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Some properties of generalized (s, k)-Bessel function in two variables

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Abstract

The devotion of this paper is to study the Bessel function of two variables in k-calculus. we discuss the generating function of k-Bessel function in two variables and develop its relations. After this we introduce the generalized (s, k)-Bessel function of two variables which help to develop its generating function. The s-analogy of k-Bessel function in two variables is also discussed. Some recurrence relations of the generalized (s, k)-Bessel function in two variables are also derived.

Keywords: k-Bessel function, generalized (s, k)-Bessel function, generalized (s, k)-Bessel function in two variables.

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1. Introduction

Many special functions of mathematical physics have been generalized to a base s which are known as special s-functions. The Bessel s-function is one of the essential special s-functions which was introduced by Jackson and Swarthrow [33]. Special functions in term of k were presented by Diaz and Parigaun [2]. Later on, the researchers introduced various types of k-special functions by following the idea of Diaz and Parigaun [2]. Kokologiannaki [12] investigated further properties of k-gamma, k-beta and k-zeta functions. Mansour [15] introduced the k-generalized gamma function by functional equation. Krasniqi [13] investigated limits for k-gamma and k-beta functions. Merovci [16] gave the power product inequalities for the k-gamma function. Mubeen and Habibullah [17] proposed the so-called k-fractional integral based on gamma k-function and its applications. In [18], Mubeen and Habibullah defined the integral representation of generalized confluent hypergeometric k-functions and hypergeometric k-functions by

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utilizing the properties of Pochhammer k-symbols, k-gamma, and k-beta functions. In [17], Mubeen et al. proposed the following second order linear differential equation for hypergergeometric k-functions as

$$k\omega(1-kx)\omega'' + [\gamma - (\alpha+\beta+k)\,kx]\,\omega' - \alpha\beta\omega = 0.$$

The solution in the form of the so-called k-hypergeometric series of k-hypergeometric differential equation by utilizing the Frobenius method can be found in the work of Mubeen et al. [23, 22]. Recently, Li and Dong [14] investigated the hypergeometric series solutions for the second-order non-homogeneous k-hypergeometric differential equation with the polynomial term. Rahman et al. [27, 21] proposed the generalization of Wright hypergeometric k-functions and derived its various basic properties.

Furthermore, Mubeen and Iqbal [19] investigated the generalized version of Grüss-type inequalities by considering k-fractional integrals. Agarwal et al. [1] established certain Hermite-Hadamard type inequalities involving k-fractional integrals. Set et al. [32] proposed generalized Hermite-Hadamard type inequalities for Riemann-Liouville k-fractional integral. Östrowski type k-fractional integral inequalities can be found in the work of Mubeen et al. [20]. Many researchers have established further the generalized version of Riemann-Liouville k-fractional integrals and defined a large numbers of various inequalities via by using different kinds of generalized fractional integrals. The interesting readers may consult [9, 26, 25, 28]. The Hadamard k-fractional integrals can be found in the work of Farid et al. [5]. In [6], Farid proposed the idea of Hadamard-type inequalities for k-fractional Riemann-Liouville integrals. In [10, 35], the authors have introduced inequalities by employing Hadamard-type inequalities for k-fractional integrals. Nisar et al. [24] investigated Gronwall type inequalities by utilizing Riemann-Liouville k- and Hadamard k-fractional derivatives [24]. In [24], they presented dependence solutions of certain k-fractional differential equations of arbitrary real order with initial conditions. Samraiz et al. [31] proposed Hadamard k-fractional derivative and properties. Recently, Rahman et al. [29] defined generalized k-fractional derivative operator. Diaz and Teruel introduced the generalized gamma and beta (s, k)-functions in 2005 [3]. They also proved various identities of gamma and beta (s, k)-functions in two parameter deformation. In this paper, the generalized (s, k)-Bessel function is introduced. Firstly, the Bessel function of two variables at level k is introduced by constructing its generating function and some recurrence relations. Secondly, the generating function of the generalized (s, k)-Bessel function is constructed and some of its recurrence relations are developed. Also, the s-analogy of the generalized k-Bessel function of two variables is given. Finally, the concluding comments on (s, k)-Bessel function are given.

2. preliminaries

In this section, we present certain well-known definition and mathematical preliminaries.

Definition 2.1 ([4])**.** The s-factorial is defined by

$$[n]_s! = \frac{(s;s)_n}{(1-s)^n}, \tag{2.1}$$

where n is any positive integer and $0 < s < 1$. Replacing n by $n+k$ in (2.1), where $k > 0$, we get

$$[n+k]_s! = \frac{(s;s)_{n+k}}{(1-s)^{n+k}}. \tag{2.2}$$

Definition 2.2 ([3])**.** The generalized (s, k)-gamma function is defined as

$$\Gamma_{s,k}(t) = \frac{(1-s^k)_{s,k}^{\frac{t}{k}-1}}{(1-s)^{\frac{t}{k}-1}}, \qquad t > 0,$$

where k is any positive real number and $0 < s < 1$.

After changing of variable t by nk, we get

$$\Gamma_{s,k}(nk) = \prod_{j=1}^{n-1}[jk]_s = \prod_{j=1}^{n-1}\frac{(1-s^{jk})}{(1-s)} = \frac{(1-s^k)_{s,k}^{n-1}}{(1-s)^{n-1}}.$$

Definition 2.3 ([34])**.** The s-Bessel function of two variables x and y is given by

$$J_{\nu.\mu}(x,y;s) = \sum_{m,n=0}^{\infty}\frac{(-1)^{m+n}(\frac{x}{2})^{2m+\nu}(\frac{yp(x)}{2})^{2n+\mu}}{[m]_s!\ [n]_s!\Gamma_s(\nu+m+1)\Gamma_s(\mu+n+1)}, \tag{2.3}$$

where ν, μ are not negative integers.

Definition 2.4. The relation between s-gamma function and (s,k)-gamma function is given by

$$\lim_{s\to 1}\Gamma_{s,k}(nk) = \lim_{s\to 1}\Gamma_{s^k}(n) = k^{n-1}\Gamma(n),$$

where $k > 0$, $0 < s < 1$ and n is positive real number.

3. The k-Bessel function and generalized (s,k)-Bessel function in two variables

In this section, we introduce k-Bessel function and generalized (s,k)-Bessel function in two variables.

Definition 3.1. The k-Bessel function in two variables is defined as

$$J^k_{\nu,\mu}(x,y) = \sum_{m,n=0}^{\infty}\frac{(-1)^{m+n}(\frac{x}{2})^{2m+\frac{\nu}{k}}(\frac{yp(x)}{2})^{2n+\frac{\mu}{k}}}{m!n!\Gamma_k(\nu+mk+k)\Gamma_k(\mu+nk+k)}. \tag{3.1}$$

If ν and μ are not negative integers, then we have

$$J^k_{-\nu,-\mu}(x,y) = (-1)^{\nu+\mu}J^k_{\nu,\mu}(x,y). \tag{3.2}$$

Definition 3.2. The generalized (s,k)-Bessel function of two variables x, y is defined by

$$J^k_{\nu,\mu}(x,y;s) = \sum_{m,n=0}^{\infty}\frac{(-k)^{m+n}(\frac{x}{2})^{2m+\frac{\nu}{k}}(\frac{yp(x)}{2})^{2n+\frac{\mu}{k}}}{[m]_{s^k}![n]_{s^k}!\Gamma_{s,k}(\nu+mk+k)\Gamma_{s,k}(\mu+nk+k)},$$

where k is any positive real number, $0 < s < 1$ and ν, μ are non negative integers.

Remark 3.3. If we let $k = 1$, then the generalized (s,k)-Bessel function reduces to s-Bessel function (2.3).

Remark 3.4. If we let $s = 1$, then the generalized (s,k)-Bessel function reduces to k-Bessel function (3.1).

Remark 3.5. If we let $s = k = 1$, then the generalized (s,k)-Bessel function reduces to the following Bessel function in two variables

$$J\nu,\mu(x,y) = \sum_{m,n=0}^{\infty}\frac{(-1)^{m+n}(\frac{x}{2})^{2m+\nu}(\frac{yp(x)}{2})^{2n+\mu}}{m!n!\Gamma(\nu+m+1)\Gamma(\mu+n+1)},$$

where ν, μ are non negative integers.

4. Properties of Bessel s-function and Bessel (s,k)-function in two variables

The study of Bessel function and s-Bessel function of two variables in k-calculus gives important theories in the filed of analysis. We discuss some important results about k-Bessel function and (s,k)-Bessel function in two variables. We derive the generating function of k-Bessel function of two variables, and also discuss the s-analogy of the generalized k-Bessel function of two variables in the form of theorems.

Lemma 4.1. *The relation between Bessel function and* k*-Bessel function in two variables is given by*

$$J^{k}_{\nu,\mu}(x,y) = k^{\frac{-(\nu+\mu)}{2k}} J_{\frac{\nu}{k},\frac{\mu}{k}}\left(\frac{x}{\sqrt{k}},\frac{y}{\sqrt{k}}\right) \tag{4.1}$$

or counter part is

$$J_{\frac{\nu}{k},\frac{\mu}{k}}(x,y) = k^{\frac{\nu+\mu}{2k}} J^{k}_{\nu,\mu}(x\sqrt{k},y\sqrt{k}), \tag{4.2}$$

where ν,μ *are non negative integers and* k *is any positive real number.*

Proof. By definition of k-Bessel function, we have

$$\begin{aligned} J^{k}_{\nu,\mu}(x,y) &= \sum_{m,n=0}^{\infty} \frac{(-1)^{m+n}(\frac{x}{2})^{2m+\frac{\nu}{k}}(\frac{yp(x)}{2})^{2n+\frac{\mu}{k}}}{[m]![n]!\Gamma_k(\nu+mk+k)\Gamma_k(\mu+nk+k)} \\ &= k^{\frac{-(\nu+\mu)}{2k}} \sum_{m,n=0}^{\infty} \frac{(-1)^{m+n}(\frac{x}{2\sqrt{k}})^{2m+\frac{\nu}{k}}(\frac{yp(x)}{2\sqrt{k}})^{2n+\frac{\mu}{k}}}{[m]![n]!\Gamma(\frac{\nu}{k}+m+1)\Gamma(\frac{\mu}{k}+n+1)} = k^{\frac{-(\nu+\mu)}{2k}} J_{\frac{\nu}{k},\frac{\mu}{k}}\left(\frac{x}{\sqrt{k}},\frac{y}{\sqrt{k}}\right). \end{aligned} \tag{4.3}$$

After replacing x by $x\sqrt{k}$ and y by $y\sqrt{k}$ in equation (4.3), we get (4.2). □

Lemma 4.2. *The following relation holds for Bessel function and* k*-Bessel function of two variables*

$$J^{k}_{\nu k,\mu k}(x,y) = k^{\frac{-(\nu+\mu)}{2}} J_{\nu,\mu}\left(\frac{x}{\sqrt{k}},\frac{y}{\sqrt{k}}\right)$$

or its counter part is

$$J_{\nu,\mu}(x,y) = k^{\frac{\nu+\mu}{2}} J^{k}_{\nu k,\mu k}(x\sqrt{k},y\sqrt{k}),$$

where ν,μ *are integers and* k *is any positive real number.*

Proof. Consider the definition of k-Bessel function in two variables, we have

$$J^{k}_{\nu,\mu}(x,y) = \sum_{m,n=0}^{\infty} \frac{(-1)^{m+n}(\frac{x}{2})^{2m+\frac{\nu}{k}}(\frac{yp(x)}{2})^{2n+\frac{\mu}{k}}}{[m]![n]!\Gamma_k(\nu+mk+k)\Gamma_k(\mu+nk+k)}. \tag{4.4}$$

After replacing ν by νk and μ by μk in equation (4.4), we have

$$\begin{aligned} J^{k}_{\nu k,\mu k}(x,y) &= \sum_{m,n=0}^{\infty} \frac{(-1)^{m+n}(\frac{x}{2})^{2m+\nu}(\frac{yp(x)}{2})^{2n+\mu}}{[m]![n]!\Gamma_k(\nu k+mk+k)\Gamma_k(\mu k+nk+k)}. \\ &= k^{\frac{-(\nu+\mu)}{2}} \sum_{m,n=0}^{\infty} \frac{(-1)^{m+n}(\frac{x}{2\sqrt{k}})^{2m+\nu}(\frac{yp(x)}{2\sqrt{k}})^{2n+\mu}}{[m]![n]!\Gamma(\nu+m+1)\Gamma(\mu+n+1)} = k^{\frac{-(\nu+\mu)}{2}} J_{\nu,\mu}\left(\frac{x}{\sqrt{k}},\frac{y}{\sqrt{k}}\right). \end{aligned} \tag{4.5}$$

After replacing x by $x\sqrt{k}$ and y by $y\sqrt{k}$ in equation (4.5), we have

$$J^{k}_{\nu k,\mu k}(x\sqrt{k},y\sqrt{k}) = k^{\frac{-(\nu+\mu)}{2}} J_{\nu,\mu}(x,y) \quad \text{or} \quad J_{\nu,\mu}(x,y) = k^{\frac{(\nu+\mu)}{2}} J^{k}_{\nu k,\mu k}(x\sqrt{k},y\sqrt{k}).$$

□

Lemma 4.3. *The* k*-Bessel function in two variables satisfies*

$$J^{k}_{-\nu,-\mu}(x,y) = (-k)^{\frac{\nu+\mu}{k}} J^{k}_{\nu,\mu}(x,y), \tag{4.6}$$

where ν,μ *are non negative integers and* k *is any positive real number.*

Proof. Replacing the value of ν, μ by $-\nu$, $-\mu$ in equation (4.1), and resulting equation is as follows

$$J^k_{-\nu,-\mu}(x,y) = k^{\frac{\nu+\mu}{2k}} J_{-\frac{\nu}{k},-\frac{\mu}{k}}\left(\frac{x}{\sqrt{k}},\frac{y}{\sqrt{k}}\right).$$

Using the value of the equation (3.2), we get

$$J^k_{-\nu,-\mu}(x,y) = (-1)^{\frac{\nu+\mu}{k}} k^{\frac{\nu+\mu}{2k}} J_{\frac{\nu}{k},\frac{\mu}{k}}\left(\frac{x}{\sqrt{k}},\frac{y}{\sqrt{k}}\right) = (-k)^{\frac{\nu+\mu}{2k}} k^{-\frac{(\nu+\mu)}{2k}} J_{\frac{\nu}{k},\frac{\mu}{k}}\left(\frac{x}{\sqrt{k}},\frac{y}{\sqrt{k}}\right) = (-k)^{\frac{\nu+\mu}{k}} J^k_{\nu,\mu}(x,y).$$

□

Theorem 4.4. *For $t \neq 0, w \neq 0$ and $t, w \in \mathbb{C}$, then the generating function of k-Bessel function in two variables is*

$$\exp\left[\frac{x}{2\sqrt{k}}\left(\frac{t}{\sqrt{k}}-\frac{\sqrt{k}}{t}\right)+\frac{yp(x)}{2\sqrt{k}}\left(\frac{w}{\sqrt{k}}-\frac{\sqrt{k}}{w}\right)\right] = \sum_{\nu,\mu=-\infty}^{\infty} t^\nu w^\mu J^k_{\nu k,\mu k}(x,y),$$

where ν, μ are non negative integers and k is any positive real number.

Proof. Let

$$A \equiv \sum_{\nu,\mu=-\infty}^{\infty} t^\nu w^\mu J^k_{\nu k,\mu k}(x,y), \equiv \sum_{\nu,\mu=-\infty}^{-1} t^\nu w^\mu J^k_{\nu k,\mu k}(x,y) + \sum_{\nu,\mu=0}^{\infty} t^\nu w^\mu J^k_{\nu k,\mu k}(x,y). \tag{4.7}$$

After replacing ν by $-\nu-1$ and μ by $-\mu-1$ in first summation of the equation (4.7), we have

$$A \equiv \sum_{\nu,\mu=0}^{\infty} t^{-\nu-1} w^{-\mu-1} J^k_{-(\nu+1)k,-(\mu+1)k}(x,y) + \sum_{\nu,\mu=0}^{\infty} t^\nu w^\mu J^k_{\nu k,\mu k}(x,y). \tag{4.8}$$

By using equation (4.6) in equation (4.8), we have

$$\begin{aligned} A &\equiv \sum_{\nu,\mu=0}^{\infty} t^{-\nu-1} w^{-\mu-1} (-k)^{\nu+\mu+2} J^k_{(\nu+1)k,(\mu+1)k}(x,y) + \sum_{\nu,\mu=0}^{\infty} t^\nu w^\mu J^k_{\nu k,\mu k}(x,y) \\ &\equiv \sum_{\nu,\mu=0}^{\infty}\sum_{m,n=0}^{\infty} t^{-\nu-1} w^{-\mu-1} (-k)^{\nu+\mu+2} \frac{(-1)^{m+n}(\frac{x}{2})^{2m+\nu+1}(\frac{yp(x)}{2})^{2n+\mu+1}}{m!n!\Gamma_k(mk+(\nu+1)k+k)\Gamma_k(nk+(\mu+1)k+k)} \\ &+ \sum_{\nu,\mu=0}^{\infty}\sum_{m,n=0}^{\infty} \frac{t^\nu w^\mu (-1)^{m+n}(\frac{x}{2})^{2m+\nu}(\frac{yp(x)}{2})^{2n+\mu}}{m!n!\Gamma_k(mk+\nu k+k)\Gamma_k(nk+\mu k+k)}. \end{aligned} \tag{4.9}$$

After replacing ν by $\nu-2m$ and μ by $\mu-2n$ in the equation (4.9), we have

$$\begin{aligned} A &\equiv \sum_{\nu=0}^{\infty}\sum_{m=0}^{\frac{\nu}{2}} \frac{t^{-\nu-2m-1}(-1)^{\nu-m+1}k^{\nu-2m+1}(\frac{x}{2})^{\nu+1}}{m!\Gamma_k(mk+(\nu-2m)k+2k)} \\ &+ \sum_{\mu=0}^{\infty}\sum_{n=0}^{\frac{\mu}{2}} \frac{w^{-\mu-2n-1}(-1)^{\mu-n+1}k^{\mu-2n+1}(\frac{yp(x)}{2})^{\mu+1}}{n!\Gamma_k(nk+(\nu-2n)k+2k)} \\ &+ \sum_{\nu=0}^{\infty}\sum_{m=0}^{\frac{\nu}{2}} \frac{t^{\nu-2m}(-1)^m(\frac{x}{2})^\nu}{m!\Gamma_k(mk+(\nu-2m)k+2k)} + \sum_{\mu=0}^{\infty}\sum_{n=0}^{\frac{\mu}{2}} \frac{w^{\mu-2n}(-1)^n(\frac{yp(x)}{2})^\mu}{n!\Gamma_k(nk+(\mu-2n)k+2k)}. \end{aligned} \tag{4.10}$$

After replacing ν by $\nu-1$ and μ by $\mu-1$ in first and second summation of the equation (4.10), we have

$$\begin{aligned}A &\equiv \sum_{\nu=1}^{\infty}\sum_{m=0}^{\frac{\nu-1}{2}}\frac{t^{-\nu+2m}(-1)^{\nu-m}k^{\nu-2m}(\frac{x}{2})^{\nu}}{m!\Gamma_k(\nu k-mk+k)}+\sum_{\mu=1}^{\infty}\sum_{n=0}^{\frac{\mu-1}{2}}\frac{w^{-\mu+2n}(-1)^{\mu-n}k^{\mu-2n}(\frac{yp(x)}{2})^{\mu}}{n!\Gamma_k(\mu k-nk+k)}+2\\&+\sum_{\nu=1}^{\infty}\sum_{m=0}^{\frac{\nu}{2}}\frac{t^{\nu-2m}(-1)^{m}(\frac{x}{2})^{\nu}}{m!\Gamma_k(\nu k-mk+k)}+\sum_{\mu=1}^{\infty}\sum_{n=0}^{\frac{\mu}{2}}\frac{w^{\mu-2n}(-1)^{n}(\frac{yp(x)}{2})^{\mu}}{n!\Gamma_k(\mu k-nk+k)},\\&\equiv \sum_{\nu=1}^{\infty}\sum_{m=0}^{\frac{\nu-1}{2}}\frac{t^{-\nu+2m}(-1)^{\nu-m}k^{-m}(\frac{x}{2})^{\nu}}{m!\Gamma(\nu-m+1)}+\sum_{\mu=1}^{\infty}\sum_{n=0}^{\frac{\mu-1}{2}}\frac{w^{-\mu+2n}(-1)^{\mu-n}k^{-n}(\frac{yp(x)}{2})^{\mu}}{n!\Gamma(\mu-n+1)}\\&+2+\sum_{\nu=1}^{\infty}\sum_{m=0}^{\frac{\nu}{2}}\frac{t^{\nu-2m}(-1)^{m}k^{-\nu+m}(\frac{x}{2})^{\nu}}{m!\Gamma(\nu-m+1)}+\sum_{\mu=1}^{\infty}\sum_{n=0}^{\frac{\mu}{2}}\frac{w^{\mu-2n}(-1)^{n}k^{-\mu+n}(\frac{yp(x)}{2})^{\mu}}{n!\Gamma(\mu-n+1)}.\end{aligned}$$

By rearranging the terms, we have

$$\begin{aligned}A &\equiv 2+\sum_{\nu=1}^{\infty}\sum_{m=0}^{\frac{\nu}{2}}\frac{(-1)^{m}(\frac{t}{\sqrt{k}})^{\nu-m-m}(\frac{x}{2\sqrt{k}})^{\nu}}{m!(\nu-m)!}+\sum_{\mu=1}^{\infty}\sum_{n=0}^{\frac{\mu}{2}}\frac{(-1)^{n}(\frac{w}{\sqrt{k}})^{\mu-n-n}(\frac{yp(x)}{2\sqrt{k}})^{\mu}}{n!(\mu-n)!}\\&+\sum_{\nu=1}^{\infty}\sum_{m=0}^{\frac{\nu-1}{2}}\frac{(-1)^{\nu-m}(\frac{t}{\sqrt{k}})^{m-(\nu-m)}(\frac{x}{2\sqrt{k}})^{\nu}}{m!(\nu-m)!}+\sum_{\mu=1}^{\infty}\sum_{n=0}^{\frac{\nu-1}{2}}\frac{(-1)^{\mu-n}(\frac{w}{\sqrt{k}})^{n-(\mu-n)}(\frac{yp(x)}{2\sqrt{k}})^{\mu}}{n!(\mu-n)!}.\end{aligned}$$

By using [30, Lemma 12, page 112], we have

$$\begin{aligned}A &\equiv 2+\sum_{\nu=1}^{\infty}\sum_{m=0}^{\nu}\frac{(-1)^{m}(\frac{t}{\sqrt{k}})(\frac{x}{2\sqrt{k}})^{\nu}}{m!(\nu-m)!}+\sum_{\mu=1}^{\infty}\sum_{n=0}^{\mu}\frac{(-1)^{n}(\frac{w}{\sqrt{k}})^{\mu-n-n}(\frac{yp(x)}{2\sqrt{k}})^{\mu}}{n!(\mu-n)!}\\&\equiv \sum_{\nu=0}^{\infty}\sum_{m=0}^{\nu}\frac{(-1)^{m}(\frac{t}{\sqrt{k}})^{\nu-m}(\frac{\sqrt{k}}{t})^{m}(\frac{x}{2\sqrt{k}})^{\nu}}{m!(\nu-m)!}+\sum_{\mu=0}^{\infty}\sum_{n=0}^{\mu}\frac{(-1)^{n}(\frac{w}{\sqrt{k}})^{\mu-n}(\frac{\sqrt{k}}{w})^{n}(\frac{yp(x)}{2\sqrt{k}})^{\mu}}{n!(\mu-n)!}\\&\equiv \sum_{\nu=0}^{\infty}\frac{(\frac{t}{\sqrt{k}}-\frac{\sqrt{k}}{t})^{\nu}(\frac{x}{2\sqrt{k}})^{\nu}}{m!(\nu)!}+\sum_{\mu=0}^{\infty}\frac{(\frac{w}{\sqrt{k}}-\frac{\sqrt{k}}{w})^{\mu}(\frac{yp(x)}{2\sqrt{k}})^{\mu}}{n!(\mu)!}\\&\equiv \exp\left[\frac{x}{2\sqrt{k}}\left(\frac{t}{\sqrt{k}}-\frac{\sqrt{k}}{t}\right)+\frac{yp(x)}{2\sqrt{k}}\left(\frac{w}{\sqrt{k}}-\frac{\sqrt{k}}{w}\right)\right],\end{aligned}$$

which is required generating function of k-Bessel function in two variables. □

Lemma 4.5. *The (s,k)-Bessel function of two variables satisfies the relation*

$$J^{k}_{\nu,\mu}(-x,y;s)=(-1)^{\frac{\nu}{k}}J^{k}_{\nu,\mu}(x,y;s),$$

where ν,μ are integers, k is any real number and $0<s<1$.

Proof. Since (s,k)-Bessel function in two variables is

$$J^{k}_{\nu,\mu}(x,y;s)=\sum_{m,n=0}^{\infty}\frac{(-k)^{m+n}(\frac{x}{2})^{2m+\frac{\nu}{k}}(\frac{yp(x)}{2})^{2n+\frac{\mu}{k}}}{\Gamma_{s,k}[\nu+mk+k][m]_{s^k}!\Gamma_{s,k}[\mu+nk+k][n]_{s^k}!},$$

by changing x by $-x$ in the above, we get

$$\begin{aligned} J^k_{\nu,\mu}(-x,y;s) &= \sum_{m,n=0^\infty} \frac{(-k)^{m+n}(\frac{-x}{2})^{2m+\frac{\nu}{k}}(\frac{yp(x)}{2})^{2n+\frac{\mu}{k}}}{\Gamma_{s,k}[\nu+mk+k][m]_{s^k}!\Gamma_{s,k}[\mu+nk+k][n]_{s^k}!} \\ &= (-1)^{2m+\frac{\nu}{k}} \sum_{m,n=0}^{\infty} \frac{(-k)^{m+n}(\frac{x}{2})^{2m+\frac{\nu}{k}}(\frac{yp(x)}{2})^{2n+\frac{\mu}{k}}}{\Gamma_{s,k}[\nu+mk+k][m]_{s^k}!\Gamma_{s,k}[\mu+nk+k][n]_{s^k}!}. \end{aligned}$$

Here, $(-1)^{2m}$ is positive for all values of m. Therefore, $(-1)^{2m}=1$, then we have

$$J^k_{\nu,\mu}(-x,y;s) = (-1)^{\frac{\nu}{k}} \sum_{m,n=0}^{\infty} \frac{(-k)^{m+n}(\frac{x}{2})^{2m+\frac{\nu}{k}}(\frac{yp(x)}{2})^{2n+\frac{\mu}{k}}}{\Gamma_{s,k}[\nu+mk+k][m]_{s}!\Gamma_{s,k}[\mu+nk+k][n]_{s}!} = (-1)^{\frac{\nu}{k}} J^k_{\nu,\mu}(x,y;s).$$

□

Lemma 4.6. *The* (s,k)*-Bessel function of two variables holds*

$$J^k_{\nu,\mu}(x,-y;s) = (-1)^{\frac{\mu}{k}} J^k_{\nu,\mu}(x,y;s),$$

where ν,μ *are non negative integers,* k *is any real positive number and* $0<s<1$*.*

Proof. The (s,k)-Bessel function is

$$J^k_{\nu,\mu}(x,y;s) = \sum_{m,n=0}^{\infty} \frac{(-k)^{m+n}(\frac{x}{2})^{2m+\frac{\nu}{k}}(\frac{yp(x)}{2})^{2n+\frac{\mu}{k}}}{\Gamma_{s,k}[\nu+mk+k][m]_{s^k}!\Gamma_{s,k}[\mu+nk+k][n]_{s^k}!}. \tag{4.11}$$

After replacing y by $-y$ in the equation (4.11), we have

$$\begin{aligned} J^k_{\nu,\mu}(x,-y;s) &= \sum_{m,n=0^\infty} \frac{(-k)^{m+n}(\frac{x}{2})^{2m+\frac{\nu}{k}}(\frac{-yp(x)}{2})^{2n+\frac{\mu}{k}}}{\Gamma_{s,k}[\nu+mk+k][m]_{s^k}!\Gamma_{s,k}[\mu+nk+k][n]_{s^k}!} \\ &= (-1)^{2n+\frac{\mu}{k}} \sum_{m,n=0}^{\infty} \frac{(-k)^{m+n}(\frac{x}{2})^{2m+\frac{\nu}{k}}(\frac{yp(x)}{2})^{2n+\frac{\mu}{k}}}{\Gamma_{s,k}[\nu+mk+k][m]_{s^k}!\Gamma_{s,k}[\mu+nk+k][n]_{s^k}!}. \end{aligned}$$

For all values of n, $(-1)^{2n}$ is positive. Therefore, $(-1)^{2n}=1$, then we have

$$J^k_{\nu,\mu}(x,-y;s) = (-1)^{\frac{\mu}{k}} \sum_{m,n=0}^{\infty} \frac{(-k)^{m+n}(\frac{x}{2})^{2m+\frac{\nu}{k}}(\frac{yp(x)}{2})^{2n+\frac{\mu}{k}}}{\Gamma_{s,k}[\nu+mk+k][m]_{s^k}!\Gamma_{s,k}[\mu+nk+k][n]_{s^k}!} = (-1)^{\frac{\mu}{k}} J^k_{\nu,\mu}(x,y;s).$$

□

Lemma 4.7. *The* (s,k)*-Bessel function of two variables holds*

$$J^k_{\nu,\mu}(-x,-y;s) = (-1)^{\frac{\nu+\mu}{k}} J^k_{\nu,\mu}(x,y;s),$$

where ν,μ *are non negative integers,* k *is any real positive number and* $0<s<1$*.*

Proof. The (s,k)-Bessel function of two variables is given by

$$J^k_{\nu,\mu}(x,y;s) = \sum_{m,n=0}^{\infty} \frac{(-k)^{m+n}(\frac{x}{2})^{2m+\frac{\nu}{k}}(\frac{yp(x)}{2})^{2n+\frac{\mu}{k}}}{\Gamma_{s,k}[\nu+mk+k][m]_{s^k}!\Gamma_{s,k}[\mu+nk+k][n]_{s^k}!}.$$

After replacing x by $-x$ and y by $-y$ in the above, we have

$$J^{k}_{\nu,\mu}(-x,-y;s) = \sum_{m,n=0^{\infty}} \frac{(-k)^{m+n}(\frac{-x}{2})^{2m+\frac{\nu}{k}}(\frac{-yp(x)}{2})^{2n+\frac{\mu}{k}}}{\Gamma_{s,k}[\nu+mk+k][m]_{s^k}!\Gamma_{s,k}[\mu+nk+k][n]_{s^k}!}$$
$$= (-1)^{2\frac{\nu+\mu}{k}}(-1)^{2m+2n} \sum_{m,n=0}^{\infty} \frac{(-k)^{m+n}(\frac{x}{2})^{2m+\frac{\nu}{k}}(\frac{yp(x)}{2})^{2n+\frac{\mu}{k}}}{\Gamma_{s,k}[\nu+mk+k][m]_{s^k}!\Gamma_{s,k}[\mu+nk+k][n]_{s^k}!}.$$

For all values of m and n, $(-1)^{2m+2n}$ is positive. Therefore, $(-1)^{2m+2n} = 1$, then we have

$$J^{k}_{\nu,\mu}(-x,-y;s) = (-1)^{\frac{\nu+\mu}{k}} \sum_{m,n=0}^{\infty} \frac{(-k)^{m+n}(\frac{x}{2})^{2m+\frac{\nu}{k}}(\frac{yp(x)}{2})^{2n+\frac{\mu}{k}}}{\Gamma_{s,k}[\nu+mk+k][m]_{s^k}!\Gamma_{s,k}[\mu+nk+k][n]_{s^k}!} = (-1)^{\frac{\nu+\mu}{k}} J^{k}_{\nu,\mu}(x,y;s).$$

□

Now, we construct the generating function of the generalized (s,k)-Bessel function of two variables.

Theorem 4.8. *Prove that the generating function of the generalized Bessel q,k-function of two variables is the expansion of*

$$E_{s^k}\left[\frac{x}{2}(t-\frac{k}{t}) + \frac{yp(x)}{2}(w-\frac{k}{w})\right] \tag{4.12}$$

where $t \neq 0, w \neq 0$, $t, w \in \mathbb{C}$, and k is any positive real number.

Proof. There are two important cases of exponential s-function which are defined by

$$E_s(x) = \sum_{r=0}^{\infty} \frac{x^r}{[r]_s!} \tag{4.13}$$

and

$$E_s(x) = \sum_{r=0}^{\infty} \frac{s^{\frac{r^2}{2}} x^r}{(s;s)_r}, \qquad |x| < 1.$$

By taking limit $s \to 1$, we get $\lim_{s\to 1}(E_s(1-s)x) = e^x$. Taking left hand side of (4.12) and using (4.13), we have

$$E_{s^k}\left[\frac{x}{2}(t-\frac{k}{t}) + \frac{yp(x)}{2}(w-\frac{k}{w})\right] = \sum_{\nu=0}^{\infty} \frac{(\frac{xt}{2})^{\nu}}{[\nu]_{s^k}!} \sum_{m=0}^{\infty} \frac{(\frac{-kx}{2t})^{m}}{[m]_{s^k}!} \sum_{\mu=0}^{\infty} \frac{(\frac{yp(x)w}{2})^{\mu}}{[\mu]_{s^k}!} \sum_{n=0}^{\infty} \frac{(\frac{-kyp(x)}{2w})^{n}}{[n]_{s^k}!}. \tag{4.14}$$

Replacing ν by $\frac{\nu}{k}+m$ and μ by $\frac{\mu}{k}+n$ in the equation (4.14), we have

$$= \sum_{\nu=-\infty}^{\infty} \frac{(\frac{xt}{2})^{\frac{\nu}{k}+m}}{[\frac{\nu}{k}+m]_{q^k}!} \sum_{m=0}^{\infty} \frac{(\frac{-kx}{2t})^{m}}{[m]_{s^k}!} \sum_{\mu=-\infty}^{\infty} \frac{(\frac{yp(x)w}{2})^{\frac{\mu}{k}+n}}{[\frac{\mu}{k}+n]_{s^k}!} \sum_{n=0}^{\infty} \frac{(\frac{-kyp(x)}{2w})^{n}}{[n]_{s^k}!}$$
$$= \sum_{\nu,\mu=-\infty}^{\infty} \sum_{m,n=0}^{\infty} \frac{(-k)^m(\frac{x}{2})^{\frac{\nu}{k}+2m}\, t^{\frac{\nu}{k}+m-m}\, (\frac{yp(x)}{2})^{\frac{\mu}{k}+2n}(-k)^n\, w^{\frac{\mu}{k}+n-n}}{\Gamma_{s^k}[\frac{\nu}{k}+m+1][m]_{s^k}!\Gamma_{q^k}[\frac{\mu}{k}+n+1][n]_{s^k}!}$$
$$= \sum_{\nu,\mu=-\infty}^{\infty} t^{\frac{\nu}{k}} w^{\frac{\mu}{k}} \sum_{m,n=0}^{\infty} \frac{(-k)^{m+n}(\frac{x}{2})^{2m+\frac{\nu}{k}}(\frac{yp(x)}{2})^{2n+\frac{\mu}{k}}}{\Gamma_{s,k}[\nu+mk+k][m]_{s^k}!\Gamma_{s,k}[\mu+nk+k][n]_{s^k}!}$$
$$= \sum_{\nu,\mu=-\infty}^{\infty} t^{\frac{\nu}{k}} w^{\frac{\mu}{k}} J^{k}_{\nu,\mu}(x,y;s),$$

which is required generating function for (s,k)-Bessel function of two variables. □

Lemma 4.9. *If the parameters* ν *and* μ *are integers then generalized* (s,k)*-Bessel function satisfies*

$$J^{k}_{-\nu,\mu}(x,y;s) = (-k)^{\frac{\nu}{k}} J^{k}_{\nu,\mu}(x,y;s).$$

Proof. Replacing ν by $-\nu$ in (s,k)-Bessel function of two variables we get

$$\begin{aligned} J^{k}_{-\nu,\mu}(x,y;s) &= \sum_{m,n=0}^{\infty} \frac{(-k)^{m+n}(\frac{x}{2})^{2m+\frac{\nu}{k}}(\frac{yp(x)}{2})^{2n+\frac{\mu}{k}}}{\Gamma_{s,k}[\nu+mk+k][m]_{s^k}!\Gamma_{s,k}[\mu+nk+k][n]_{s^k}!} \\ &= \sum_{m,n=0}^{\infty} \frac{(-k)^{m+n}(\frac{x}{2})^{2m+\frac{\nu}{k}}(\frac{yp(x)}{2})^{2n+\frac{\mu}{k}}}{\Gamma_{s^k}[\frac{\nu}{k}+m+1][m]_{s^k}!\Gamma_{s,k}[\mu+nk+k][n]_{s^k}!}. \end{aligned} \tag{4.15}$$

Substituting ν by $-\nu$ in the equation (4.15), we get

$$J^{k}_{-\nu,\mu}(x,y;s) = \sum_{m,n=0}^{\infty} \frac{(-k)^{m+n}(\frac{x}{2})^{2m-\frac{\nu}{k}}(\frac{yp(x)}{2})^{2n+\frac{\mu}{k}}}{\Gamma_{s^k}[\frac{-\nu}{k}+m+1][m]_{s^k}!\Gamma_{s,k}[\mu+nk+k][n]_{s^k}!}. \tag{4.16}$$

After replacing m by $\frac{\nu}{k}+r$ in equation (4.16), we have

$$\begin{aligned} J^{k}_{-\nu,\mu}(x,y;s) &= \sum_{r,n=0}^{\infty} \frac{(-k)^{\frac{\nu}{k}+r+n}(\frac{x}{2})^{\frac{2\nu}{k}+2r-\frac{\nu}{k}}(\frac{yp(x)}{2})^{2n+\frac{\mu}{k}}}{\Gamma_{s^k}[\frac{-\nu}{k}+\frac{\nu}{k}+r+1][\frac{\nu}{k}+r]_{s^k}!\Gamma_{s,k}[\mu+nk+k][n]_{s^k}!} \\ &= \sum_{r,n=0}^{\infty} \frac{(-k)^{\frac{\nu}{k}}(-k)^{r+n}(\frac{x}{2})^{2r+\frac{\nu}{k}}(\frac{yp(x)}{2})^{2n+\frac{\mu}{k}}}{\Gamma_{s^k}[r+1]\Gamma_{s^k}[\frac{\nu}{k}+r+1]\Gamma_{s,k}[\mu+nk+k][n]_{s^k}!} \\ &= (-k)^{\frac{\nu}{k}} \sum_{r,n=0}^{\infty} \frac{(-k)^{r+n}(\frac{x}{2})^{2r+\frac{\nu}{k}}(\frac{yp(x)}{2})^{2n+\frac{\mu}{k}}}{\Gamma_{s,k}[\nu+rk+k][r]_{s^k}!\Gamma_{s,k}[\mu+nk+k][n]_{s^k}!} = (-k)^{\frac{\nu}{k}} J^{k}_{\nu,\mu}(x,y;s), \end{aligned}$$

which is required recurrence relation. □

Lemma 4.10. *If the parameters* ν *and* μ *are integers, then* (s,k)*-Bessel function of two variables satisfies the relation*

$$J^{k}_{\nu,-\mu}(x,y;s) = (-k)^{\frac{\mu}{k}} J^{k}_{\nu,\mu}(x,y;s).$$

Proof. Consider the (s,k)-Bessel function of two variables

$$\begin{aligned} J^{k}_{\nu,\mu}(x,y;s) &= \sum_{m,n=0}^{\infty} \frac{(-k)^{m+n}(\frac{x}{2})^{2m+\frac{\nu}{k}}(\frac{yp(x)}{2})^{2n+\frac{\mu}{k}}}{\Gamma_{s,k}[\nu+mk+k][m]_{s^k}!\Gamma_{s,k}[\nu+nk+k][n]_{s^k}!} \\ &= \sum_{m,n=0}^{\infty} \frac{(-k)^{m+n}(\frac{x}{2})^{2m+\frac{\nu}{k}}(\frac{yp(x)}{2})^{2n+\frac{\mu}{k}}}{\Gamma_{s,k}[\nu+mk+k][m]_{s^k}![n+\frac{\mu}{k}]_{s^k}![n]_{s^k}!}. \end{aligned} \tag{4.17}$$

Replacing μ by $-\mu$ in equation (4.17), we have

$$J^{k}_{\nu,-\mu}(x,y;s) = \sum_{m,n=0}^{\infty} \frac{(-k)^{m+n}(\frac{x}{2})^{2m+\frac{\nu}{k}}(\frac{yp(x)}{2})^{2n-\frac{\mu}{k}}}{\Gamma_{s,k}[\nu+mk+k][m]_{s^k}![n-\frac{\mu}{k}]_{s^k}![n]_{s^k}!}. \tag{4.18}$$

Replacing n by $s+\frac{\mu}{k}$ in equation (4.18), we get

$$J^{k}_{\nu,-\mu}(x,y;s) = \sum_{m,s=0}^{\infty} \frac{(-k)^{m+s+\frac{\mu}{k}}(\frac{x}{2})^{2m+\frac{\nu}{k}}(\frac{yp(x)}{2})^{2s+\frac{2\mu}{k}-\frac{\mu}{k}}}{\Gamma_{s,k}[\nu+mk+k][m]_{s^k}![s+\frac{\mu}{k}-\frac{\mu}{k}]_{s^k}![s+\frac{\mu}{k}]_{s^k}!}$$

$$= (-k)^{\frac{\mu}{k}} \sum_{m,s=0}^{\infty} \frac{(-k)^{m+s}(\frac{x}{2})^{2m+\frac{\nu}{k}}(\frac{yp(x)}{2})^{2s+\frac{\mu}{k}}}{\Gamma_{s,k}[\nu+mk+k][m]_{s^k}![s+\frac{\mu}{k}]_{s^k}![s]_{s^k}!}$$

$$= (-k)^{\frac{\mu}{k}} \sum_{m,s=0}^{\infty} \frac{(-k)^{m+s}(\frac{x}{2})^{2m+\frac{\nu}{k}}(\frac{yp(x)}{2})^{2s+\frac{\mu}{k}}}{\Gamma_{q,k}[\nu+mk+k][m]_{q^k}!\Gamma_{q,k}[\mu+sk+k][s]_{q^k}!} = (-k)^{\frac{\mu}{k}} J^k_{\nu,\mu}(x,y;s).$$

□

Theorem 4.11. *The* (s,k)*-Bessel function in two variables is* s*-analogy of* k*-Bessel function in two variables,*

$$\lim_{s\to 1} J^k_{\nu,\mu}[(1-s)x,(1-s)y;s] = J^k_{\nu,\mu}(x,y),$$

where ν,μ *are non negative integers,* k *is any positive real number and* $0 < s < 1$.

Proof. Consider the (s,k)-Bessel function is in two variables

$$\begin{aligned} J^k_{\nu,\mu}(x,y;s) &= \sum_{m,n=0}^{\infty} \frac{(-k)^{m+n}(\frac{x}{2})^{2m+\frac{\nu}{k}}(\frac{yp(x)}{2})^{2n+\frac{\mu}{k}}}{\Gamma_{s,k}(\nu+mk+k)[m]_{s^k}!\Gamma_{s,k}(\mu+nk+k)[n]_{s^k}!} \\ &= \sum_{m,n=0}^{\infty} \frac{(-k)^{m+n}(\frac{x}{2})^{2m+\frac{\nu}{k}}(\frac{yp(x)}{2})^{2n+\frac{\mu}{k}}}{[m]_{s^k}![n]_{s^k}!\Gamma_{s^k}(\frac{\nu}{k}+m+1)\Gamma_{s^k}(\frac{\mu}{k}+n+1)} \\ &= \sum_{m,n=0}^{\infty} \frac{(-k)^{m+n}(\frac{x}{2})^{2m+\frac{\nu}{k}}(\frac{yp(x)}{2})^{2n+\frac{\mu}{k}}}{[m+\frac{\nu}{k}]_{s^k}![m]_{s^k}![n+\frac{\mu}{k}]_{s^k}![n]_{s^k}!}. \end{aligned} \tag{4.19}$$

By taking left hand side of the equation (4.19) and using the equation (2.2), we have

$$\begin{aligned} &\lim_{s\to 1} J^k_{\nu,\mu}((1-s)x,(1-s)y;s) \\ &= \lim_{s\to 1} \sum_{m,n=0}^{\infty} \frac{(-k)^{m+n}(1-s)^{m+\frac{\nu}{k}}(1-s)^m(1-s)^{n+\frac{\mu}{k}}(1-s)^n(\frac{x}{2})^{2m+\frac{\nu}{k}}(\frac{yp(x)}{2})^{2n+\frac{\mu}{k}}}{(s^k;s^k)_{m+\frac{\nu}{k}}(s^k;s^k)_m(s^k;s^k)_{n+\frac{\mu}{k}}(s^k;s^k)_n}. \end{aligned}$$

Gaspor [7] has given the relation

$$((s;s))_{n+r} = (s;s)_r(s^{r+1};s)_n. \tag{4.20}$$

By using the relation defined in the equation (4.20),

$$\begin{aligned} &= \lim_{s\to 1} \frac{(1-s)^{\frac{\nu}{k}}(1-s)^{\frac{\mu}{k}}}{(s^k;s^k)_{\frac{\nu}{k}}(s^k;s^k)_{\frac{\mu}{k}}} \sum_{m,n=0}^{\infty} \frac{(-k)^{m+n}(1-s)^{2m}(1-s)^{2n}(\frac{x}{2})^{2m+\frac{\nu}{k}}(\frac{yp(x)}{2})^{2n+\frac{\mu}{k}}}{((s^k)^{\frac{\nu}{k}+1};s^k)_m(s^k;s^k)_m((s^k)^{\frac{\mu}{k}+1};s^k)_n(s^k;s^k)_n} \\ &= \frac{1}{\Gamma(\frac{\nu}{k}+1)\Gamma(\frac{\mu}{k}+1)} \sum_{m,n=0}^{\infty} \frac{(-1)^{m+n}(\frac{x}{2})^{2m+\frac{\nu}{k}}(\frac{yp(x)}{2})^{2n+\frac{\mu}{k}}}{k^{\frac{\nu+\mu}{k}+m+n}(\Gamma(1)(1)_m)(\frac{\nu}{k}+1)_m(\frac{\mu}{k}+1)_n(\Gamma(1)(1)_n)} \\ &= \sum_{m,n=0}^{\infty} \frac{(-1)^{m+n}(\frac{x}{2})^{2m+\frac{\nu}{k}}(\frac{yp(x)}{2})^{2n+\frac{\mu}{k}}}{m!n!\Gamma_k(\nu+mk+k)\Gamma_k(\mu+nk+k)} = J^k_{\nu,\mu}(x,y). \end{aligned}$$

□

5. Conclusion

In our work, the two parameter deformation of classical Bessel function is introduced. We discussed some important relations between k-Bessel function and simple Bessel function in two variables. Also, we developed the generating functions which satisfies the k-Bessel function and (s,k)-Bessel function in two variables. Moreover, we established a result in which (s,k)-Bessel function is s-analogy of k-Bessel function. If $k=1$, generalized (s,k)-Bessel function reduces to s-Bessel function in two variables. By taking $s=1$ in (s,k)-Bessel function, we get k-Bessel function in two variables. For $s=1, k=1$, the generalized (s,k)-Bessel function reduces to simple classical Bessel function.

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