



On hyperstability of Cauchy functional equation in $(2, \gamma)$ -Banach spaces



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Abstract

In this paper, we generalize the recent hyperstability results obtained by Brzdęk and concerning the Cauchy functional equation

$$f(x_1 + x_2) = f(x_1) + f(x_2).$$

The obtained results are in $(2, \gamma)$ -Banach spaces. The main tool used in the analysis is some fixed point theorem.

Keywords: Hyperstability, Cauchy equation, additive function, restricted domain, 2-Banach spaces.

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1. Introduction

People working in almost all areas of mathematical analysis can ask the following question: When is it true that a mathematical object (e.g. function) satisfying a certain property approximately must be close to some object satisfying the property exactly? In the case of equations (e.g., difference, differential, functional, integral), we can particularly ask whether functions satisfying an equation in an approximate way must be (somehow) close to a solution of the equation. This could be the starting point of the stability problem of various equations.

Stability theory popped up as a result of Ulam's famous talk in 1940 at the University of Wisconsin (see [23]), where (among others) he presented an open problem (see, e.g., [24] for more details). It should be noted that many famous mathematicians interacted with Ulam's problem. In particular, in 1941, Hyers (see [23]) gave some answer to Ulam's problem concerning Banach spaces. The result obtained by Hyers was generalized further by many authors see e.g. [7, 9–13, 15, 23–26, 28, 30, 31].

In particular, in 1950, Aoki in [2] generalized the result obtained by Hyers for approximate additive mappings and subsequently in 1978 Rassias [34] generalized the result further for approximate linear mappings. The author in [17] investigates the stability of the functional equation of the p-Wright affine functions in $(2, \alpha)$ -Banach spaces. In [29] the authors investigate some stability and hyperstability results

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for some Cauchy-Jensen functional equation in 2-Banach spaces by using Brzdęk's fixed point approach. In 2015, Zhang in [36] obtained hypersatbility results of the generalized linear functional equation by employing Brzdęk's fixed point theory. In [33] the authors generalized the results that have been obtained in [36]. In other words, they obtained hyperstability results of the generalized linear functional equation in several variables. The author in [35] proved some hyperstability results for the Drygas functional equation on a restricted domain.

Roughly speaking, a given functional equation (see, e.g., [1, 16]) \mathcal{FE} is called stable in some class of functions if any function from that class, satisfying \mathcal{FE} approximately (in some sense), is near (in some way) to an exact solution of \mathcal{FE} . It is well-known that the concept of an approximate solution and the idea of nearness of two functions can be understood in numerous nonstandard ways, depending on the needs and tools available in a particular situation. One of such non-classical measures of a distance can be introduced by the notion of a 2-norm. As far as we know Gähler in [19] pioneered the notion of the 2-normed space, and it seems that the first work on the Hyers-Ulam stability of functional equations in complete 2-normed spaces (that is, 2-Banach spaces) is [21] (see also, e.g., [14, 32] for some details concerning such stability in 2-Banach spaces).

This article is organized as follows. In Section 2 we recall some basic definitions and present the functional equation of our interest; in Section 3 we introduce the fixed point theorem that is our main tool in the proofs; in Section 4 we investigate the hyperstability of the Cauchy functional equation, and in Section 5 we conclude our work.

2. Preliminaries

Throughout the article, we use \mathbb{R} to denote the set of reals, \mathbb{R}_+ the set of nonnegative reals, \mathbb{N} to denote the set of positive integers, \mathbb{N}_0 to denote $\mathbb{N} \cup \{0\}$, and A^B denotes the family of all functions mapping a set B into a set A . Let us recall first (see, for instance, [18]) some definitions.

Definition 2.1. By a linear 2-normed space we mean a pair $(X, \|\cdot, \cdot\|)$ such that X is at least a two-dimensional real linear space and

$$\|\cdot, \cdot\| : X \times X \rightarrow \mathbb{R}$$

is a function satisfying the following conditions:

- (N1) $\|a, b\| = 0$ if and only if a and b are linearly dependent;
- (N2) $\|a, b\| = \|b, a\|$ for $a, b \in X$;
- (N3) $\|a, b + c\| \leq \|a, b\| + \|a, c\|$ for $a, b, c \in X$;
- (N4) $\|\lambda a, b\| = |\lambda| \|a, b\|$ for $\lambda \in \mathbb{R}$ and $a, b \in X$.

A generalized version of a linear 2-normed spaces is the $(2, \gamma)$ -normed space defined in the following manner.

Definition 2.2. Let γ be a fixed real number with $0 < \gamma \leq 1$, and let X be a linear space over \mathbb{K} with $\dim X > 1$. A function

$$\|\cdot, \cdot\|_\gamma : X \times X \rightarrow \mathbb{R}_+$$

is called a $(2, \gamma)$ -norm on X if and only if it satisfies the following conditions:

- (G1) $\|x_1, x_2\|_\gamma = 0$ if and only if x_1 and x_2 are linearly dependent;
- (G2) $\|x_1, x_2\|_\gamma = \|x_2, x_1\|_\gamma$ for $x_1, x_2 \in X$;
- (G3) $\|x_1, x_2 + x_3\|_\gamma \leq \|x_1, x_2\|_\gamma + \|x_1, x_3\|_\gamma$ for $x_i \in X, i = 1, 2, 3$;
- (G4) $\|\lambda x_1, x_2\|_\gamma = |\lambda|^\gamma \|x_1, x_2\|_\gamma$ for $\lambda \in \mathbb{R}$ and $x_1, x_2 \in X$.

The pair $(X, \|\cdot, \cdot\|_\gamma)$ is called a $(2, \gamma)$ -normed space.

Definition 2.3. A sequence $(x_n)_{n \in \mathbb{N}}$ of elements of a linear $(2, \gamma)$ -normed space X is called a *Cauchy*

sequence if there are linearly independent $y, z \in X$ such that

$$\lim_{n,m \rightarrow \infty} \|x_n - x_m, z\|_\gamma = 0 = \lim_{n,m \rightarrow \infty} \|x_n - x_m, y\|_\gamma,$$

whereas $(x_n)_{n \in \mathbb{N}}$ is said to be convergent if there exists an $x \in X$ (called a limit of this sequence and denoted by $\lim_{n \rightarrow \infty} x_n$) with

$$\lim_{n,m \rightarrow \infty} \|x_n - x, y\|_\gamma = 0, y \in X.$$

Definition 2.4. A linear $(2, \gamma)$ -normed space in which every Cauchy sequence is convergent is called a $(2, \gamma)$ -Banach space.

Let us also mention that in linear $(2, \gamma)$ -normed spaces, every convergent sequence has exactly one limit and the standard properties of the limit of a sum and a scalar product are valid. Next, it is easily seen that we have the following property.

Lemma 2.5. If X is a linear $(2, \gamma)$ -normed space, $x, y, z \in X$, y, z are linearly independent, and

$$\|x, y\|_\gamma = 0 = \|x, z\|_\gamma,$$

then $x = 0$.

Let us yet recall a version of a lemma from [32].

Lemma 2.6. If X is a linear $(2, \gamma)$ -normed space and $(x_n)_{n \in \mathbb{N}}$ is a convergent sequence of elements of X , then

$$\lim_{n \rightarrow \infty} \|x_n, z\|_\gamma = \|\lim_{n \rightarrow \infty} x_n, z\|_\gamma, z \in X.$$

We introduce a simple example of a $(2, \gamma)$ -normed space.

Example 2.7. For $x = (x_1, x_2), y = (y_1, y_2) \in X = \mathbb{R}^2$, the $(2, \gamma)$ -norm on X is defined by

$$\|x, y\|_\gamma = |x_1 y_2 - x_2 y_1|^\gamma,$$

where γ is a fixed real number with $0 < \gamma \leq 1$.

In this article our target is to generalize the following two results obtained in [5, 6] and concerning the hyperstability of the Cauchy functional equation

$$f(x_1 + x_2) = f(x_1) + f(x_2). \quad (2.1)$$

Theorem 2.8. Let E_1 and E_2 be normed spaces, $X \subset E_1 \setminus \{0\}$ be nonempty, $c \geq 0$ and $p < 0$. Assume that there exists a positive integer m_0 with

$$-x, nx \in X, \quad x \in X, n \in \mathbb{N}, n \geq m_0.$$

Then every operator $g : E_1 \rightarrow E_2$ with

$$\|g(x_1 + x_2) - g(x_1) - g(x_2)\| \leq c(\|x_1\|^p + \|x_2\|^p), \quad x_1, x_2 \in X, x_1 + x_2 \in X,$$

is additive on X , i.e.,

$$g(x_1 + x_2) = g(x_1) + g(x_2), \quad x_1, x_2 \in X, x_1 + x_2 \in X. \quad (2.2)$$

Theorem 2.9. Let E_1 and E_2 be normed spaces, and $X \subset E_1 \setminus \{0\}$ be nonempty. Take $c \geq 0$ and let p, q be real numbers with $p + q < 0$. Assume that there exists a positive integer m_0 with

$$x_1, nx_1 \in X, \quad n \in \mathbb{N}, n \geq m_0.$$

Then every operator $g : E_1 \rightarrow E_2$, satisfying the inequality

$$\|g(x_1 + x_2) - g(x_1) - g(x_2)\| \leq c\|x_1\|^p\|x_2\|^q, \quad x_1, x_2 \in X, x_1 + x_2 \in X,$$

is additive on X , that is, fulfills condition (2.2).

That is, we obtain analogous results but in $(2, \gamma)$ -Banach spaces. The method of the proof of the main result corresponds to some observations in [9] and the main tool in it is a fixed point theorem.

3. Fixed point theorem

In order to use the fixed point approach we need to assume the following three assumptions.

(A1) E is a nonempty set, $(Y, \|\cdot, \cdot\|_\gamma)$ is a $(2, \gamma)$ -Banach space, Y_0 is a subset of Y containing two linearly independent vectors, $j \in \mathbb{N}$,

$$f_i : E \rightarrow E, \quad g_i : Y_0 \rightarrow Y_0,$$

and

$$L_i : E \times Y_0 \rightarrow \mathbb{R}_+ \text{ for } i = 1, \dots, j.$$

(A2) $T : Y^E \rightarrow Y^E$ is an operator satisfying the inequality

$$\|T\xi(x) - T\mu(x), y\|_\gamma \leq \sum_{i=1}^j L_i(x, y) \|\xi(f_i(x)) - \mu(f_i(x)), g_i(y)\|_\gamma,$$

where $\xi, \mu \in Y^E$, $x \in E$, and $y \in Y_0$.

(A3) $\Lambda : \mathbb{R}^{E \times Y_0} \rightarrow \mathbb{R}^{E \times Y_0}$ is an operator defined by

$$\Lambda\delta(x, y) := \sum_{i=1}^j L_i(x, y) \delta(f_i(x), g_i(y)), \quad \delta \in \mathbb{R}^{E \times Y_0}, x \in E, y \in Y_0.$$

Now, its the position to present the main tool used in the investigation of the hyperstability, namely, a version of the fixed point theorem which is introduced in [9].

Theorem 3.1. *Let assumptions (A1)-(A3) hold and functions*

$$\varepsilon : E \times Y_0 \rightarrow \mathbb{R}_+ \text{ and } \varphi : E \rightarrow Y$$

fulfill the following two conditions:

$$\|T\varphi(x) - \varphi(x), y\|_\gamma \leq \varepsilon(x, y), \quad x \in E, y \in Y_0, \quad \varepsilon^*(x, y) := \sum_{i=1}^{\infty} (\Lambda^i \varepsilon)(x, y) < \infty, \quad x \in E, y \in Y_0.$$

Then, there exists a unique fixed point ψ of T for which

$$\|\varphi(x) - \psi(x), y\|_\gamma \leq \varepsilon^*(x, y), \quad x \in E, y \in Y_0.$$

Moreover,

$$\psi(x) = \lim_{l \rightarrow \infty} (T^l \varphi)(x), \quad x \in E.$$

4. Hyperstability results in $(2, \gamma)$ -Banach spaces

A functional equation \mathcal{D} is called *hyperstable* if any function f satisfying \mathcal{D} approximately (in some sense) must be actually a solution to \mathcal{D} (see e.g. [8, 22]). It should be noted that the first *hyperstability* result was published in [3] and concerned ring homomorphisms. However, the term *hyperstability* was used for the first time in [27] by Maksa and Páles. It should be noted that the most classical result concerning the hyperstability of the Cauchy equation (2.1) is Theorem 1.1 in both [5, 6], see also [2, 20] for more details. In this article one propose is to generalize the results obtained in [5] in $(2, \gamma)$ -Banach spaces in the following way.

Theorem 4.1. *Let $E_1 = E_2$ be $(2, \gamma)$ -Banach spaces, $X \subset E_1 \setminus \{0\}$ be nonempty, $c \geq 0, 0 < \gamma \leq 1$ and $p < 0$. Assume that there exists a positive integer m_0 with*

$$-x, nx \in X, \quad x \in X, n \in \mathbb{N}, n \geq m_0.$$

Then every operator $g : E_1 \rightarrow E_2$ with

$$\|g(x_1 + x_2) - g(x_1) - g(x_2), y\|_\gamma \leq c(\|x_1, y\|_\gamma^p + \|x_2, y\|_\gamma^p), \quad x_1, x_2 \in X, x_1 + x_2 \in X, y \in Y_0 \quad (4.1)$$

is additive on X , i.e.,

$$g(x_1 + x_2) = g(x_1) + g(x_2), \quad x_1, x_2 \in X, x_1 + x_2 \in X.$$

Proof. Let f denote the restriction of g to the set X . Take $m \in \mathbb{N}$ such that

$$(m + 1)^{\gamma p} + m^{\gamma p} < 1$$

and $m \geq m_0$. Note that (4.1), with x_1 replaced by $(m + 1)x_1$ and $x_2 = -mx_1$, gives

$$\|f(x_1) - f((m + 1)x_1) - f(-mx_1), y\|_\gamma \leq c((m + 1)^{\gamma p} + m^{\gamma p})\|x_1, y\|_\gamma^p, \quad x_1 \in X, y \in Y_0. \quad (4.2)$$

Define operators $T : E_2^X \rightarrow E_2^X$ by

$$T\xi(x) := \xi((m + 1)x) + \xi(-mx), \quad x \in X, \xi \in E_2^X,$$

and $\Lambda : \mathbb{R}_+^X \rightarrow \mathbb{R}_+^X$ by

$$\Lambda\delta(x) := \delta((m + 1)x) + \delta(-mx), \quad x \in X, \delta \in \mathbb{R}_+^X.$$

Then it is easily seen that Λ has the form described in (A3) with $j = 2$ and

$$f_1(x) = (m + 1)x, \quad f_2(x) = -mx, \quad L_1(x) = L_2(x) = 1$$

for $x \in X$. Further, (4.2) can be written in the form

$$\|Tf(x_1) - f(x_1), y\|_\gamma \leq c((m + 1)^{\gamma p} + m^{\gamma p})\|x_1, y\|_\gamma^p := \varepsilon(x_1, y), \quad x_1 \in X, y \in Y_0$$

and assumption (A2) holds, too. Note yet that we have

$$\begin{aligned} \varepsilon^*(x_1, y) &:= \sum_{n=0}^{\infty} \Lambda^n \varepsilon(x_1, y) \leq c((m + 1)^{\gamma p} + m^{\gamma p})\|x_1, y\|_\gamma^p \sum_{n=0}^{\infty} ((m + 1)^{\gamma p} + m^{\gamma p})^n \\ &= \frac{c((m + 1)^{\gamma p} + m^{\gamma p})\|x_1, y\|_\gamma^p}{1 - (m + 1)^{\gamma p} - m^{\gamma p}}, \quad x_1 \in X, y \in Y_0. \end{aligned}$$

Consequently, in view of Theorem 3.1, there exists a fixed point $T_* : X \rightarrow E_2$ of operator T such that

$$\|f(x_1) - T_*(x_1), y\|_\gamma \leq \frac{c((m + 1)^{\gamma p} + m^{\gamma p})\|x_1, y\|_\gamma^p}{1 - (m + 1)^{\gamma p} - m^{\gamma p}}, \quad x_1 \in X, y \in Y_0, \quad (4.3)$$

moreover, T_* is given by the formula

$$T_*(x_1) := \lim_{n \rightarrow \infty} (T^n f)(x_1), \quad x_1 \in X.$$

Clearly T_* is a solution to the equation

$$T(x) = T((m + 1)x) + T(-mx). \quad (4.4)$$

Now, we show that

$$\|T^n f(x_1 + x_2) - T^n f(x_1) - T^n f(x_2), y\|_\gamma \leq c((m + 1)^{\gamma p} + m^{\gamma p})(\|x_1, y\|_\gamma^p + \|x_2, y\|_\gamma^p), \quad y \in Y_0, \quad (4.5)$$

for every $n \in \mathbb{N}_0$ and $x_1, x_2 \in X$ with $x_1 + x_2 \in X$. Clearly, if $n = 0$, then (4.5) is simply (4.1). So, take $k \in \mathbb{N}_0$ and suppose that (4.5) holds for $n = k$ and every $x_1, x_2 \in X$ with $x_1 + x_2 \in X$. Then, for every $x_1, x_2 \in X$ with $x_1 + x_2 \in X$

$$\begin{aligned}
 & \|T^{k+1}f(x_1 + x_2) - T^{k+1}f(x_1) - T^{k+1}f(x_2), y\|_\gamma \\
 &= \|T^k f((m + 1)(x_1 + x_2)) + T^k f(-m(x_1 + x_2)) - T^k f((m + 1)x_1) \\
 &\quad - T^k f(-mx_1) - T^k f((m + 1)x_2) - T^k f(-mx_2), y\|_\gamma \\
 &\leq \|T^k f((m + 1)x_1 + (m + 1)x_2) - T^k f((m + 1)x_1) - T^k f((m + 1)x_2), y\|_\gamma \\
 &\quad + \|T^k f(-mx_1 - mx_2) - T^k f(-mx_1) - T^k f(-mx_2), y\|_\gamma \\
 &\leq c((m + 1)^{\gamma p} + m^{\gamma p})^k (\|(m + 1)x_1, y\|_\gamma^p + \|(m + 1)x_2, y\|_\gamma^p) \\
 &\quad + c((m + 1)^{\gamma p} + m^{\gamma p})^k (\|mx_1, y\|_\gamma^p + \|mx_2, y\|_\gamma^p) \\
 &= c((m + 1)^{\gamma p} + m^{\gamma p})^{k+1} (\|x_1, y\|_\gamma^p + \|x_2, y\|_\gamma^p).
 \end{aligned} \tag{4.6}$$

Thus, by induction we have shown that (4.6) holds for every $x_1, x_2 \in X$ with $x_1 + x_2 \in X$, $y \in Y_0$, and $n \in \mathbb{N}_0$. Letting $n \rightarrow \infty$ in (4.6), we obtain that T_* is additive on X . Next we prove that T_* is the unique operator that is additive on X and satisfies the condition

$$\sup_{x_1 \in X} \|f(x_1) - T_*(x_1), y\|_\gamma \|x_1, y\|_\gamma^{-p} < \infty.$$

So let $T_{**} : X \rightarrow E_2$ be additive on X and

$$\sup_{x_1 \in X} \|f(x_1) - T_{**}(x_1), y\|_\gamma \|x_1, y\|_\gamma^{-p} < \infty.$$

Then there is a real constant A such that

$$\|T_*(x_1) - T_{**}(x_1), y\|_\gamma \leq \|T_*(x_1) - f(x_1), y\|_\gamma + \|f(x_1) - T_{**}(x_1), y\|_\gamma < A\|x_1, y\|_\gamma^p, \tag{4.7}$$

where $x_1 \in X$ and $y \in Y_0$. We show that, for each $j \in \mathbb{N}_0$,

$$\|T_*(x_1) - T_{**}(x_1), y\|_\gamma \leq A\|x_1, y\|_\gamma^p \sum_{n=j}^{\infty} (m^{\gamma p} + (1 + m)^{\gamma p})^n, \quad x_1 \in X. \tag{4.8}$$

The case $j = 0$ is exactly (4.7). So fix $l \in \mathbb{N}_0$ and assume that (4.8) holds for $j = l$. Then, in view of (4.7),

$$\begin{aligned}
 \|T_* - T_{**}, y\|_\gamma &= \|T_*((m + 1)(x_1)) + T_*(-mx_1) - T_{**}((m + 1)(x_1)) - T_{**}(-mx_1), y\|_\gamma \\
 &\leq \|T_*((m + 1)(x_1)) - T_{**}((m + 1)(x_1)), y\|_\gamma + \|T_*(-mx_1) - T_{**}(-mx_1), y\|_\gamma \\
 &\leq A(\|(m + 1)x_1, y\|_\gamma^p + \|-mx_1, y\|_\gamma^p) \sum_{n=l}^{\infty} ((m + 1)^{\gamma p} + m^{\gamma p})^n \\
 &= A\|x_1, y\|_\gamma^p \sum_{n=l+1}^{\infty} ((m + 1)^{\gamma p} + m^{\gamma p})^n, \quad x_1 \in X, y \in Y_0,
 \end{aligned}$$

because T_* and T_{**} are solutions to equation (4.4). Thus we have shown that (4.8) holds true for every $j \in \mathbb{N}_0$. Now, letting $j \rightarrow \infty$ in (4.8) we get $T_* = T_{**}$. In this way we have proved that for each $m \in \mathbb{N}$, $m \geq m_0$, there exists a unique operator $T_* : X \rightarrow Y$ that is additive on X and satisfies (4.3). Note that the uniqueness of T_* means that for $x_1 \in X$, $k, n \in \mathbb{N}$, with $k, n \geq m_0$

$$\|f(x_1) - T_k(x_1), y\|_\gamma \leq \frac{c(n^{\gamma p} + (1 + n)^{\gamma p})\|x_1, y\|_\gamma^p}{1 - n^{\gamma p} - (n + 1)^{\gamma p}}, \quad y \in Y_0.$$

In fact, if $k, n \in \mathbb{N}, n \geq k \geq m_0$, then

$$\|f(x_1) - T_n(x_1), y\|_\gamma \leq \frac{c(n^{\gamma p} + (1+n)^{\gamma p})\|x_1, y\|_\gamma^p}{1 - n^{\gamma p} - (n+1)^{\gamma p}}, \leq \frac{c(k^{\gamma p} + (1+k)^{\gamma p})\|x_1, y\|_\gamma^p}{1 - k^{\gamma p} - (k+1)^{\gamma p}}, \quad x_1 \in X, y \in Y_0, \quad (4.9)$$

whence $T_n = T_k$, which implies (4.9). Letting $n \rightarrow \infty$ and fixing k in (4.9), we get $f = T_k$, which implies that f is additive on X . □

In the following theorem we obtain also a generalized version of Theorem 1.3 in [6]. The theorem in the generalized version takes the form.

Theorem 4.2. *Let $E_1 = E_2$ be $(2, \gamma)$ -Banach spaces, and $X \subset E_1 \setminus \{0\}$ be nonempty. Take $c \geq 0, 0 < \gamma < 1$ and let p, q be real numbers with $p + q < 0$. Assume that there exists a positive integer m_0 with*

$$x_1, nx_1 \in X; \quad n \in \mathbb{N}; n \geq m_0.$$

Then every operator $g : E_1 \rightarrow E_2$, satisfying the inequality

$$\|g(x_1 + x_2) - g(x_1) - g(x_2), y\|_\gamma \leq c\|x_1, y\|_\gamma^p \|x_2, y\|_\gamma^q, \quad (4.10)$$

is additive on X , that is, fulfils the condition

$$g(x_1 + x_2) = g(x_1) + g(x_2), \quad x_1, x_2 \in X, x_1 + x_2 \in X, y \in Y_0.$$

Proof. Because of the assumption that $p + q < 0$, this means that we have either $p < 0$ or $q < 0$. Therefore, it is sufficient to consider only the case where $q < 0$.

Let f denote the restriction of g to the set X . Fix $m \in \mathbb{N}$ with $m \geq m_0$ and

$$m^{\gamma(p+q)} + (1+m)^{\gamma(p+q)} < 1.$$

Taking $x_2 = mx_1$ in (4.10),

$$\|f((m+1)x_1) - f(x_1) - f(mx_1), y\|_\gamma \leq cm^{\gamma q} \|x_1, y\|_\gamma^{p+q}, \quad x_1 \in X, y \in Y_0. \quad (4.11)$$

Define operators $T : E_2^X \rightarrow E_2^X$ by

$$T\xi(x) := \xi((m+1)x) + \xi(-mx), \quad x \in X, \xi \in E_2^X,$$

and $\Lambda : \mathbb{R}_+^X \rightarrow \mathbb{R}_+^X$ by

$$\Lambda\delta(x) := \delta((m+1)x) + \delta(-mx), \quad x \in X, \delta \in \mathbb{R}_+^X.$$

Then Λ has the form described in (A3) with $j = 2, f_1(x) = (m+1)x, f_2(x) = mx, L_1(x) = L_2(x) = 1$ for $x \in X$ and (4.11) can be written as

$$\|Tf(x_1) - f(x_1), y\|_\gamma \leq cm^{\gamma q} \|x_1, y\|_\gamma^{p+q} =: \varepsilon(x, y), \quad x_1 \in X, y \in Y_0.$$

Furthermore, (A2) is also valid. Since

$$\varepsilon^*(x_1, y) := \sum_{n=0}^{\infty} \Lambda^n \varepsilon(x_1, y) \leq cm^{\gamma q} \|x_1, y\|_\gamma^{p+q} \sum_{n=0}^{\infty} (m^{\gamma(p+q)} + (1+m)^{\gamma(p+q)})^n, \quad x_1 \in X, y \in Y_0,$$

we have

$$\varepsilon^*(x_1, y) \leq \frac{cm^{\gamma q} \|x_1, y\|_\gamma^{p+q}}{1 - m^{\gamma(p+q)} - (1+m)^{\gamma(p+q)}}, \quad x_1 \in X, y \in Y_0.$$

Hence, using Theorem 3.1, there is a solution $T_m : X \rightarrow E_2$ of the equation

$$T(x) = T((1 + m)x) - T(mx), \tag{4.12}$$

such that

$$\|f(x_1) - T_m(x_1), y\|_\gamma \leq \frac{cm^{\gamma q} \|x_1, y\|_\gamma^{p+q}}{1 - m^{\gamma(p+q)} - (1 + m)^{\gamma(p+q)}}, \quad x_1 \in X, y \in Y_0. \tag{4.13}$$

Moreover,

$$T_m(x_1) := \lim_{n \rightarrow \infty} T^n f(x_1), \quad x_1 \in X.$$

Next, it can be easily shown by induction that, for every $x_1, x_2 \in X$ with $x_1 + x_2 \in X$ and $n \in \mathbb{N}_0$,

$$\begin{aligned} & \|T^n f(x_1 + x_2) - T^n f(x_1) - T^n f(x_2), y\|_\gamma \\ & \leq c((m + 1)^{\gamma(p+q)} + m^{\gamma(p+q)})^n (\|x_1, y\|_\gamma^p \|x_2, y\|_\gamma^q), \quad x_1, x_2 \in X, y \in Y_0. \end{aligned} \tag{4.14}$$

To this end, it is enough to observe that the case $n = 0$ is just (4.10) and, for every $k \in \mathbb{N}_0$ and $x_1, x_2 \in X$ with $x_1 + x_2 \in X$,

$$\begin{aligned} & \|T^{k+1} f(x_1 + x_2) - T^{k+1} f(x_1) - T^{k+1} f(x_2), y\|_\gamma \\ & \leq \|T^k f((m + 1)x_1 + (m + 1)x_2) - T^k f((m + 1)x_1) \\ & \quad - T^k f((m + 1)x_2), y\|_\gamma + \|T^k f(mx_1 + mx_2) - T^k f(mx_1) - T^k f(mx_2), y\|_\gamma, \quad x_1, x_2 \in X, y \in Y_0. \end{aligned}$$

Letting $n \rightarrow \infty$ in (4.14), we obtain that

$$T_m(x_1 + x_2) = T_m(x_1) + T_m(x_2), \quad x_1, x_2 \in X, x_1 + x_2 \in X. \tag{4.15}$$

Next, we prove that T_m is the unique function mapping X into E_2 that is additive on X and such that

$$\sup_{x_1 \in X} \|f(x_1) - T_m(x_1), y\|_\gamma \|x_1, y\|_\gamma^{-p-q} < \infty.$$

So, suppose that $T_0 : X \rightarrow Y$ is additive on X and satisfies

$$\sup_{x_1 \in X} \|f(x_1) - T_0(x_1), y\|_\gamma \|x_1, y\|_\gamma^{-p-q} < \infty.$$

Then there is a positive real constant B with

$$\|T_m(x_1) - T_0(x_1), y\|_\gamma \leq B \|x_1, y\|_\gamma^{p+q}, \quad x_1 \in X. \tag{4.16}$$

We can easily show by induction that, for each $j \in \mathbb{N}_0$,

$$\|T_m(x_1) - T_0(x_1), y\|_\gamma \leq B \|x_1, y\|_\gamma^{p+q} \sum_{n=j}^{\infty} (m^{\gamma(p+q)} + (1 + m)^{\gamma(p+q)})^n, \quad x_1 \in X, y \in Y_0. \tag{4.17}$$

It is enough to note that the case $j = 0$ follows from (4.16) and, for each $l \in \mathbb{N}_0$,

$$\|T_m(x_1) - T_0(x_1), y\|_\gamma \leq \|T_m((m + 1)x_1) - T_0((m + 1)x_1), y\|_\gamma + \|T_m(mx_1) - T_0(mx_1), y\|_\gamma, \quad x_1 \in X, y \in Y_0,$$

because T_m and T_0 are solutions to (4.12). Hence, letting $j \rightarrow \infty$ in (4.17), we get $T_m = T_0$. Thus we have proved that, for each $m \in \mathbb{N}$, $m \geq m_0$, there exists a unique solution $T_m : X \rightarrow Y$ to (4.15) satisfying (4.13). The uniqueness of T_m means that

$$\|f(x_1) - T_k(x_1), y\|_\gamma \leq \frac{cn^q \|x_1, y\|_\gamma^{p+q}}{1 - n^{\gamma(p+q)} - (1 + n)^{\gamma(p+q)}}, \quad x_1 \in X, y \in Y_0, \tag{4.18}$$

for every $x_1 \in X$ and $k, n \in \mathbb{N}$, $n \geq m_0$ and $k \geq m_0$. In fact, if $k, n \in \mathbb{N}$, $n \geq k \geq m_0$, then

$$\|f(x_1) - T_n(x_1), y\|_\gamma \leq \frac{cn^q \|x_1, y\|_\gamma^{p+q}}{1 - n^{\gamma(p+q)} - (1 + n)^{\gamma(p+q)}} \leq \frac{ck^q \|x_1, y\|_\gamma^{p+q}}{1 - k^{\gamma(p+q)} - (1 + k)^{\gamma(p+q)}}, \quad x_1 \in X, y \in Y_0,$$

whence $T_n = T_k$, which yields (4.18). Fixing k and letting $n \rightarrow \infty$ in (4.18), we get $f = T_k$. This implies that f is additive on the set X . □

5. Conclusion

In this article, we managed to generalize some of the recent results concerning the hyperstability of the Cauchy functional equation. The main tool used is a fixed point. The results obtained in this article may be further generalized to be in (n, γ) -Banach spaces for some $n \in \mathbb{N}$. This could be a potential future work.

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