligned} &\sum_{i=1}^{n} G(p_i q) + \sum_{i=1}^{n} \overline{B}_1(p_i q) + \overline{A}_1(1)q - (G(q) - G(0)) \sum_{i=1}^{n} M_1(p_i) - M_2(q) \left[ \sum_{i=1}^{n} G(p_i) \right. \\ &\quad \left. - \sum_{i=1}^{n} \overline{B}_2(p_i) + \sum_{i=1}^{n} \overline{B}_1(p_i) - \overline{A}_2(1) + \overline{A}_1(1) \right] - nG(0) - n\overline{B}_1(0) \\ &= \overline{A}_3(p_1, \ldots, p_n; q) + \overline{B}_3(p_1, \ldots, p_n; q) \end{aligned} \tag{3.6}$$

for all $q \in I$, $(p_1,\dots,p_n) \in \Gamma_n$. Let $x \in I$, $(s_1,\dots,s_n) \in \Gamma_n$ and write $q = s_t x$, $t = 1,\dots,n$ consecutively in (3.6); summing up the resulting $n$ equations so obtained and then substituting the expression $\sum_{t=1}^{n} G(s_t x)$ calculated from (3.6), it follows that

$$\begin{aligned}
&\sum_{i=1}^{n}\sum_{t=1}^{n} G(p_i s_t x) + \sum_{i=1}^{n}\sum_{t=1}^{n} \overline{B}_1(p_i s_t x) + \overline{A}_1(1)x - n^2 G(0) - n^2\overline{B}_1(0) - (G(x)-G(0))\sum_{i=1}^{n} M_1(p_i)\sum_{t=1}^{n} M_1(s_t)\\
&\quad = \overline{A}_3(p_1,\dots,p_n;x) + \sum_{t=1}^{n}\overline{B}_3(p_1,\dots,p_n;s_t x)\\
&\qquad + M_2(x)\left\{\sum_{i=1}^{n} M_1(p_i)\left[\sum_{t=1}^{n} G(s_t) - \sum_{t=1}^{n}\overline{B}_2(s_t) + \sum_{t=1}^{n}\overline{B}_1(s_t) - \overline{A}_2(1) + \overline{A}_1(1)\right]\right.\\
&\qquad \left. + \sum_{t=1}^{n} M_2(s_t)\left[\sum_{i=1}^{n} G(p_i) - \sum_{i=1}^{n}\overline{B}_2(p_i) + \sum_{i=1}^{n}\overline{B}_1(p_i) - \overline{A}_2(1) + \overline{A}_1(1)\right]\right\}\\
&\qquad + \sum_{i=1}^{n} M_1(p_i)\left[\overline{A}_3(s_1,\dots,s_n;x) + \overline{B}_3(s_1,\dots,s_n;x) - \sum_{t=1}^{n}\overline{B}_1(s_t x) - \overline{A}_1(1)x + n\overline{B}_1(0)\right]
\end{aligned}$$

for all $x \in I$, $(p_1,\dots,p_n) \in \Gamma_n$, $(s_1,\dots,s_n) \in \Gamma_n$. Clearly the left hand side of the above equation is commutative in $p_i$ and $s_t$, $i = 1,\dots,n$; $t = 1,\dots,n$ (see p. 59. Aczél [1]). Thus the commutativity of $p_i$ and $s_t$, $i = 1,\dots,n$; $t = 1,\dots,n$ on the right hand side gives

$$\begin{aligned}
&\overline{A}_3(p_1,\dots,p_n;x)\left[1 - \sum_{t=1}^{n} M_1(s_t)\right] - \overline{A}_3(s_1,\dots,s_n;x)\left[1 - \sum_{i=1}^{n} M_1(p_i)\right]\\
&\quad = \sum_{i=1}^{n}\overline{B}_3(s_1,\dots,s_n;p_i x) - \sum_{t=1}^{n}\overline{B}_3(p_1,\dots,p_n;s_t x)\\
&\qquad + \sum_{t=1}^{n} M_1(s_t)\left[\overline{B}_3(p_1,\dots,p_n;x) - \sum_{i=1}^{n}\overline{B}_1(p_i x) - \overline{A}_1(1)x + n\overline{B}_1(0)\right]\\
&\qquad - \sum_{i=1}^{n} M_1(p_i)\left[\overline{B}_3(s_1,\dots,s_n;x) - \sum_{t=1}^{n}\overline{B}_1(s_t x) - \overline{A}_1(1)x + n\overline{B}_1(0)\right]\\
&\qquad + M_2(x)\left\{\left[\sum_{t=1}^{n} M_1(s_t) - \sum_{t=1}^{n} M_2(s_t)\right]\left[\sum_{i=1}^{n} G(p_i) - \sum_{i=1}^{n}\overline{B}_2(p_i) + \sum_{i=1}^{n}\overline{B}_1(p_i)\right.\right.\\
&\qquad \left. - \overline{A}_2(1) + \overline{A}_1(1)\right] - \left[\sum_{i=1}^{n} M_1(p_i) - \sum_{i=1}^{n} M_2(p_i)\right]\left[\sum_{t=1}^{n} G(s_t) - \sum_{t=1}^{n}\overline{B}_2(s_t)\right.\\
&\qquad \left.\left. + \sum_{t=1}^{n}\overline{B}_1(s_t) - \overline{A}_2(1) + \overline{A}_1(1)\right]\right\}.
\end{aligned} \tag{3.7}$$

Here, we observe that the proof depends on the mappings $M_1$ and $M_2$. So, we divide our discussion into two cases.

**Case 1.** $M_1 = M_2$. In this case, equation (3.7) gives

$$\begin{aligned}
&\overline{A}_3(p_1,\dots,p_n;x)\left[1 - \sum_{t=1}^{n} M_2(s_t)\right] - \overline{A}_3(s_1,\dots,s_n;x)\left[1 - \sum_{i=1}^{n} M_2(p_i)\right]\\
&\quad = \sum_{i=1}^{n}\overline{B}_3(s_1,\dots,s_n;p_i x) - \sum_{t=1}^{n}\overline{B}_3(p_1,\dots,p_n;s_t x)
\end{aligned} \tag{3.8}$$

$$+\sum_{t=1}^{n} M_2(s_t)\left[\overline{B}_3(p_1,\dots,p_n;x)-\sum_{i=1}^{n}\overline{B}_1(p_ix)-\overline{A}_1(1)x+n\overline{B}_1(0)\right]$$
$$-\sum_{i=1}^{n} M_2(p_i)\left[\overline{B}_3(s_1,\dots,s_n;x)-\sum_{t=1}^{n}\overline{B}_1(s_tx)-\overline{A}_1(1)x+n\overline{B}_1(0)\right].$$

For fixed $(p_1,\dots,p_n)\in\Gamma_n$ and $(s_1,\dots,s_n)\in\Gamma_n$, the right hand side of (3.8) is bounded on I whereas its left hand side is additive in $x\in I$, consequently by Result 2.5, it follows that

$$\begin{aligned}&[\overline{A}_3(p_1,\dots,p_n;x)-x\overline{A}_3(p_1,\dots,p_n;1)]\left[1-\sum_{t=1}^{n}M_2(s_t)\right]\\ &=[\overline{A}_3(s_1,\dots,s_n;x)-x\overline{A}_3(s_1,\dots,s_n;1)]\left[1-\sum_{i=1}^{n}M_2(p_i)\right].\end{aligned}\tag{3.9}$$

Since the multiplicative mapping $M_2: I\to\mathbb{R}$ is different from identity mapping, it follows from Results 2.1 and 2.3, that $\left[1-\sum\limits_{t=1}^{n}M_2(s_t)\right]$ does not vanish identically on $\Gamma_n$. Thus there exists a probability distribution $(s_1^*,\dots,s_n^*)\in\Gamma_n$ so that $1-\sum\limits_{t=1}^{n}M_2(s_t^*)\neq 0$. Using this in (3.9), we get

$$\overline{A}_3(p_1,\dots,p_n;x)=a_0(x)\left[1-\sum_{i=1}^{n}M_2(p_i)\right]+x\overline{A}_3(p_1,\dots,p_n;1),\tag{3.10}$$

where $a_0:\mathbb{R}\to\mathbb{R}$ is a mapping defined as

$$a_0(x)=[\overline{A}_3(s_1^*,\dots,s_n^*;x)-x\overline{A}_3(s_1^*,\dots,s_n^*;1)]\left[1-\sum_{t=1}^{n}M_2(s_t^*)\right]^{-1}.$$

Clearly the mapping $a_0:\mathbb{R}\to\mathbb{R}$ is additive with $a_0(1)=0$. On substituting the value of '$\overline{A}_3(p_1,\dots,p_n;1)$' calculated from (3.6) (for $q=1$) in (3.10), it follows that

$$\begin{aligned}\overline{A}_3(p_1,\dots,p_n;x)=a_0(x)\left[1-\sum_{i=1}^{n}M_2(p_i)\right]+x\Bigg[&mG(0)\sum_{i=1}^{n}M_2(p_i)+\sum_{i=1}^{n}\overline{B}_2(p_i)\\ &+\overline{A}_2(1)-nG(0)-n\overline{B}_1(0)-\overline{B}_3(p_1,\dots,p_n;1)\Bigg].\end{aligned}\tag{3.11}$$

From (3.11) and (3.8), we obtain

$$\begin{aligned}&\Bigg\{\overline{B}_3(s_1,\dots,s_n;x)-\sum_{t=1}^{n}\overline{B}_1(s_tx)+n\overline{B}_1(0)+x\Bigg[\sum_{t=1}^{n}\overline{B}_2(s_t)+\overline{A}_2(1)-\overline{A}_1(1)\\ &\quad+(m-n)G(0)-n\overline{B}_1(0)-\overline{B}_3(s_1,\dots,s_n;1)\Bigg]\Bigg\}\sum_{i=1}^{n}M_2(p_i)\\ &=\Bigg\{\overline{B}_3(p_1,\dots,p_n;x)-\sum_{i=1}^{n}\overline{B}_1(p_ix)+n\overline{B}_1(0)+x\Bigg[\sum_{i=1}^{n}\overline{B}_2(p_i)+\overline{A}_2(1)-\overline{A}_1(1)+(m-n)G(0)\\ &\quad-n\overline{B}_1(0)-\overline{B}_3(p_1,\dots,p_n;1)\Bigg]\Bigg\}\sum_{t=1}^{n}M_2(s_t)+\sum_{i=1}^{n}\overline{B}_3(s_1,\dots,s_n;p_ix)-\sum_{t=1}^{n}\overline{B}_3(p_1,\dots,p_n;s_tx)\\ &\quad+x\left[\overline{B}_3(p_1,\dots,p_n;1)-\overline{B}_3(s_1,\dots,s_n;1)-\sum_{i=1}^{n}\overline{B}_2(p_i)+\sum_{t=1}^{n}\overline{B}_2(s_t)\right]\end{aligned}\tag{3.12}$$

for all $x\in I$, $(p_1,\dots,p_n)\in\Gamma_n$ and $(s_1,\dots,s_n)\in\Gamma_n$. Now, we divide our discussion into two cases.

**Case 1.1.** Coefficient of $\sum_{i=1}^{n} M_2(p_i)$ vanishes identically. In this case, functional equation (3.12) yields

$$\begin{aligned}\overline{B}_3(s_1,\ldots,s_n;x)=&\sum_{t=1}^{n}\overline{B}_1(s_tx)-n\overline{B}_1(0)-x\Bigg[\sum_{t=1}^{n}\overline{B}_2(s_t)+\overline{A}_2(1)-\overline{A}_1(1)\\&+(m-n)G(0)-n\overline{B}_1(0)-\overline{B}_3(s_1,\ldots,s_n;1)\Bigg].\end{aligned}\tag{3.13}$$

With the aid of (3.11) and (3.13), functional equation (3.6) reduces to

$$\begin{aligned}&\sum_{i=1}^{n}\overline{G}(p_iq)-\overline{G}(q)\sum_{i=1}^{n}M_2(p_i)-M_2(q)\sum_{i=1}^{n}\overline{G}(p_i)+M_2(q)\Bigg[\sum_{i=1}^{n}\overline{B}_2(p_i)\\&\quad-\sum_{i=1}^{n}\overline{B}_1(p_i)+\overline{A}_2(1)-\overline{A}_1(1)+(m-n)g(0)\Bigg]=0,\end{aligned}\tag{3.14}$$

where $\overline{G}: I\to\mathbb{R}$ is defined as

$$\overline{G}(x)=G(x)-G(0)-a_0(x)+mG(0)x\tag{3.15}$$

for all $x\in I$. Clearly $\overline{G}(0)=0$ and $\overline{G}(1)=0$. Applying Result 2.1 on (3.14), there exists a mapping $E:\mathbb{R}\times I\to\mathbb{R}$, additive in the first variable such that

$$\begin{aligned}&\overline{G}(pq)-M_2(p)\overline{G}(q)-M_2(q)\overline{G}(p)+M_2(q)\big[B_2^*(p)-B_1^*(p)+[\,\overline{A}_2(1)\\&\quad-\overline{A}_1(1)+(m-n)g(0)]p\big]=E(p;q)\end{aligned}\tag{3.16}$$

with $E(1;q)=-nM_2(q)[g(0)-h(0)]$ (follows by using the expressions $\overline{B}_1(x)=B_1^*(x)+f(0)-g(0), B_1^*(0)=0$ and $\overline{B}_2(x)=B_2^*(x)+f(0)-h(0), B_2^*(0)=0$). Also for $q=1$, (3.16) gives

$$B_2^*(p)-B_1^*(p)+[\,\overline{A}_2(1)-\overline{A}_1(1)+(m-n)g(0)]p=E(p;1).\tag{3.17}$$

The left hand side of (3.17) is bounded on I, while the right hand side is additive in first variable 'p'. Thus by Result 2.5, it follows that $E(p;1)=pE(1;1)$. Consequently we get, $B_2^*(p)-B_1^*(p)=p(B_2^*(1)-B_1^*(1))$. Thus (3.16) can be written in the form

$$\overline{G}(pq)-M_2(p)\overline{G}(q)-M_2(q)\overline{G}(p)=E(p;q)-pM_2(q)[B_2^*(1)-B_1^*(1)+\overline{A}_2(1)-\overline{A}_1(1)+(m-n)g(0)].\tag{3.18}$$

Also, it can be easily verified from (3.18) that

$$\begin{aligned}&E\,(r;pq)-rM_2(pq)[B_2^*(1)-B_1^*(1)+\overline{A}_2(1)-\overline{A}_1(1)+(m-n)g(0)]\\&\quad+M_2(r)\{E\,(p;q)-pM_2(q)[B_2^*(1)-B_1^*(1)+\overline{A}_2(1)-\overline{A}_1(1)+(m-n)g(0)]\}\\&=E\,(rp;q)-rpM_2(q)[B_2^*(1)-B_1^*(1)+\overline{A}_2(1)-\overline{A}_1(1)+(m-n)g(0)]\\&\quad+M_2(q)\{E\,(r;p)-rM_2(p)[B_2^*(1)-B_1^*(1)+\overline{A}_2(1)-\overline{A}_1(1)+(m-n)g(0)]\}\end{aligned}\tag{3.19}$$

for all $p\in I$, $q\in I$, $r\in I$. Now, we assert that the right hand side of (3.18), i.e., $E(p;q)-pM_2(q)[B_2^*(1)-B_1^*(1)+\overline{A}_2(1)-\overline{A}_1(1)+(m-n)g(0)]$ vanishes identically on $I\times I$. To the contrary suppose that it does not vanish and there exists some $p^*\in I$, $q^*\in I$ such that $E(p^*;q^*)-p^*M_2(q^*)[B_2^*(1)-B_1^*(1)+\overline{A}_2(1)-\overline{A}_1(1)+(m-n)g(0)]\neq 0$. Then from (3.19), it follows that

$$\begin{aligned}M_2(r)=&\Big\{E\,(p^*;q^*)-p^*M_2(q^*)[B_2^*(1)-B_1^*(1)+\overline{A}_2(1)-\overline{A}_1(1)+(m-n)g(0)]\Big\}^{-1}\\&\times\Big\{E\,(rp^*;q^*)-rp^*M_2(q^*)[B_2^*(1)-B_1^*(1)+\overline{A}_2(1)-\overline{A}_1(1)+(m-n)g(0)]\end{aligned}$$

$$+M_2(q^*)\{E(r;p^*) - rM_2(p^*)[B_2^*(1) - B_1^*(1) + \overline{A}_2(1) - \overline{A}_1(1) + (m-n)g(0)]\}$$
$$\left. - E(r;p^*q^*) + rM_2(p^*q^*)[B_2^*(1) - B_1^*(1) + \overline{A}_2(1) - \overline{A}_1(1) + (m-n)g(0)]\right\}$$

for all $r \in I$. The additivity on the right hand side implies that mapping $M_2 : I \to \mathbb{R}$ is additive. Using this additivity we obtain $1 \neq \sum_{t=1}^{n} M_2(s_t) = 1$, a contradiction and so our assertion follows. Consequently (3.18) reduces to $\overline{G}(pq) - M_2(p)\overline{G}(q) - M_2(q)\overline{G}(p) = 0$ whose general solution is $\overline{G}(p) = M_2(p)\ell(p)$ for all $p \in I$; $\ell : I \to \mathbb{R}$ being a logarithmic mapping. Thus, we obtain ($\beta_1$) (ii) from (3.5) and (3.15) by defining an additive mapping $a_2 : \mathbb{R} \to \mathbb{R}$ as $a_2(x) = a_0(x) - mg(0)x$ and bounded mapping $b_2 : \mathbb{R} \to \mathbb{R}$ as $b_2(x)=0$. Further from ($\beta_1$) (ii), (3.1) and (3.3) (with $M_1 = M_2$); ($\beta_1$) (i) and ($\beta_1$) (iii) follows by defining additive mappings $a_1 : \mathbb{R} \to \mathbb{R}$ as $a_1(x) = a_2(x) + \overline{A}_1(x)$; $a_3 : \mathbb{R} \to \mathbb{R}$ as $a_3(x) = a_2(x) - \overline{A}_2(x) + \overline{A}_1(1)x$ and bounded mappings $b_1 : \mathbb{R} \to \mathbb{R}$ as $b_1(x) = b_2(x) + B_1^*(x)$ where $|b_1(x)| \leqslant 18\varepsilon$ with $b_1(0) = 0$ and $b_3 : \mathbb{R} \to \mathbb{R}$ as $b_3(x) = b_2(x) - B_2^*(x) + B_1^*(x)$ where $|b_3(x)| \leqslant 36\varepsilon$ with $b_3(0) = 0$.

**Case 1.2.** Coefficient of $\sum_{i=1}^{n} M_2(p_i)$ does not vanish identically. In this case, the boundedness of the mappings $\overline{B}_3$, $\overline{B}_2$ and $\overline{B}_1$ in functional equation (3.12) yields $\left|\sum_{i=1}^{n} M_2(p_i)\right| \leqslant \varepsilon$ for all $(p_1, \dots, p_n) \in \Gamma_n$ and some positive real number $\varepsilon$. Then by Result 2.2 followed by Result 2.3, we get $M_2(p) = p^\beta$ for some positive real number $\beta \neq 1$ satisfying (1.3). On substituting $M_1(p) = M_2(p) = p^\beta$ in functional inequality (B) and proceeding similarly the functional equations (3.6) and (3.11) with $a_0(1) = 0$, gives

$$\begin{aligned}\sum_{i=1}^{n} \overline{G}(p_i q) - q^\beta \sum_{i=1}^{n} \overline{G}(p_i) - \overline{G}(q) \sum_{i=1}^{n} p_i^\beta &= \overline{B}_3(p_1, \dots, p_n; q) - \sum_{i=1}^{n} \overline{B}_1(p_i q) \\ &+ n\overline{B}_1(0) + q^\beta \left[ -\sum_{i=1}^{n} \overline{B}_2(p_i) + \sum_{i=1}^{n} \overline{B}_1(p_i) - \overline{A}_2(1) + \overline{A}_1(1) - (m-n)g(0) \right] \\ &+ q \left[ \sum_{i=1}^{n} \overline{B}_2(p_i) + \overline{A}_2(1) - \overline{A}_1(1) + (m-n)g(0) - n\overline{B}_1(0) - \overline{B}_3(p_1, \dots, p_n; 1) \right],\end{aligned} \tag{3.20}$$

where $\overline{G} : I \to \mathbb{R}$ is a mapping defined as in (3.15). Further we observe that the right hand side of functional equation (3.20) is bounded by $36\varepsilon(3 + 2n)$ (follows from (3.3) and the expressions $\overline{B}_1(x) = B_1^*(x) + f(0) - g(0)$, $B_1^*(0) = 0$ with $|B_1^*(x)| \leqslant 18\varepsilon$; $\overline{B}_2(x) = B_2^*(x) + f(0) - h(0)$, $B_2^*(0) = 0$ with $|B_2^*(x)| \leqslant 18\varepsilon$). By Result 2.2, there exists a mapping $\overline{A}_4 : \mathbb{R} \times I \to \mathbb{R}$, additive in the first variable and a mapping $\overline{B}_4 : \mathbb{R} \times I \to \mathbb{R}$, bounded in the first variable by $648\varepsilon(3 + 2n)$ with $\overline{B}_4(0; q) = 0$ satisfying

$$\overline{G}(pq) - q^\beta\, \overline{G}(p) - p^\beta\, \overline{G}(q) = \overline{A}_4(p; q) + \overline{B}_4(p; q) \tag{3.21}$$

for all $p \in I$, $q \in I$. Define a mapping $\overline{E} : I \times I \to \mathbb{R}$ as

$$\overline{E}(p; q) = \overline{G}(pq) - q^\beta\, \overline{G}(p) - p^\beta\, \overline{G}(q) \tag{3.22}$$

for all $p \in I$, $q \in I$. It can easily be verified from (3.22) that

$$r^\beta\, \overline{E}(p; q) + \overline{E}(r; pq) = q^\beta\, \overline{E}(r; p) + \overline{E}(rp; q) \tag{3.23}$$

for all $p \in I$, $q \in I$ and $r \in I$. From (3.21), (3.22), and (3.23), it follows that

$$\begin{aligned}&\overline{A}_4(r; pq) - q^\beta\, \overline{A}_4(r; p) - \overline{A}_4(rp; q) \\ &\quad = q^\beta\, \overline{B}_4(r; p) + \overline{B}_4(rp; q) - r^\beta\, \overline{A}_4(p; q) - r^\beta\, \overline{B}_4(p; q) - \overline{B}_4(r; pq).\end{aligned} \tag{3.24}$$

The left hand side is additive in $r \in I$, while its right hand side is bounded on I. Consequently by Result 2.5, we find that left hand side is linear, i.e.,

$$\overline{A}_4(r;pq) - q^{\beta}\,\overline{A}_4(r;p) - \overline{A}_4(rp;q) = r[\,\overline{A}_4(1;pq) - q^{\beta}\,\overline{A}_4(1;p) - \overline{A}_4(p;q)]. \tag{3.25}$$

Now, on replacing $r$ by 1 in (3.24) and using $1^{\beta} := 1$, we get

$$\overline{A}_4(1;pq) - q^{\beta}\,\overline{A}_4(1;p) = q^{\beta}\,\overline{B}_4(1;p) - \overline{B}_4(1;pq) \tag{3.26}$$

for all $p \in I$, $q \in I$. From (3.24), (3.25), and (3.26), we obtain

$$\begin{aligned}(r^{\beta} - r)\overline{A}_4(p;q) = r\overline{B}_4(1;pq) - rq^{\beta}\,\overline{B}_4(1;p) - \overline{B}_4(r;pq)\\ - r^{\beta}\,\overline{B}_4(p;q) + \overline{B}_4(rp;q) + q^{\beta}\,\overline{B}_4(r;p)\end{aligned} \tag{3.27}$$

for all $p \in I$, $q \in I$ and $r \in I$. In view of our assumption that $0 < \beta \in \mathbb{R}$ with $\beta \neq 1$, (3.27) yield that additive mapping $\overline{A}_4(p;q)$ is bounded in the first variable on I. Thus by Result 2.5, we conclude that mapping $\overline{A}_4(p;q)$ must be linear which implies $\overline{A}_4(p;q) = p\overline{A}_4(1;q)$ for all $p \in I$, $q \in I$. However equation (3.26) with $p = 1$ implies that mapping $q \to \overline{A}_4(1;q)$ is bounded by $648\varepsilon(3+2n)$ on I. Consequently (3.21) gives

$$|\overline{G}(pq) - q^{\beta}\,\overline{G}(p) - p^{\beta}\,\overline{G}(q)| \leqslant 1296\varepsilon(3+2n)$$

for all $p \in I$, $q \in I$. By Result 2.6, there exists a logarithmic mapping $\ell : I \to \mathbb{R}$ and a bounded mapping $b_2 : \mathbb{R} \to \mathbb{R}$ satisfying $|b_2(p)| \leqslant 4e\{1296\varepsilon(3+2n)\}$ such that $\overline{G}(p) = p^{\beta}\ell(p) + b_2(p)$. Hence from (3.5) and (3.15) we obtain ($\beta_1$) (ii) (with $M_2(p) = p^{\beta}$) again by defining an additive mapping $a_2 : \mathbb{R} \to \mathbb{R}$ as $a_2(x) = a_0(x) - mg(0)x$. Moreover from ($\beta_1$) (ii), (3.1) and (3.3) (with $M_1 = M_2$); ($\beta_1$) (i) and ($\beta_1$) (iii) (with $M_2(p) = p^{\beta}$) follows by defining the additive mappings as in the previous case and the bounded mappings as $b_1 : \mathbb{R} \to \mathbb{R}$ as $b_1(x) = b_2(x) + B_1^*(x)$ where $|b_1(x)| \leqslant 4e\{1296\varepsilon(3+2n)\} + 18\varepsilon$; $b_3 : \mathbb{R} \to \mathbb{R}$ as $b_3(x) = b_2(x) - B_2^*(x) + B_1^*(x)$, where $|b_3(x)| \leqslant 4e\{1296\varepsilon(3+2n)\} + 36\varepsilon$.

**Case 2.** $M_1 \neq M_2$. In this case, there will be no loss of generality in assuming $n \geqslant m$. Letting $p_{m+1} = \cdots = p_n = 0$ in (3.4). We get

$$\begin{aligned}&\Bigg|\sum_{i=1}^{m}\sum_{j=1}^{m} G(p_iq_j) - \sum_{i=1}^{m} M_1(p_i)\sum_{j=1}^{m} G(q_j) - \sum_{j=1}^{m} M_2(q_j)\sum_{i=1}^{m} G(p_i) + \sum_{j=1}^{m} M_2(q_j)\\ &\times\left[\sum_{i=1}^{m}\overline{B}_2(p_i) - \sum_{i=1}^{m}\overline{B}_1(p_i) + \overline{A}_2(1) - \overline{A}_1(1) + (m-n)\left[G(0) - \overline{B}_2(0) + \overline{B}_1(0)\right]\right]\\ &+\sum_{i=1}^{m}\sum_{j=1}^{m}\overline{B}_1(p_iq_j) + \overline{A}_1(1) + m(n-m)\left[G(0) + \overline{B}_1(0)\right]\Bigg| \leqslant \varepsilon\end{aligned} \tag{3.28}$$

for all $(p_1, \ldots, p_m) \in \Gamma_m$, $(q_1, \ldots, q_m) \in \Gamma_m$. Now on interchanging $p_i$ and $q_j$, $i = 1, \ldots, m$; $j = 1, \ldots, m$ in the functional inequality (3.28), we have

$$\begin{aligned}&\Bigg|\sum_{i=1}^{m}\sum_{j=1}^{m} G(p_iq_j) - \sum_{j=1}^{m} M_1(q_j)\sum_{i=1}^{m} G(p_i) - \sum_{i=1}^{m} M_2(p_i)\sum_{j=1}^{m} G(q_j) + \sum_{i=1}^{m} M_2(p_i)\\ &\times\left[\sum_{j=1}^{m}\overline{B}_2(q_j) - \sum_{j=1}^{m}\overline{B}_1(q_j) + \overline{A}_2(1) - \overline{A}_1(1) + (m-n)\left[G(0) - \overline{B}_2(0) + \overline{B}_1(0)\right]\right]\\ &+\sum_{i=1}^{m}\sum_{j=1}^{m}\overline{B}_1(p_iq_j) + \overline{A}_1(1) + m(n-m)\left[G(0) + \overline{B}_1(0)\right]\Bigg| \leqslant \varepsilon.\end{aligned} \tag{3.29}$$

Applying triangle inequality to functional inequalities (3.28) and (3.29), we obtain

$$\begin{aligned}\Bigg|&\left[\sum_{j=1}^{m} M_1(q_j)-\sum_{j=1}^{m} M_2(q_j)\right]\sum_{i=1}^{m} G(p_i)-\left[\sum_{i=1}^{m} M_1(p_i)-\sum_{i=1}^{m} M_2(p_i)\right]\sum_{j=1}^{m} G(q_j)\\ &+\sum_{i=1}^{m} M_2(p_i)\left[\sum_{j=1}^{m} \overline{B}_1(q_j)-\sum_{j=1}^{m} \overline{B}_2(q_j)\right]-\sum_{j=1}^{m} M_2(q_j)\left[\sum_{i=1}^{m} \overline{B}_1(p_i)-\sum_{i=1}^{m} \overline{B}_2(p_i)\right]\\ &+\Big[-\overline{A}_2(1)+\overline{A}_1(1)-(m-n)\left[G(0)-\overline{B}_2(0)+\overline{B}_1(0)\right]\Big]\left[\sum_{i=1}^{m} M_2(p_i)-\sum_{j=1}^{m} M_2(q_j)\right]\Bigg| \leqslant 2\varepsilon.\end{aligned} \tag{3.30}$$

Since $M_1 \neq M_2$, therefore we have $\left[\sum_{j=1}^{m} M_1(q_j)-\sum_{j=1}^{m} M_2(q_j)\right]$ does not vanish identically on $\Gamma_m$ (follows from Results 2.1 and 2.4). Consequently there exists a probability distribution $(q_1^*,\dots,q_m^*)\in\Gamma_m$ such that $\left[\sum_{j=1}^{m} M_1(q_j^*)-\sum_{j=1}^{m} M_2(q_j^*)\right]\neq 0$. Then (3.30) gives

$$\left|\sum_{i=1}^{m} G(p_i)-c_1\sum_{i=1}^{m} M_1(p_i)+c_2\sum_{i=1}^{m} M_2(p_i)-c_3\left[\sum_{i=1}^{m} \overline{B}_1(p_i)-\sum_{i=1}^{m} \overline{B}_2(p_i)\right]-c_4\right| \leqslant 2\varepsilon c,$$

where $0\neq c=\left[\sum_{j=1}^{m} M_1(q_j^*)-\sum_{j=1}^{m} M_2(q_j^*)\right]^{-1}\in\mathbb{R}$; $c_1, c_2, c_3, c_4$ are arbitrary real constants and $(p_1,\dots,p_m)\in\Gamma_m$.

By Result 2.2, there exists an additive mapping $\overline{A}_5:\mathbb{R}\to\mathbb{R}$ and a mapping $\overline{B}_5:\mathbb{R}\to\mathbb{R}$ such that $|\overline{B}_5(p)|\leqslant 36\varepsilon c$, $\overline{B}_5(0)=0$, and

$$G(p)-G(0)=c_1M_1(p)-c_2M_2(p)+c_3\left[\overline{B}_1(p)-\overline{B}_2(p)-\overline{B}_1(0)+\overline{B}_2(0)\right]+c_4p+\overline{A}_5(p)+\overline{B}_5(p).$$

Then again on using the expressions $\overline{B}_1(x)=B_1^*(x)+f(0)-g(0), B_1^*(0)=0$ with $|B_1^*(x)|\leqslant 18\varepsilon$; $\overline{B}_2(x)=B_2^*(x)+f(0)-h(0), B_2^*(0)=0$ with $|B_2^*(x)|\leqslant 18\varepsilon$ and taking $d:=c_1$, $\bar{d}:=d-c_2$, the solution ($\beta_2$) (ii) holds from (3.5) by defining an additive mapping $a_5:\mathbb{R}\to\mathbb{R}$ as $a_5(x)=\overline{A}_5(x)+c_4x$; a bounded mapping $b_5:\mathbb{R}\to\mathbb{R}$ as $b_5(x)=\overline{B}_5(x)+c_3[B_1^*(x)-B_2^*(x)]+\bar{d}M_2(x)$ where $b_5(0)=0$ and $|b_5(x)|\leqslant 36\varepsilon(c+c_3)+|\bar{d}M_2(x)|\leqslant 36\varepsilon(c+c_3)+|\bar{d}|$ (follows from Result 2.3). Further from (3.1) and (3.3), ($\beta_2$) (i) and ($\beta_2$) (iii) hold by defining the additive mappings $a_4:\mathbb{R}\to\mathbb{R}$ as $a_4(x)=a_5(x)+\overline{A}_1(x)$; $a_6:\mathbb{R}\to\mathbb{R}$ as $a_6(x)=a_5(x)-\overline{A}_2(x)+\overline{A}_1(1)x$ and bounded mappings $b_4:\mathbb{R}\to\mathbb{R}$ as $b_4(x)=b_5(x)+B_1^*(x)$ where $b_4(0)=0$ and $|b_4(x)|\leqslant 18\varepsilon(2c+2c_3+1)+|\bar{d}|$; $b_6:\mathbb{R}\to\mathbb{R}$ as $b_6(x)=b_5(x)-B_2^*(x)+B_1^*(x)$ where $b_6(0)=0$ and $|b_6(x)|\leqslant 36\varepsilon(c+c_3+1)+|\bar{d}|$. This completes the proof of Theorem. □

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