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Some new results of fixed point in dislocated quasi-metric spaces

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Abstract

In this paper, we introduce some new fixed point theorems in a dislocated quasi-metric space. We present several fixed point theorems, which generalize and improve some comparable fixed point results. Moreover, we provide some examples to illustrate our results.

Keywords: Fixed point, dislocated quasi-metric spaces, contraction mapping.

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1. Introduction

The Banach contraction principle in metric space is the first important result in fixed point theory [4]. Since then, various generalizations have been made in many different forms with different types of spaces [1–3] and [5–7, 10–15]. Some of the well-known generalizations, with useful applications in logical programming and electronics engineering [9], are obtained in the framework of dislocated metric spaces [8], and dislocated quasi-metric spaces [15]. The present paper provides new generalizations of fixed point theorem in the setting of dislocated quasi-metric spaces, which generalize, improve, and fuse the results founded in [2, 12–14] by using a new contraction type and without any continuity requirement.

2. Preliminaries

We introduce here some basic concepts of the theory of dislocated quasi-metric spaces [15].

Definition 2.1. Let X be a nonempty set and $d : X \times X \to \mathbb{R}^+$ be a function such that

1. $d(x, y) = d(y, x) = 0$ implies $x = y$;
2. $d(x, y) \leqslant d(x, z) + d(z, y)$, for all x, y, $z \in X$.

Then, d is called dislocated quasi-metric (or simply dq-metric) on X.

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Clearly, for a dq-metric, the self distance of points need not to be zero necessarily and the usual property of symmetry is no longer valid. As example for dq-metric space, we can consider the set $X = [0, 1]$ endowed with the following dq-metric

$$d : X \times X \to \mathbb{R}^+, \quad d(x, y) = |x - y| + |x|.$$

Definition 2.2. A sequence $\{x_n\}$ in a dq-metric space (X, d) is called a Cauchy sequence if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that,

$$d(x_m, x_n) \leqslant \epsilon \quad \text{or} \quad d(x_n, x_m) \leqslant \epsilon, \quad \forall\, m, n \geqslant N.$$

Definition 2.3. A sequence $\{x_n\}$ is said to be dq-convergent to x in a dq-metric space X, if

$$\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x, x_n) = 0.$$

Here, x is called dq-limit of sequence $\{x_n\}$ and we write $x_n \to x$, as $n \to \infty$.

Definition 2.4. Let (X, d_1) and (Y, d_2) be two dq-metric spaces, the function $f : X \to Y$ is said to be continuous if for each sequence $\{x_n\} \subset X$ which dq-converges to x in X, the sequence $\{f(x_n)\}$ is dq-converges to $f(x)$ in Y.

Definition 2.5. A dq-metric space (X, d) is called complete if every Cauchy sequence in X is dq-convergent.

We remark here that, in dq-metric space, the dq-limit is unique and a Cauchy sequence which processes a dq-convergent subsequence, is also dq-converges. In the sequel, for simplicity, we omit the prefix "dq" to indicate the limit and convergence.

3. Main results

First, we start with the following lemma which we will use in the sequel.

Lemma 3.1. *If x is a limit of some sequence $\{x_n\}$ in a dq-metric space (X, d), then*

$$d(x, x) = 0.$$

Proof. Let $x \in X$, and $\{x_n\} \subset X$ a sequence which converges to x. Then

$$d(x, x) \leqslant d(x, x_n) + d(x_n, x), \quad \forall\, n \in \mathbb{N}.$$

Passing to limit, when $n \to \infty$, we obtain $d(x, x) \leqslant 0$, and therefore

$$d(x, x) = 0. \qquad \square$$

Next, we state and prove our main fixed point result in complete dq-metric spaces. Unlike various papers which impose the contraction continuity condition [2], the following result provides the same result without continuity condition and under less restrictive condition.

Theorem 3.2. *Let (X, d) be a complete dq-metric space and T a self-mapping of X such that*

$$d(Tx, Ty) \leqslant \lambda \max \left\{ \begin{array}{l} 2\, d(x, y),\ \frac{2\, d(x,Tx)\, d(y,Ty)}{d(x,y)},\ [d(x, Tx) + d(y, Ty)], \\ \frac{[d(x,Ty)+d(y,Tx)]}{2},\ [d(x, Tx) + d(x, y)], \\ [d(y, Ty) + d(x, y)],\ \frac{2\,[d(x,Ty)+d(x,y)]}{3} \end{array} \right\}, \tag{3.1}$$

for all $x, y \in X$ with $d(x, y) \neq 0$, and $\lambda \in [0, \frac{1}{2})$. Then, T has a unique fixed point in X.

Proof. Assume $T : X \to X$ verifies the condition (3.1), we consider

$$M(x,y) = \max \left\{ \begin{array}{l} 2\,d(x,y)\,,\ \frac{2\,d(x,Tx)\,d(y,Ty)}{d(x,y)}\,,\ d(x,Tx)+d(y,Ty), \\ \frac{d(x,Ty)+d(y,Tx)}{2}\,,\ d(x,Tx)+d(x,y), \\ d(y,Ty)+d(x,y)\,,\ \frac{2\,[d(x,Ty)+d(x,y)]}{3} \end{array} \right\}.$$

Then, we distinguish the following different cases.

◇ **Case** 1 **:** If $M(x,y) = \frac{2\,d(x,Tx)\,d(y,Ty)}{d(x,y)}$, then

$$d(Tx,Ty) \leqslant \lambda\,\frac{2\,d(x,Tx)\,d(y,Ty)}{d(x,y)}, \quad \forall\, x,y \in X.$$

Taking $y = Tx$, respectively $x = Ty$, in the previous inequality, we find

$$d(Tx,T^2x) \leqslant 2\lambda\,d(Tx,T^2x), \quad \forall\, x \in X, \tag{3.2}$$

$$d(T^2y,Ty) \leqslant \lambda\,\frac{2\,d(Ty,T^2y)\,d(y,Ty)}{d(Ty,y)}, \quad \forall\, y \in X. \tag{3.3}$$

Since $2\lambda \in [0,1)$, the inequality (3.2) implies that $d(Tx,T^2x) = 0$, for all $x \in X$, which leads also, by taking $y = x$ in the inequality (3.3) to $d(T^2x,Tx) = 0$, for all $x \in X$. Therefore, we conclude that $T^2x = Tx$ and thus the mapping T has a fixed point.

◇ **Case** 2 **:** If $M(x,y) = 2\,d(x,y)$, then

$$d(Tx,Ty) \leqslant 2\lambda\,d(x,y), \quad \forall\, x,y \in X. \tag{3.4}$$

We consider a Picard sequence $x_{n+1} = Tx_n$ with initial guess $x_0 \in X$. We will show that $\{x_n\}$ is a Cauchy sequence in X. For that, let $n \in \mathbb{N}^*$ and use (3.4) to get

$$d(x_n,x_{n+1}) = d(Tx_{n-1},Tx_n) \leqslant 2\lambda\,d(x_{n-1},x_n) = h\,d(x_{n-1},x_n).$$

We reiterate this process to find $d(x_n,x_{n+1}) \leqslant h^n d(x_0,x_1)$ and then we conclude

$$\begin{aligned} d(x_n,x_m) &\leqslant d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}) + \cdots + d(x_{m-1},x_m) \\ &\leqslant (h^n + h^{n+1} + \cdots + h^{m-1})\,d(x_0,x_1) \\ &\leqslant \frac{h^n}{1-h}\,d(x_0,x_1). \end{aligned} \tag{3.5}$$

Since $h = 2\lambda \in [0,1)$, it follows from (3.5) that $\{x_n\}$ is a Cauchy sequence in a complete dq-metric space, and therefore there exists $u \in X$ such that

$$\lim_{n\to\infty} x_n = u \quad \text{and} \quad \lim_{n\to\infty} Tx_n = u.$$

From the inequality (3.4), we deduce

$$d(Tx_n,Tu) \leqslant 2\,\lambda\,d(x_n,u), \quad \forall\, n \in \mathbb{N}.$$

Then, since $d(\cdot,Tu),\ d(\cdot,u) : X \to \mathbb{R}$ are continuous, we deduce

$$d(u,Tu) = \lim_{n\to\infty} d(Tx_n,Tu) \leqslant 2\lambda \lim_{n\to\infty} d(x_n,u) = 2\lambda\,d(u,u).$$

From Lemma 3.1, $d(u,u) = 0$ and thus $d(u,Tu) = 0$. On the other hand, we have

$$\begin{aligned} d(Tu,u) &\leqslant d(Tu,x_n) + d(x_n,u) = d(Tu,Tx_{n-1}) + d(x_n,u) \\ &\leqslant 2\lambda\,d(u,x_{n-1}) + d(x_n,u) \underset{n\to\infty}{\longrightarrow} 0, \end{aligned}$$

which leads to $d(Tu,u) = 0$ and then $d(u,Tu) = d(Tu,u) = 0$. Therefore, we conclude that $Tu = u$ and hence T has a fixed point.

◇ **Case** 3 **:** If $M(x, y) = [d(x, Tx) + d(y, Ty)]$, then

$$d(Tx, Ty) \leqslant \lambda \, [d(x, Tx) + d(y, Ty)], \quad \forall \, x, y \in X. \tag{3.6}$$

We consider a sequence $x_{n+1} = Tx_n$ with initial guess $x_0 \in X$. We will prove that $\{x_n\}$ is a Cauchy sequence. For that, we use (3.6) to deduce

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\leqslant \lambda \, [d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)] \\ &\leqslant \lambda \, d(x_{n-1}, x_n) + \lambda \, d(x_n, x_{n+1}), \quad \text{for } n \in \mathbb{N}, \end{aligned}$$

which implies

$$d(x_n, x_{n+1}) \leqslant \frac{\lambda}{1-\lambda} \, d(x_{n-1}, x_n), \quad \text{for } n \in \mathbb{N}.$$

Since $h = \frac{\lambda}{1-\lambda} \in [0, 1)$, then $\{x_n\}$ is a Cauchy sequence in the complete space X, and therefore there exists $u \in X$ such that

$$\lim_{n\to\infty} x_n = u \quad \text{and} \quad \lim_{n\to\infty} Tx_n = u.$$

Next, to prove that u is the fixed point of T, we use (3.6) to obtain

$$\begin{aligned} d(Tx_n, Tu) &\leqslant \lambda \, [d(x_n, Tx_n) + d(u, Tu)] \\ &\leqslant \lambda \, [d(x_n, u) + d(u, Tx_n) + d(u, Tu)], \quad \forall \, n \in \mathbb{N}. \end{aligned}$$

Then, by passing to limit for in the above inequality, we find

$$d(u, Tu) \leqslant \frac{2\lambda}{1-\lambda} \, d(u, u).$$

Using Lemma 3.1, we get $d(u, Tu) = 0$. On the other hand, we have

$$\begin{aligned} d(Tu, u) \leqslant d(Tu, x_n) + d(x_n, u) &= d(Tu, Tx_{n-1}) + d(x_n, u) \\ &\leqslant \lambda \, [d(u, Tu) + d(x_{n-1}, Tx_{n-1})] + d(x_n, u) \\ &\leqslant \lambda \, [d(x_{n-1}, u) + d(u, Tx_{n-1})] + d(x_n, u) \\ &= \lambda \, [d(x_{n-1}, u) + d(u, x_n)] + d(x_n, u) \underset{n\to\infty}{\longrightarrow} 0, \end{aligned}$$

which implies that $d(Tu, u) = 0$ and then $d(u, Tu) = d(Tu, u) = 0$. Therefore, we conclude that $Tu = u$ and hence T has a fixed point.

◇ **Case** 4 **:** If $M(x, y) = \frac{d(x,Ty)+d(y,Tx)}{2}$, then

$$d(Tx, Ty) \leqslant \frac{\lambda}{2} [d(x, Ty) + d(y, Tx)], \quad \forall \, x, y \in X. \tag{3.7}$$

We consider a sequence $x_{n+1} = Tx_n$ with initial guess $x_0 \in X$. Then, we have

$$\begin{aligned} d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) &\leqslant \frac{\lambda}{2} [d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})] \\ &\leqslant \frac{\lambda}{2} [d(x_{n-1}, x_{n+1}) + d(x_n, x_n)] \\ &\leqslant \frac{\lambda}{2} [d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + d(x_n, x_n)], \end{aligned} \tag{3.8}$$

for all $n \in \mathbb{N}^*$. Moreover, we have

$$\begin{aligned} d(x_n, x_n) = d(Tx_{n-1}, Tx_{n-1}) &\leqslant \frac{\lambda}{2}\,[d(x_{n-1}, Tx_{n-1}) + d(x_{n-1}, Tx_{n-1})] \\ &= \lambda\, d(x_{n-1}, x_n) \\ &\leqslant d(x_{n-1}, x_n), \quad \forall\, n \in \mathbb{N}^*. \end{aligned} \tag{3.9}$$

Then, we combine the two inequalities (3.8) and (3.9) to find

$$\begin{aligned} d(x_n, x_{n+1}) &\leqslant \frac{\lambda}{2}\,[d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + d(x_{n-1}, x_n)] \\ &= \frac{\lambda}{2}\,[2\,d(x_{n-1}, x_n) + d(x_n, x_{n+1})], \quad \forall\, n \in \mathbb{N}^*. \end{aligned}$$

Next, this inequality can be reformulated as follows

$$d(x_n, x_{n+1}) \leqslant \frac{\lambda}{1-\frac{\lambda}{2}}\, d(x_{n-1}, x_n), \quad \forall\, n \in \mathbb{N}^*.$$

We can easily see that $h = \frac{\lambda}{1-\frac{\lambda}{2}} \in [0, 1)$, and then $\{x_n\}$ is a Cauchy sequence in the complete space X. Therefore, there exists $u \in X$ such that

$$\lim_{n\to\infty} x_n = u \quad \text{and} \quad \lim_{n\to\infty} Tx_n = u.$$

From the inequality (3.7), we deduce

$$d(Tx_n, Tu) \leqslant \frac{\lambda}{2}\,[d(x_n, Tu) + d(u, Tx_n)]\,, \quad \forall\, n \in \mathbb{N}^*.$$

Passing to limit, as $n \to \infty$, we obtain

$$d(u, Tu) \leqslant \frac{\lambda}{2}\,[d(u, Tu) + d(u, u)]\,. \tag{3.10}$$

Using Lemma 3.1, it follows from (3.10) that

$$d(u, Tu) \leqslant \frac{\lambda}{2}\, d(u, Tu),$$

which implies that $d(u, Tu) = 0$, since $\frac{\lambda}{2} \in [0, 1)$. On the other hand, we have

$$\begin{aligned} d(Tu, u) \leqslant d(Tu, x_n) + d(x_n, u) &= d(Tu, Tx_{n-1}) + d(x_n, u) \\ &\leqslant \frac{\lambda}{2}\,[d(u, Tx_{n-1}) + d(x_{n-1}, Tu)] + d(x_n, u) \\ &\leqslant \frac{\lambda}{2}\,[d(u, x_n) + d(x_{n-1}, u) + d(u, Tu)] + d(x_n, u) \\ &= \frac{\lambda}{2}\,[d(u, x_n) + d(x_{n-1}, u)] + d(x_n, u) \underset{n\to\infty}{\longrightarrow} 0, \end{aligned}$$

which implies that $d(Tu, u) = 0$ and then $d(u, Tu) = d(Tu, u) = 0$. Therefore, we conclude that $Tu = u$ and hence T has a fixed point.

◇ **Case** 5 **:** If $M(x, y) = d(x, Tx) + d(x, y)$, then

$$d(Tx, Ty) \leqslant \lambda\,[d(x, Tx) + d(x, y)]\,, \quad \forall\, x, y \in X.$$

We define a sequence $x_{n+1} = Tx_n$ with initial guess $x_0 \in X$. Then, we have

$$\begin{aligned} d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) &\leqslant \lambda\left[d(x_{n-1}, Tx_{n-1}) + d(x_{n-1}, x_n)\right] \\ &= \lambda\left[d(x_{n-1}, x_n) + d(x_{n-1}, x_n)\right] \\ &= 2\lambda\, d(x_{n-1}, x_n), \quad \forall\, n \in \mathbb{N}^*. \end{aligned}$$

Since $2\lambda \in [0,1)$, $\{x_n\}$ is a Cauchy sequence in the complete space X. Then, there exists $u \in X$ such that $\{x_n\}$ and $\{Tx_n\}$ converge to u in X. Next, for all $n \in \mathbb{N}^*$, we have

$$\begin{aligned} d(Tx_n, Tu) &\leqslant \lambda\left[d(x_n, Tx_n) + d(x_n, u)\right] \\ &\leqslant \lambda\left[d(x_n, u) + d(u, Tx_n) + d(x_n, u)\right], \end{aligned} \tag{3.11}$$

$$d(Tu, Tx_n) \leqslant \lambda\left[d(u, Tu) + d(u, x_n)\right]. \tag{3.12}$$

We pass to limit in (3.11) to get $d(u, Tu) = 0$ and then in (3.12) to conclude

$$d(u, Tu) = d(Tu, u) = 0.$$

Therefore, we have $Tu = u$ and thus u is a fixed point of T.

◇ **Case** 6 **:** If $M(x, y) = d(y, Ty) + d(x, y)$, then

$$d(Tx, Ty) \leqslant \lambda\left[d(y, Ty) + d(x, y)\right], \quad \forall\, x, y \in X.$$

We consider a sequence $x_{n+1} = Tx_n$ with a given $x_0 \in X$. Then, we have

$$\begin{aligned} d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) &\leqslant \lambda\ \left[d(x_n, Tx_n) + d(x_{n-1}, x_n)\right] \\ &\leqslant \lambda\left[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)\right], \end{aligned}$$

for all $n \in \mathbb{N}^*$. Then, for $h = \frac{\lambda}{1-\lambda} \in [0,1)$, we conclude that

$$d(x_n, x_{n+1}) \leqslant h\, d(x_{n-1}, x_n), \quad \forall\, n \in \mathbb{N}^*.$$

Therefore, $\{x_n\}$ is a Cauchy sequence in the complete space X, and then there exists $u \in X$ such that $\{x_n\}$ and $\{Tx_n\}$ converge to u in X. Furthermore, we have

$$\begin{aligned} d(Tu, Tx_n) &\leqslant \lambda\left[d(x_n, Tx_n) + d(u, x_n)\right] \\ &\leqslant \lambda\left[d(x_n, u) + d(u, Tx_n) + d(u, x_n)\right], \end{aligned} \tag{3.13}$$

$$d(Tx_n, Tu) \leqslant \lambda\left[d(u, Tu) + d(x_n, u)\right]. \tag{3.14}$$

We pass to limit in (3.13) to get $d(Tu, u) = 0$, and then in (3.14) to conclude

$$d(u, Tu) \leqslant \lambda\, d(u, Tu).$$

Therefore, since $\lambda \in [0,1)$, we deduce that $d(u, Tu) = 0$. Finally, we conclude that

$$d(Tu, u) = d(u, Tu) = 0,$$

which leads to $Tu = u$ and thus the self-mapping T has a fixed point u.

◇ **Case** 7 **:** If $M(x, y) = \frac{2}{3}\left[d(x, Ty) + d(x, y)\right]$, then

$$d(Tx, Ty) \leqslant \frac{2\lambda}{3}\left[d(x, Ty) + d(x, y)\right], \quad \forall\, x, y \in X. \tag{3.15}$$

Let $x_0 \in X$ given, we consider a sequence $x_{n+1} = Tx_n$, then from (3.15), it follows

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\leqslant \frac{2\lambda}{3}\left[d(x_{n-1}, Tx_n) + d(x_{n-1}, x_n)\right] \\ &= \frac{2\lambda}{3}\left[d(x_{n-1}, x_{n+1}) + d(x_{n-1}, x_n)\right] \\ &\leqslant \frac{2\lambda}{3}\left[d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + d(x_{n-1}, x_n)\right], \end{aligned}$$

for all $x, y \in X$ and $n \in \mathbb{N}^*$. This inequality implies that

$$d(x_n, x_{n+1}) \leqslant \frac{\frac{4\lambda}{3}}{1 - \frac{2\lambda}{3}}\, d(x_{n-1}, x_n), \quad \forall\, x, y \in X. \tag{3.16}$$

Since $h = \frac{\frac{4\lambda}{3}}{1-\frac{2\lambda}{3}} \in [0, 1)$, the inequality (3.16) implies that $\{x_n\}$ is a Cauchy sequence in the complete space X. Therefore, there exists $u \in X$ such that,

$$\lim_{n\to\infty} x_n = u \quad \text{and} \quad \lim_{n\to\infty} Tx_n = u. \tag{3.17}$$

Thus, the continuity of $d(\cdot, u)$, $d(\cdot, Tu) : X \to \mathbb{R}$ and (3.15) lead to

$$\begin{aligned} d(u, Tu) &= \lim_{n\to\infty} d(Tx_n, Tu) \\ &\leqslant \lim_{n\to\infty} \frac{2\lambda}{3}\left[d(x_n, Tu) + d(x_n, u)\right] = \frac{2\lambda}{3} d(u, Tu), \end{aligned}$$

which implies that $d(u, Tu) = 0$, since $\frac{2\lambda}{3} \in [0, 1)$. Moreover, we have

$$d(Tu, Tx_n) \leqslant \frac{2\lambda}{3}\left[d(u, Tx_n) + d(u, x_n)\right].$$

Keeping in mind (3.17), we deduce that $d(Tu, u) = 0$ and finally, we conclude that

$$d(u, Tu) = d(Tu, u) = 0,$$

which implies that $Tu = u$, and hence u is a fixed point of T. Therefore, the existence part of Theorem 3.2 has been established. For the uniqueness part, we consider two fixed points $u, v \in X$ of a self-mapping T where u is the unique limit of the Picard sequence $x_{n+1} = Tx_n$ with a given initial guess $x_0 \in X$. Then, it comes from (3.1) that

$$d(Tu, Tv) \leqslant \lambda \max \left\{ \begin{array}{l} 2\, d(u, v),\ \frac{2\, d(u,Tu)\, d(v,Tv)}{d(u,v)},\ d(u, Tu) + d(v, Tv), \\ \frac{d(u,Tv)+d(v,Tu)}{2},\ d(u, Tu) + d(u, v), \\ d(v, Tv) + d(u, v),\ \frac{2\,[d(u,Tv)+d(u,v)]}{3} \end{array} \right\},$$

Using Lemma 3.1 and the fact $\lambda \in [0, 1/2)$, we deduce the following results.

- ⋄ **Case** 1 **:** Recalling that $1 - 2\lambda > 0$, then we have

$$\begin{aligned} d(u, v) = d(Tu, Tv) \leqslant 2\lambda\, d(u, v) &\Rightarrow (1 - 2\lambda)\, d(u, v) \leqslant 0 \\ &\Rightarrow d(u, v) = 0. \end{aligned}$$

- ⋄ **Case** 2 **:** Keeping in mind that $d(u, u) = 0$, then we have

$$\begin{aligned} d(u, v) = d(Tu, Tv) &\leqslant \frac{2\lambda\, d(u, Tu)\, d(v, Tv)}{d(u, v)} = \frac{2\lambda\, d(u, u)\, d(v, v)}{d(u, v)} = 0 \\ &\Rightarrow d(u, v) = 0. \end{aligned}$$

◇ **Case** 3 **:** Keeping in mind that $d(u,u) = 0$ and $1-\lambda > 0$, then we have

$$\begin{aligned} d(u,v) = d(Tu,Tv) &\leqslant \lambda[d(u,Tu) + d(v,Tv)] = \lambda\,[d(u,u) + d(u,v)] = 0 \\ &\Rightarrow\ (1-\lambda)\, d(u,v) \leqslant 0 \\ &\Rightarrow\ d(u,v) = 0. \end{aligned}$$

◇ **Case** 4 **:** Recalling that $\frac{\lambda/2}{1-\lambda/2} \in [0,1)$, then we have

$$\begin{aligned} d(u,v) = d(Tu,Tv) &\leqslant \lambda\,\frac{d(u,Tv) + d(v,Tu)}{2} = \lambda\,\frac{d(u,v) + d(v,u)}{2} \\ &\Rightarrow\ d(u,v) \leqslant \frac{\lambda/2}{1-\lambda/2}\, d(v,u) \\ &\Rightarrow\ d(u,v) \leqslant (\frac{\lambda/2}{1-\lambda/2})^2\, d(u,v) \\ &\Rightarrow\ d(u,v) = 0. \end{aligned}$$

◇ **Case** 5 **:** Since $d(u,u) = 0$ and $1-\lambda > 0$, then we have

$$\begin{aligned} d(u,v) = d(Tu,Tv) &\leqslant \lambda\,[d(u,Tu) + d(u,v)] = \lambda\,[d(u,u) + d(u,v)] \\ &\Rightarrow\ (1-\lambda)\, d(u,v) \leqslant 0 \\ &\Rightarrow\ d(u,v) = 0. \end{aligned}$$

◇ **Case** 6 **:** Recalling that $\frac{\lambda}{1-2\lambda} \in [0,1)$, then we have

$$\begin{aligned} d(u,v) = d(Tu,Tv) &\leqslant \lambda\,[d(v,Tv) + d(u,v)] = \lambda\,[d(v,v) + d(u,v)] \\ &\leqslant \lambda\,[d(v,u) + d(u,v) + d(u,v)] \\ &\Rightarrow\ d(u,v) \leqslant \frac{\lambda}{1-2\lambda}\, d(v,u) \\ &\Rightarrow\ d(u,v) \leqslant \left(\frac{\lambda}{1-2\lambda}\right)^2 d(u,v) \\ &\Rightarrow\ d(u,v) = 0. \end{aligned}$$

◇ **Case** 7 **:** Keeping in mind that $1 - \frac{4\lambda}{3} > 0$, then we have

$$\begin{aligned} d(u,v) = d(Tu,Tv) &\leqslant \frac{2\lambda}{3}\,[d(u,Tv) + d(u,v)] = \frac{2\lambda}{3}\,[d(u,v) + d(u,v)] \\ &\Rightarrow (1 - \frac{4\lambda}{3})\, d(u,v) \leqslant 0 \\ &\Rightarrow\ d(u,v) = 0. \end{aligned}$$

Hence, we have proved that in all the cases, that $d(u,v) = 0$. In addition, by using the same techniques, we can show that $d(v,u) = 0$, and therefore, we can conclude that $u = v$. □

Now, we illustrate our result by the following example.

Example 3.3. Consider the set $X = \{0, 10, \frac{1}{5}\}$ endowed with the metric d defined by

$$d(x,y) = x + 2y, \quad \forall\, x, y \in X.$$

We construct a self-mapping T by $T(0) = 0$, $T(10) = \frac{1}{5}$ and $T(\frac{1}{5}) = 0$. For $\lambda = \frac{1}{3}$, we can easily see that all the assumptions of Theorem 3.2 are satisfied, and then 0 is the unique fixed point of a mapping T .

As a consequence of Theorem 3.2, we may state the following corollary.

Corollary 3.4 ([12, Theorem 3.1]). *Let* (X, d) *be a complete dq-metric space and* $T : X \to X$ *a continuous self-mapping. If the following condition holds*

$$\begin{aligned} d(Tx, Ty) \leqslant a_1\, d(x, y) + a_2 \frac{d(x, Tx)\, d(y, Ty)}{d(x, y)} + a_3\, [d(x, Tx) + d(y, Ty)] \\ + a_4\, [d(x, Ty) + d(y, Tx)] + a_5\, [d(x, Tx) + d(x, y)] \\ + a_6\, [d(y, Ty) + d(x, y)] + a_7\, [d(x, Ty) + d(y, Tx)]\,, \end{aligned}$$

for all $x, y \in X$ *with* $d(x, y) \neq 0$*, and where* $\{a_i\}_{i=1,\cdots,7} \subset \mathbb{R}^+$ *satisfying*

$$0 < a_1 + a_2 + 2a_3 + 4a_4 + 2a_5 + 2a_6 + 3a_7 < 1,$$

then, the self-mapping T *has a unique fixed point.*

Note here that the above corollary requires continuity of a mapping T. In the next theorem, we provide a comparable result without any the continuity condition.

Theorem 3.5. *Let* (X, d) *be a complete dq-metric space and* $T : X \to X$ *a self-mapping. If*

$$\begin{aligned} d(Tx, Ty) \leqslant a_1\, d(x, y) + a_2 \frac{d(x, Tx)\, d(y, Ty)}{d(x, y)} + a_3\, [d(x, Tx) + d(y, Ty)] \\ + a_4\, [d(x, Ty) + d(y, Tx)] + a_5\, [d(x, Tx) + d(x, y)] \\ + a_6\, [d(y, Ty) + d(x, y)] + a_7\, [d(x, Ty) + d(y, Tx)]\,, \end{aligned} \tag{3.18}$$

holds for all $x, y \in X$ *with* $d(x, y) \neq 0$*, and where* $\{a_i\}_{i=1,\cdots,7} \subset \mathbb{R}^+$ *satisfying*

$$0 < a_1 + a_2 + 2a_3 + 4a_4 + 2a_5 + 2a_6 + 3a_7 < 1,$$

then, the self-mapping T *has a unique fixed point.*

Proof. Let T be self-mapping of X verifying assumptions of Theorem 3.5 and consider

$$M(x, y) = \max \left\{ \begin{array}{l} 2d(x, y),\ \frac{2\, d(x, Tx)\, d(y, Ty)}{d(x, y)},\ d(x, Tx) + d(y, Ty), \\ \frac{d(x, Ty) + d(y, Tx)}{2},\ d(x, Tx) + d(x, y), \\ d(y, Ty) + d(x, y),\ \frac{2\,[d(x, Ty) + d(y, Tx)]}{3} \end{array} \right\}. \tag{3.19}$$

Using the inequality (3.18) and the definition (3.19) of M, we find

$$\begin{aligned} d(Tx, Ty) &\leqslant \frac{a_1}{2} M(x, y) + \frac{a_2}{2} M(x, y) + a_3 M(x, y) \\ &\quad + 2a_4 M(x, y) + a_5 M(x, y) + a_6 M(x, y) + \frac{3a_7}{2} M(x, y) \\ &\leqslant \left(\frac{a_1}{2} + \frac{a_2}{2} + a_3 + 2a_4 + a_5 + a_6 + \frac{3a_7}{2} \right) M(x, y). \end{aligned}$$

We set $\lambda = \frac{a_1}{2} + \frac{a_2}{2} + a_3 + 2a_4 + a_5 + a_6 + \frac{3a_7}{2} \in [0, \frac{1}{2})$, then, the previous inequality implies

$$d(Tx, Ty) \leqslant \lambda\, M(x, y), \quad \text{with} \quad \lambda \in [0, 1).$$

Hence, Theorem 3.2 concludes the proof of Theorem 3.5.

□

We now provide another result, which generalizes our main result stated Theorem 3.2.

Theorem 3.6. *Let* (X, d) *be a complete dq-metric space and* T *a self-mapping of* X *such that*

$$d(Tx, Ty) \leqslant \lambda \max \left\{ \begin{array}{l} \alpha\, d(x,y),\ \frac{\alpha\, d(x,Tx)\, d(y,Ty)}{d(x,y)},\ d(x,Tx) + d(y,Ty), \\ \frac{d(x,Ty)+d(y,Tx)}{\alpha},\ d(x,Tx) + d(x,y), \\ d(y,Ty) + d(x,y),\ \frac{\alpha\,[d(x,Ty)+d(y,Tx)]}{3} \end{array} \right\},$$

for all $x, y \in X$ *with* $d(x,y) \neq 0$, $\lambda \in [0, \frac{1}{\alpha})$ *and* $\alpha \geqslant 2$. *Then,* T *has a unique fixed point in* X.

Proof. Theorem 3.6 can be proved in a similar way to the proof of Theorem 3.2. □

Here, an illustrative example for which our main result Theorem 3.6 is applicable.

Example 3.7. Consider the set $X = \left\{0, \frac{1}{7}, 30\right\}$ endowed with the following dq-metric

$$d(x,y) = x + 2y, \quad \forall\, x,y \in X.$$

Next, we construct a self-mapping T given by $T(0) = 0$, $T(30) = \frac{1}{7}$ and $T(\frac{1}{7}) = 0$. For $\lambda = \frac{1}{3}$ and $\alpha = 3$, we can see that all the assumptions of Theorem 3.6 are satisfied, and hence 0 is the unique fixed point of the mapping T.

4. Conclusion

In this paper, we gave some new results of fixed point theorems in complete dislocated quasi-metric space. These results generalize the results founded in [12]. However, their proof of their results seems not to be correct, see page 4698. More precisely, the authors have considered the inequality

$$d(\xi_n, \xi_n) \leqslant d(\xi_{n-1}, \xi_n) + d(\xi_n, \xi_{n+1}),$$

which is not always true in dq-metric spaces. As example, we consider the set $X = \{0, 1, 2\}$ and the mapping $d : X \times X \to \mathbb{R}^+$ defined by

$$\begin{array}{lll} d(0,0) = 0\,, & d(1,1) = 7\,, & d(2,2) = 6, \\ d(0,1) = 5\,, & d(1,0) = 2\,, & d(1,2) = 3, \\ d(2,1) = 4\,, & d(0,2) = 1\,, & d(2,0) = 5. \end{array}$$

We can easily verify that d is a dq-metric on X for which the previous inequality is not always valid, as we can see for $\xi_{n-1} = 2$, $\xi_n = 1$ and $\xi_{n+1} = 0$.

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