J. Math. Computer Sci., 24 (2022), 59–72

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

On right chain ordered semihypergroups

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Abstract

The purposes of this paper are to introduce generalizations of the right chain ordered semigroups to the context of the right chain ordered semihypergroups. Furthermore, we present the concepts of prime, completely prime, semiprime, and completely semiprime right hyperideals of ordered semihypergroups. We also introduce the idea of associated prime right hyperideals. Moreover, we give some characterizations of prime, completely prime, semiprime, and completely semiprime right hyperideals of ordered semihypergroups. Finally, we obtain necessary and sufficient prime right hyperideal conditions to be a semiprime right hyperideal.

Keywords: Ordered semihypergroup, prime right hyperideal, completely prime right hyperideal, semiprime right hyperideal, completely semiprime right hyperideal.

2020 MSC: 20N20, 06F05.

1. Introduction

The notion of ordered semihypergroups introduced by Heidari and Davvaz [9] in 2011, which is a generalization of ordered semigroups. In 2014, Changphas and Davvaz [1] introduced and studied the notion of right simple elements in ordered semihypergroups as a generalization of right simple elements in ordered semigroups. In 2015, Ghazavi et al. [6] introduced and studied the notion of prime hyperideals in ordered semihypergroups as a generalization of prime ideals. A proper hyperideal I is called prime if $A \circ B \subseteq P$ implies that $A \subseteq P$ or $B \subseteq P$ for any hyperideals A and B of ordered semihypergroup S. In [12], Tang et al. introduced and studied the notion of completely prime hyperideals in ordered semihypergroups. A hyperideal I of an ordered semihypergroup S is called completely prime if for any $x, y \in S$ such that $x \circ y \cap I \neq \emptyset$, then $x \in I$ or $y \in I$. Davvaz et al. [3] introduced and studied the notion of homomorphism theory of ordered semihypergroups. In 2016, Gu and Tang [7, 8] introduced and studied the notion of ordered regular (strongly ordered regular) equivalence relation in an ordered semihypergroup. In 2017, Tang et al. [13] introduced and studied the notion of hyper S-posets over an ordered semihypergroup. Omidi and Davvaz [10] introduced and studied the notion of k-hyperideals in ordered

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doi: 10.22436/jmcs.024.01.06

Received: 2020-10-29 *Revised:* 2020-11-25 *Accepted:* 2020-12-02

semihyperrings as a generalization of k-hyperideals. In [15], Tang and Xie introduced and studied the notion of weakly prime, quasi-prime, quasi-semiprime and weakly quasi-prime left hyperideals in ordered semihypergroups. Omidi et al. [11] introduced and studied the notion of simple ordered semihypergroups. In 2018, Farooq et al. [4] introduced and studied the notion of int-soft generalized bi-hyperideals in ordered semihypergroups. Tang et al. [14] introduced and studied the notion of order-congruences and strong order-congruences in ordered semihypergroups. Changphas et al. [2] introduced and studied the notion of right chain ordered semigroups.

Motivated and inspired by the above works, the aim of this paper is to extend the concept of chain ordered semihypergroups in multiplicative hyperrings given by Changphas et al. [4] to the context of right chain ordered semihypergroups. Furthermore, we introduce the concepts of prime, completely prime, semiprime and completely semiprime right hyperideals of ordered semihypergroups. We also introduce the concept of associated prime right hyperideals. Moreover, we give some characterizations of prime, completely prime, semiprime and completely semiprime right hyperideals of ordered semihypergroups. Finally, we obtain necessary and sufficient conditions of prime right hyperideal in order to be a semiprime right hyperideal.

2. Preliminaries

A mapping $\circ : S \times S \to \mathcal{P}^*(S)$, where $\mathcal{P}^*(S)$ denotes the family of all non empty subsets of S, is called a *hyperoperation* on S. An image of the pair (x, y) is denoted by $x \circ y$. The couple $(S, \circ)$ is called a *hypergroupoid*. A hypergroupoid $(S, \circ)$ is called a *semihypergroup* if $\circ$ is associative, that is $x \circ (y \circ z) = (x \circ y) \circ z$ for every $x, y, z \in S$. If $x \in S$ and A, B are non empty subsets of S, then we denote $A \circ B = \bigcup\limits_{a \in A, b \in B} a \circ b, x \circ B = \{x\} \circ B$ and $A \circ x = A \circ \{x\}$. Clearly, if $(S, \circ)$ is a semihypergroup, then $\bigcup\limits_{a \in y \circ z} x \circ a = \bigcup\limits_{b \in x \circ y} b \circ z$ for all $x, y, z \in S$.

We now recall the notion of ordered semihypergroups from [9].

Definition 2.1 ([9])**.** An algebraic hyperstructure $(S, \circ, \leqslant)$ is called an *ordered semihypergroup* if $(S, \circ)$ is a semihypergroup and $(S, \leqslant)$ is a partially ordered set such that: for any $x, y, a \in S, x \leqslant y$ implies $a \circ x \leqslant a \circ y$ and $x \circ a \leqslant y \circ a$.

If $A, B \in \mathcal{P}^*(S)$, then we say that $A \preceq B$ if for every $a \in A$ there exists $b \in B$ such that $a \leqslant b$ see [15]. In particular, if $A = \{a\}$, then we write $a \preceq B$ instead of $\{a\} \preceq B$. Clearly, every ordered semigroup can be regarded as an ordered semihypergroup. Also see [12]. Throughout this paper, unless stated otherwise, S stands for an ordered semihypergroup.

Definition 2.2 ([9])**.** Let A be a non empty subset of an ordered semihypergroup S is called a *subsemihypergroup* of S if $A \circ A \subseteq A$.

For a subsemihypergroup A of an ordered semihypergroup S and a subset B of A, we denote by $(B]_A$ the subset of A defined by $(B]_A = \{a \in A : a \leqslant b \text{ for some } b \in B\}$. In particular, for $A = S$, we write $(B]$ instead of $(B]_S$. Clearly, $(B] = \{a \in S : a \leqslant b \text{ for some } b \in B\}$.

Lemma 2.3 ([9])**.** *Let S be an ordered semihypergroup and $\emptyset \neq A, B \subseteq S$. Then the following statements hold.*

1. $A \subseteq (A]$ *and* $((A]] = (A]$.
2. *If* $A \subseteq B$, *then* $(A] \subseteq (B]$.
3. $((A] \circ (B]] = ((A] \circ B] = (A \circ (B]] = (A \circ B]$.
4. $(A] \circ (B] \subseteq (A \circ B], (A] \circ B \subseteq (A \circ B]$ *and* $A \circ (B] \subseteq (A \circ B]$.
5. $((A^m]^n] = (A^{mn}]$ *for all* $m, n \in \mathbf{Z}^+$.
6. *For any* $x, y \in S$ *if* $x \leqslant y$, *then* $(x \circ A] \subseteq (y \circ A]$ *and* $(A \circ x] \subseteq (A \circ y]$.

Lemma 2.4. *Let* S *be an ordered semihypergroup. If* $\{A_i\}i \in I$ *is a family of non empty subsets of* S, *then* $\left(\bigcup_{i\in I} A_i\right] \subseteq \bigcup_{i\in I}(A_i]$ *and* $\left(\bigcap_{i\in I} A_i\right] \subseteq \bigcap_{i\in I}(A_i]$.

Proof. The proof is straightforward. □

Definition 2.5 ([9])**.** A non empty subset A of an ordered semihypergroup S is called a *left (resp. right) hyperideal* of S if

1. $S \circ A \subseteq A$ (resp: $A \circ S \subseteq A$);
2. for every $s \in S, a \in A$ if $s \leqslant a$, then $s \in A$.

Clearly, A is a left hyperideal (right hyperideal) of an ordered semihypergroup S if and only if $S \circ A \subseteq A$ (resp: $A \circ S \subseteq A$) and $(A] = A$. If A is both a left and a right hyperideal of S, then it is called a *(two-sided) hyperideal* of S [9].

Lemma 2.6 ([15])**.** *Let* S *be an ordered semihypergroup and let* $\{A_i : i \in I\}$ *be a family of left hyperideals (resp. right, hyperideal) of* S. *Then the following statements hold.*

1. $\bigcup_{i\in I} A_i$ *is a left hyperideal (resp. right, hyperideal) of* S.
2. $\bigcap_{i\in I} A_i (\neq \emptyset)$ *is a left hyperideal (resp. right, hyperideal) of* S.

Lemma 2.7 ([9])**.** *Let* S *be an ordered semihypergroup and* $\emptyset \neq A, B \subseteq S$. *Then the following statements hold.*

1. *If* A, B *are left hyperideals (resp. right, hyperideal) of* S, *then* $(A \circ B]$ *is a left hyperideal (resp. right, hyperideal) of* S.
2. *For any non empty subset* C *of* S *if* $A \preceq B$, *then* $C \circ A \preceq C \circ B$ *and* $A \circ C \preceq B \circ C$.

Proposition 2.8. *If* I *is a right hyperideal of an ordered semihypergroup* S, *then* $(X \circ I]$ *is a right hyperideal of* S *where* $\emptyset \neq X \subseteq S$.

Proof. Let $x \in X, r \in S$ and $a \in I$. Then $(x \circ a) \circ r = x \circ (a \circ r) \subseteq X \circ I \subseteq (X \circ I]$. Hence $(X \circ I]$ is a right hyperideal of S. □

Proposition 2.9. *Let* I *be a left hyperideal of an ordered semihypergroup* S. *If* J *is a right hyperideal of* S, *then* $(I \circ X \circ J]$ *is a hyperideal of* S *where* $\emptyset \neq X \subseteq S$.

Proof. Let $x \in X, r, s \in S$ and $a \in I, b \in J$. Then $r \circ (a \circ x \circ b) \circ r = (r \circ a) \circ x \circ (b \circ s) \subseteq I \circ X \circ J \subseteq (I \circ X \circ J]$. Hence $(I \circ X \circ J]$ is a hyperideal of S. □

An element e of an ordered semihypergroup S is called a *scalar identity element* of S if $e \circ s = \{s\} = s \circ e$ for any $s \in S$. An element 0 of S is called a *scalar zero element* of S if $0 \circ s = \{0\} = 0 \circ s$ for any $s \in S$. We shall use the notation S^0 with the following meaning:

$$S^0 = \begin{cases} S, & \text{if } S \text{ has a scalar zero element,} \\ S \cup \{0\}, & \text{otherwise.} \end{cases}$$

Here is an example of a scalar identity element in an ordered semihypergroup.

Example 2.10. Let $S = \{a, b, c, d\}$ with the operation $\circ$ and the order relation $\leqslant$ below:

$\circ$	a	b	c	d
a	{a}	{b}	{c}	{d}
b	{b}	S	{b, c, d}	{b, c, d}
c	{c}	{b, c, d}	S	{b, c, d}
d	{d}	{b, c, d}	{b, c, d}	S

$\leqslant := \{(a,a),(b,b),(c,c),(d,d),(a,b),(a,c),(a,d),(c,b)\}$. It is easy to see that a is a scalar identity element of an ordered semihypergroup S.

Here is an example of a scalar zero element in an ordered semihypergroup.

Example 2.11. Let $S = \{a,b,c,d\}$ with the operation $\circ$ and the order relation $\leqslant$ below:

$\circ$	a	b	c	d
a	{a}	{a}	{a}	{a}
b	{a}	{b}	{a, c}	{a}
c	{a}	{a}	{a}	{a}
d	{a}	{a, d}	{a}	{a}

$\leqslant := \{(a,a),(b,b),(c,c),(d,d),(a,b),(a,c),(a,d),(c,b),(d,b)\}$. It is easy to see that a is a scalar zero element of an ordered semihypergroup S.

Corollary 2.12. *If S is an ordered semihypergroup with scalar identity and $\emptyset \neq X \subseteq S$, then*

1. $(X \circ S]$ *is a right hyperideal of* S *containing* X*;*
2. $(S \circ X]$ *is a left hyperideal of* S *containing* X*;*
3. $(S \circ X \circ S]$ *is a hyperideal of* S *containing* X.

Proof. 1. By Proposition 2.8, $(X \circ S]$ is a right hyperideal of S. Clearly, $X = X \circ e \subseteq X \circ S \subseteq (X \circ S]$. Hence $(X \circ S]$ is a right hyperideal of S containing X.

2 and 3 can be proved similarly. □

3. Hoehnke hyperideals

Let P be a proper right hyperideal (hyperideal) of an ordered semihypergroup S. Then P is called *prime right hyperideal* (hyperideal) if for any right hyperideals (hyperideal) A, B of $S, A \circ B \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$, and P is called *completely prime right hyperideal* (hyperideal) if for any elements a, b of $S, a \circ b \subseteq P$ implies $a \in P$ or $b \in P$. Also, P is called *semiprime right hyperideal* (hyperideal) if for any right hyperideal (hyperideal) A of $S, A^2 \subseteq P$ implies $A \subseteq P$, and P is said to be *completely semiprime right hyperideal* (hyperideal) if for any element $a \in S, a^2 \subseteq P$ implies $a \in P$.

Remark 3.1. From the above definitions we obtain immediately the following implication chart for the considered types of right hyperideals (hyperideal):

$$\begin{array}{rcr} \text{completely prime} & \Rightarrow & \text{completely semiprime} \\ \Downarrow & & \Downarrow \\ \text{prime} & \Rightarrow & \text{semiprime} \end{array}$$

The following example shows that the converse of Remark is not true.

Example 3.2. Let $S = \{a,b,c,d\}$ with the operation $\circ$ and the order relation $\leqslant$ below:

$\circ$	a	b	c	d
a	{a, d}	{a, d}	{a, d}	{a}
b	{a, d}	{b}	{a, d}	{a, d}
c	{a, d}	{a, d}	{c}	{a, d}
d	{a}	{a, d}	{a, d}	{d}

$\leqslant := \{(a,a),(b,b),(c,c),(d,d),(a,b),(a,c),(d,c),(d,b)\}$. Clearly, $(S, \circ, \leqslant)$ is an ordered semihypergroup. With a small amount of effort one can verify following.

1. $\{a,d\}, \{a,b,d\}$ and $\{a,c,d\}$ are all right hyperideals of S.

2. It is easy to see that $\{a, d\}$ is a semiprime right hyperideal of S, but it is not a prime right hyperideal of S. Indeed: $\{a, b, d\} \circ \{a, c, d\} \subseteq \{a, d\}$, but $\{a, b, d\} \nsubseteq \{a, d\}$ and $\{a, c, d\} \nsubseteq \{a, d\}$.
3. Clearly, $\{a, d\}$ is a completely semiprime right hyperideal of S, but it is not a completely prime right hyperideal of S. Indeed: $d \circ c \subseteq \{a, d\}$, but $d \notin \{a, d\}$ and $c \notin \{a, d\}$.

We continue this section with the following lemma.

Lemma 3.3. *Let P be a completely prime right hyperideal of an ordered semihypergroup S. If $A, B \subseteq S$ such that $A \circ B \subseteq S$, then $A \subseteq P$ or $B \subseteq P$.*

Proof. Let $a \in A$ and $b \in B$. Then $a \circ b \in A \circ B \subseteq P$. Since P is a completely prime right hyperideal of S, we have $a \in P$ or $b \in P$. Hence $A \subseteq P$ or $B \subseteq P$. □

Corollary 3.4. *Let P be a completely semiprime right hyperideal of an ordered semihypergroup S. If $A \subseteq S$ such that $A^2 \subseteq P$, then $A \subseteq P$.*

Proof. Similar to the proof of Lemma 3.3. □

The following theorem show that a hyperideal P of an ordered semihypergroup S with scalar identity is completely prime right hyperideal of S if and only if P is prime and completely semiprime right hyperideals.

Theorem 3.5. *For a hyperideal P of an ordered semihypergroup S with scalar identity, the following conditions are equivalent.*

1. *P is completely prime right hyperideal of S.*
2. *P is prime and completely semiprime right hyperideals.*

Proof. First assume that P is completely prime right hyperideal of S. Hence it follows from Lemma 3.3 that P is prime and completely semiprime right hyperideals.

Conversely, assume that (2) holds. Let a and b be any elements of S such that $a \circ b \subseteq P$. By Lemma 2.3,

$$(b \circ S \circ a]^2 = (b \circ S \circ a] \circ (b \circ S \circ a] \subseteq (b \circ S \circ (a \circ b) \circ S \circ a] \subseteq (S \circ P \circ S] \subseteq (P] = P.$$

Since P is a completely semiprime right hyperideal of S, we have $(b \circ S \circ a] \subseteq P$. Clearly, $b \circ S \circ a \subseteq P$. Thus $(b \circ S] \circ (a \circ S] \subseteq (b \circ S \circ a \circ S] \subseteq (P \circ S] \subseteq (P] = P$. By assumption, $a \in (a \circ S] \subseteq P$ or $b \in (b \circ S] \subseteq P$. Therefore P is a completely prime right hyperideal of S. □

Next, we proved that every hyperideal P of an ordered semihypergroup S with scalar identity is completely prime hyperideal of S if and only if P is prime and completely semiprime hyperideals.

Theorem 3.6. *Let P be a hyperideal of an ordered semihypergroup S with scalar identity. Then P is completely prime hyperideal of S if and only if P is prime and completely semiprime hyperideals.*

Proof. The necessity is clear. Now let us show the sufficiency. For $a, b \in S$ such that $a \circ b \subseteq P$, by Lemma 2.3,

$$(S \circ b \circ S \circ a \circ S]^2 = (S \circ b \circ S \circ a \circ S] \circ (S \circ b \circ S \circ a \circ S] = (S \circ b \circ S \circ (a \circ b) \circ S \circ a \circ S] \subseteq (S \circ P \circ S] \subseteq (P] = P.$$

Since P is a completely semiprime hyperideal of S, we have $(S \circ b \circ S \circ a \circ S] \subseteq P$. Clearly, $(S \circ b \circ S] \circ (S \circ a \circ S] \subseteq P$. By assumption, $a \in (S \circ a \circ S] \subseteq P$ or $b \in (S \circ b \circ S] \subseteq P$. Therefore P is a completely prime hyperideal of S. □

The following concept will be useful in constructing prime and semiprime right hyperideals of ordered semihypergroups. For any proper right hyperideal A of S we define the Hoehnke hyperideal of S associated with A to be a set

$$H_A(S) = \{s \in S : r \notin (r \circ s \circ S] \text{ for all } r \in S - A\}.$$

If A is a proper ideal of a semigroup S, then the Hoehnke ideal coincides with the set $H_A(S) = \{s \in S : r \notin rsS \text{ for all } r \in S - A\}$ which was defined and studied in [5]. Moreover, If A is a proper right ideal of an ordered semigroup S, then the Hoehnke ideal coincides with the set $H_A(S) = \{s \in S : r \notin (rsS] \text{ for all } r \in S - A\}$ which was defined and studied in [2].

Theorem 3.7. *Let S be an ordered semighyperroup. Then the following statements hold.*

1. *If A and B are proper subsets of S and $A \subseteq B$, then $H_A(S) \subseteq H_B(S)$.*
2. *If A and B are proper subsets of S, then $H_A(S) \cap H_B(S) = H_{A\cap B}(S)$.*
3. *If A and B are proper subsets of S, then $H_A(S) \cup H_B(S) \subseteq H_{A\cup B}(S)$.*

Proof.

1. Let a be an element of S such that $a \in H_A(S)$. Then $r \notin (r \circ a \circ S]$ for all $r \in S - A$. Since $A \subseteq B$, we have $S - B \subseteq S - A$. Clearly, $r \notin r \circ a \circ S$ for all $r \in S - B$. Therefore $a \in H_A(S)$ and hence $H_A(S) \subseteq H_B(S)$.

2. By (1), it is easy to see that $H_{A\cap B}(S) \subseteq H_A(S) \cap H_B(S)$. Suppose that $H_A(S) \cap H_B(S) \nsubseteq H_{A\cap B}(S)$. There exists $a \in H_A(S) \cap H_B(S)$ such that $a \notin H_{A\cap B}(S)$. Then $r \in r \circ a \circ S$ for some $r \in S - A \cap B$. Since $S-(A\cap B) = (S-A) \cup (S-B)$, we have $a \notin H_A(S)$ or $a \notin H_B(S)$, a contradiction. Hence $H_A(S) \cap H_B(S) \subseteq H_{A\cap B}(S)$.

3. It is straightforward. □

Lemma 3.8. *Let A be a proper right hyperideal of an ordered semihypergroup S. If $H_A(S) \neq \emptyset$, then $H_A(S)$ is a proper hyperdeal of S.*

Proof. Let a, s_1 and s_2 be any elements of S such that $a \in H_A(S)$. Then $r \notin (r \circ a \circ S]$ for all $r \in S - A$. Suppose that $s_1 \circ a \circ s_2 \nsubseteq H_A(S)$. Then there exists $r \in S - A$ such that $r \in (r \circ (s_1 \circ a \circ s_2) \circ S]$. Clearly, $r \circ s_1 \subseteq (r \circ (s_1 \circ a \circ s_2) \circ S] \circ S \subseteq ((r \circ s_1) \circ a \circ S]$. Since $r \notin (r \circ a \circ S]$ for all $r \in S - A$, we have $r \circ s_1 \subseteq A$. Therefore $r \in (r \circ (s_1 \circ a \circ s_2) \circ S] \subseteq (A \circ a \circ s_2 \circ S] \subseteq (A] = A$, a contradiction. Then $H_A(S)$ is a hyperdeal of S. If $H_A(S) = S$, then $a \in H_A(S)$ for all $a \in S$. It follows that $r \notin (r \circ a \circ S]$ for all $r \in S - A$. Consider $r \notin (r \circ a \circ S] \subseteq (r \circ S \circ S] \subseteq (S] = S$, a contradiction. Hence $H_A(S) \neq S$. □

Corollary 3.9. *If A is a proper right hyperideal of an ordered semihypergroup S^0, then $H_A(S^0)$ is a proper hyperdeal of S^0.*

Proof. Let r be an elements of S^0. It is easy to see that $\left(r \circ 0 \circ S^0\right] = (\{0\}] \subseteq (0 \circ a] \subseteq (A] = A$ for all $a \in A$. Thus $r \notin \left(r \circ 0 \circ S^0\right]$ for all $r \in S^0 - A$. This implies that $0 \in H_A(S^0) \neq \emptyset$. Therefore by Lemma 3.8, $H_A(S^0)$ is a proper hyperdeal of S^0. □

Below we extend [2] to ordered semihypergroups, showing in particular that indeed $H_A(S^0)$ is a semiprime hyperideal of an ordered semihypergroup S^0.

Theorem 3.10. *Let A be a proper right hyperideal of an ordered semihypergroup S^0 with scalar identity. Then the following statements hold.*

1. *For any right hyperideal B of S^0, $B \subseteq H_A(S^0)$ if and only if $s \notin (s \circ B]$ for all $s \in S^0 - A$.*
2. *$H_A(S^0)$ is a semiprime right hyperideal of S^0.*
3. *If A is a left hyperideal of S^0, then $A \subseteq H_A(S^0)$.*

Proof.

1. First assume that B is a right hyperideal of S^0 such that $B \subseteq H_A(S^0)$. Assume that $s \in (s \circ B] = (s \circ B \circ e] \subseteq \left(s \circ B \circ S^0\right]$ for all $s \in S^0 - A$. Since $B \subseteq H_A(S^0)$, we have $r \notin \left(r \circ B \circ S^0\right]$ for all $r \in S^0 - A$, a contradiction. Hence $s \notin (s \circ B]$ for all $s \in S^0 - A$.

Conversely, assume that $s \notin (s \circ B]$ for all $s \in S^0 - A$. Let b be any element of S^0 such that $b \in B$. Since B is a right hyperideal of S^0, we have $b \circ S^0 \subseteq B \circ S^0 \subseteq B$. It is easy to see that $(s \circ b \circ S^0] \subseteq (s \circ B]$. Clearly, $s \notin (s \circ b \circ S^0]$. Therefore $b \in H_A(S^0)$ and hence $B \subseteq H_A(S^0)$.

2. By Corollary 3.9, $H_A(S^0)$ is a proper right hyperideal of S^0. Let B be a right hyperideal of S^0 such that $B^2 \subseteq H_A(S^0)$. Suppose that $B \nsubseteq H_A(S^0)$. Then by (1), $s \in (s \circ B]$ for some $s \in S^0 - A$. Clearly, $s \in (s \circ B] \subseteq ((s \circ B] \circ B] = (s \circ B \circ B] = (s \circ B^2] \subseteq (s \circ H_A(S^0)]$. Thus by (1), $s \in A$, a contradiction. Therefore $B \subseteq H_A(S^0)$ and hence $H_A(S^0)$ is a semiprime right hyperideal of S^0.

3. Let $s \in S^0 - A$. If $s \in (s \circ A]$, then $s \in (s \circ A] \subseteq (S \circ A] \subseteq (A] = A$, a contradiction. Therefore $s \notin (s \circ A]$. Thus by (1), $A \subseteq H_A(S^0)$. □

By Lemma 3.10, we immediately obtain the following corollary:

Corollary 3.11. *Let A be a proper hyperideal of an ordered semihypergroup S^0 with scalar identity. Then the following statements hold.*

1. *For any hyperideal B of S^0, $B \subseteq H_A(S^0)$ if and only if $s \notin (s \circ B]$ for all $s \in S^0 - A$.*
2. *$H_A(S^0)$ is a semiprime hyperideal of S^0.*
3. *$A \subseteq H_A(S^0)$.*

This concept of an associated prime right hyperideal in ordered semihypergroups is a generalization of the concept of an associated prime right ideal in ordered semigroup.

Definition 3.12. Let A be a proper right hyperideal of an ordered semihypergroup S. Then the set $P_r(A) = \{s \in S : x \circ s \subseteq A \text{ for some } x \in S - A\}$ is called the associated prime right hyperideal of A.

Below we extend [2] to ordered semihypergroups, showing in particular that indeed $P_r(A)$ is a proper right hyperideal of an ordered semihypergroup S containing A.

Lemma 3.13. *Let S be an ordered semihypergroup with scalar identity. If A be a proper right hyperideal of S, then $P_r(A)$ is a proper right hyperideal of S containing A.*

Proof. Let a be any element of S such that $a \in A$. Then $a \in \{a\} = e \circ a \subseteq A$. By Definition 3.12, $a \in P_r(A)$. Thus $A \subseteq P_r(A) \neq \emptyset$. To show that $P_r(A)$ is a proper right hyperideal of S. Let $a, s \in S$ such that $a \in P_r(A)$. Then there exists $r \in S - A$ such that $r \circ a \subseteq A$. Clearly, $r \circ (a \circ s) \subseteq A \circ s \subseteq A$. It is easy to see that $a \circ s \subseteq P_r(A)$. Therefore $P_r(A)$ is a right hyperideal of S. If $e \in P_r(A)$, then there exists $x \in S - A$ such that $x \in \{x\} = x \circ e \subseteq A$, a contradiction. Therefore $P_r(A) \neq S$ and hence $P_r(A)$ is a proper right hyperideal of S containing A. □

The following theorem show that if A is a proper right hyperideal of an ordered semihypergroup S with scalar identity, then $P_r(A)$ is a completely prime right hyperideal of S containing A.

Theorem 3.14. *Let S be an ordered semihypergroup with scalar identity. If A is a proper right hyperideal of S, then $P_r(A)$ is a completely prime right hyperideal of S containing A.*

Proof. By Lemma 3.13, $P_r(A)$ is a proper right hyperideal of S containing A. Let a and b be any elements of S such that $a \circ b \subseteq P_r(A)$. Then $x \circ (a \circ b) \subseteq A$ for some $x \in S - A$. If $x \circ a \subseteq A$, then $a \in P_r(A)$. Now if $x \circ a \nsubseteq A$, then there exists $r \in x \circ a$ such that $r \in S - A$. Clearly $r \circ b \subseteq (x \circ a) \circ b \subseteq A$. Therefore $b \in P_r(A)$ and hence $P_r(A)$ is a completely prime right hyperideal of S containing A. □

4. Prime and completely prime hyperideals

In this section, we introduce the concepts of prime, semiprime, completely prime and completely semiprime right hyperideals of ordered semihypergroups, and investigate their related properties. In particular, we discuss the relationship between the completely semiprime and the completely prime right hyperideals in right chain ordered semihypergroups.

Definition 4.1. An ordered semihypergroup S is called a right chain ordered semihypergroup if the right hyperideals of S form a chain, i.e., for any right hyperideals A, B of S we have $A \subseteq B$ or $B \subseteq A$.

Now we are giving some basic properties of right chain ordered semihypergroups, which will be very helpful in later section.

Theorem 4.2. *Let S be a right chain ordered semihypergroup. Then the following statements hold.*

1. *A right hyperideal P of S is a semiprime right hyperideal if and only if P is a prime right hyperideal of S.*
2. *A right hyperideal P of an ordered semihypergroup with scalar identity S is a completely semiprime right hyperideal if and only if P is a completely prime right hyperideal of S.*

Proof.

1. Assume that P is a semiprime right hyperideal of S. Let A and B be two right hyperideals of S such that $A \circ B \subseteq P$. Since S is a right chain semihypergroup, we have $A \subseteq B$ or $B \subseteq A$. If $A \subseteq B$, then $A^2 = A \circ A \subseteq A \circ B \subseteq P$. By assumption, $A \subseteq P$. Similarly, if $B \subseteq A$, then $B \subseteq P$. Hence P is a prime right hyperideal of S.

2. Assume that P is a completely semiprime right hyperideal of S. If P is a completely semiprime right hyperideal of S, then P is a semiprime right hyperideal of S. By (1), P is a prime right hyperideal of S. Thus by Theorem 3.5, P is a completely prime right hyperideal of S. □

By Theorem 4.2, we immediately obtain the following corollary.

Corollary 4.3. *Let S be a right chain ordered semihypergroup. Then the following statements hold.*

1. *A hyperideal P of S is a semiprime hyperideal if and only if P is a prime hyperideal of S.*
2. *A hyperideal P of a semihypergroup with scalar identity S is a completely semiprime hyperideal if and only if P is a completely prime hyperideal of S.*

Let A be a right hyperideal (hyperideal) of an ordered semihypergroup S. We adopt from [2] the following two useful notions.

Definition 4.4. A right hyperideal (hyperideal) B of an ordered semihypergroup S is said to be *right A-nilpotent* (A-nilpotent) if $B^n \subseteq A$ for some $n \in \mathbf{Z}^+$. An element a of S is said to be *right A-nilpotent* (A-nilpotent) if $a^n \subseteq A$ for some $n \in \mathbf{Z}^+$.

The Hoehnke hyperideal is a useful tool in the study of right chain ordered semihypergroups. The following result extends [2] to right chain ordered semihypergroups.

Theorem 4.5. *Let A be a proper right hyperideal of a right chain ordered semihypergroup S^0. Then the following statements hold.*

1. *If B is a right hyperideal of an ordered semihypergroup with scalar identity S^0 such that $B \subseteq H_A(S^0)$ and B is not right A-nilpotent, then $\bigcap_{n\in\mathbf{Z}^+} (B^n] (\neq \emptyset)$ is a completely prime right hyperideal of S^0.*
2. *If b is an element of an ordered semihypergroup with scalar identity S^0 such that $b \in H_A(S^0)$ and b is not right A-nilpotent, then $\bigcap_{n\in\mathbf{Z}^+} (b^n \circ S] (\neq \emptyset)$ is a prime right hyperideal of S.*

Proof.

1. Let a be an element of S such that $a^2 \subseteq \bigcap_{n\in\mathbf{Z}^+} (B^n]$. Suppose that $a \notin \bigcap_{n\in\mathbf{Z}^+} (B^n]$. Then there exists $m \in \mathbf{Z}^+$ such that $a \notin (B^m]$. Since S^0 is a right chain ordered semihypergroup, we have $(B^m] \subseteq \left(a \circ S^0\right]$. Clearly,

$$a^2 \subseteq \left(B^{2m+1}\right] = \left(B^m \circ B^{m+1}\right] \subseteq \left((a \circ S] \circ B^{m+1}\right]$$

$$\subseteq \left((a \circ S] \circ \left(B^{m+1}\right]\right]$$
$$\subseteq \left(\left(a \circ S \circ B^{m+1}\right]\right]$$
$$\subseteq \left(a \circ B^{m+1}\right] = (a \circ B^{m} \circ B] \subseteq (a \circ (a \circ S] \circ B] \subseteq \left(a^2 \circ B\right].$$

By Theorem 3.10 (1), $a^2 \subseteq A$. Thus $B^{2m+1} \subseteq \left(B^{2m+1}\right] \subseteq \left(a^2 \circ B\right] \subseteq (A] = A$. It is easy to see that B is right A-nilpotent. This is a contradiction. Therefore $\bigcap_{n\in \mathbf{Z}^+} (B^n]$ is a completely semiprime right hyperideal of S^0. By Theorem 4.2 (2), $\bigcap_{n\in \mathbf{Z}^+} (B^n]$ is a completely prime right hyperideal of S^0.

2. By Proposition 2.8, $\left(b^n \circ S^0\right]$ is a right hyperideal of S^0. Thus by assumption, $\bigcap_{n\in \mathbf{Z}^+} (b^n \circ S^0] \subseteq \left(b \circ S^0\right] \subseteq H_A(S^0)$. Let C be a right hyperideal of S^0 such that $C^2 \subseteq \bigcap_{n\in \mathbf{Z}^+} (b^n \circ S^0]$. Suppose that $C \nsubseteq \bigcap_{n\in \mathbf{Z}^+} (b^n \circ S^0]$. Then there exists $m \in \mathbf{Z}^+$ such that $C \nsubseteq (b^m \circ S^0]$. Since S^0 a right chain ordered semihypergroup, we have $\left(b^m \circ S^0\right] \subseteq C$. Clearly,

$$b^{2m} = b^m \circ b^m \subseteq \left(b^m \circ S^0\right] \circ \left(b^m \circ S^0\right] \subseteq C \circ C \subseteq \bigcap_{n\in \mathbf{Z}^+} (b^n \circ S^0] \subseteq \left(b^{2m+1} \circ S^0\right] = \left(b^{2m} \circ (b \circ S^0)\right].$$

By Theorem 3.10 (1), $b^{2m} \subseteq A$. Thus $b^{2m} \subseteq \left(b^{2m} \circ (b \circ S^0)\right] \subseteq (A] = A$. It is easy to see that b is right A-nilpotent. This is a contradiction. Therefore $\bigcap_{n\in \mathbf{Z}^+} (b^n \circ S^0]$ is a semiprime right hyperideal of S^0. By Theorem 4.2 (3), $\bigcap_{n\in \mathbf{Z}^+} (b^n \circ S^0]$ is a prime right hyperideal of S^0. □

Corollary 4.6. *Let A be a proper hyperideal of a right chain ordered semihypergroup S^0. Then the following statements hold.*

1. *If B is a hyperideal of an ordered semihypergroup with scalar identity S^0 such that $B \subseteq H_A(S^0)$ and B is not A-nilpotent, then $\bigcap_{n\in \mathbf{Z}^+} (B^n] (\neq \emptyset)$ is a completely prime hyperideal of S^0.*
2. *If b is an element of an ordered semihypergroup with scalar identity S^0 such that $b \in H_A(S^0)$ and b is not A-nilpotent, then $\bigcap_{n\in \mathbf{Z}^+} (b^n \circ S](\neq \emptyset)$ is a prime hyperideal of S.*

Proof. Similar to the proof of Lemma 4.5. □

Corollary 4.7. *Let S^0 be an ordered semihypergroup with scalar identity. Then the following statements hold.*

1. *If B is a hyperideal of a right chain ordered semihypergroup S^0 such that $(B^n] \neq \left(B^{n+1}\right]$ for all $n \in \mathbf{Z}^+$, then $\bigcap_{n\in \mathbf{Z}^+} (B^n] (\neq \emptyset)$ is a completely prime hyperideal of S^0.*
2. *If B is a right hyperideal of a right chain ordered semihypergroup S^0 such that $(B^n] \neq \left(B^{n+1}\right]$ for all $n \in \mathbf{Z}^+$, then $\bigcap_{n\in \mathbf{Z}^+} (B^n] (\neq \emptyset)$ is a completely prime right hyperideal of S^0.*

Proof.

1. By lemma 2.6, $\bigcap_{n\in \mathbf{Z}^+} (B^n]$ is a hyperideal of S^0. If $\bigcap_{n\in \mathbf{Z}^+} (B^n] = S^0$, then $(B^n] \supseteq \bigcap_{n\in \mathbf{Z}^+} (B^n] = S^0$ for all $n \in \mathbf{Z}^+$. Thus $(B^n] = \left(B^{n+1}\right]$, a contradiction. Suppose that $\bigcap_{n\in \mathbf{Z}^+} (B^n] \neq S^0$. Let $b \in S^0 - \bigcap_{n\in \mathbf{Z}^+} (B^n]$. It is

easy to see that $b \notin (B^m]$ for some $m \in \mathbf{Z}^+$. If $b \in (b \circ B]$, then

$$b \in (b \circ B] \subseteq ((b \circ B] \circ B] = ((b \circ B] \circ (B]] \subseteq (b \circ B \circ B] \subseteq \left(b \circ B^2\right] \subseteq \cdots \subseteq (b \circ B^m].$$

This means that $b \in (b \circ B^m] \subseteq (B^m] = B^m$, a contradiction. Now, let $b \notin (b \circ B]$. By Corollary 3.11 (1), $B \subseteq H_{\bigcap_{n \in \mathbf{Z}^+} (B^n]}(S^0)$. To show that B is not $\bigcap_{n \in \mathbf{Z}^+} (B^n]$-nilpotent. Assume that $B^k \subseteq \bigcap_{n \in \mathbf{Z}^+} (B^n]$ for some $k \in \mathbf{Z}^+$. Consider $\left(B^k\right] \subseteq \bigcap_{n \in \mathbf{Z}^+} (B^n] \subseteq \left(B^{k+1}\right]$. This implies that $\left(B^k\right] = \left(B^{k+1}\right]$, a contradiction. Therefore B is not $\bigcap_{n \in \mathbf{Z}^+} (B^n]$-nilpotent. By Corollary 4.6 (1), $\bigcap_{n \in \mathbf{Z}^+} (B^n]$ is a completely prime hyperideal of S^0.

2. Similar to the proof of (1). □

Later on we will need the following generalization of [2].

Theorem 4.8. *Let A be a prime right hyperideal of an ordered semihypergroup S with scalar identity. If B is a right hyperideal of a right chain ordered semihypergroup S, then $B \subseteq A$ or $P_r(A) \subseteq B$.*

Proof. Let B be a left hyperideal of S such that $P_r(A) \nsubseteq B$. Then there exists $b \in P_r(A) - B$. Clearly, $a \circ b \subseteq A$ for some $a \in S - A$. Since S is a right chain ordered semihypergroup, we have $B \subseteq (b \circ S]$. Consider,

$$(a \circ S] \circ B \subseteq (a \circ S \circ B] \subseteq (a \circ B] \subseteq (a \circ (b \circ S]] = (a \circ b \circ S] \subseteq (A \circ S] \subseteq (A] = A.$$

By assumption, $(a \circ S] \subseteq A$ or $B \subseteq A$. It is easy to see that $B \subseteq A$. □

Corollary 4.9. *Let A be a prime hyperideal of an ordered semihypergroup with with scalar identity S. If B is a hyperideal of a right chain ordered semihypergroup S, then $B \subseteq A$ or $P_r(A) \subseteq B$.*

Proof. Let B be a hyperideal of S such that $P_r(A) \nsubseteq B$. Then there exists $b \in P_r(A) - B$. Clearly, $a \circ b \subseteq A$ for some $a \in S - A$. Since S is a right chain ordered semihypergroup, we have $B \subseteq (b \circ S]$. Consider,

$$(S \circ a \circ S] \circ B \subseteq (S \circ a \circ S \circ B] \subseteq (S \circ a \circ B] \subseteq (S \circ a \circ (b \circ S]] = (S \circ a \circ b \circ S] \subseteq (S \circ A \circ S] \subseteq (A] = A.$$

By assumption, $(S \circ a \circ S] \subseteq A$ or $B \subseteq A$. It is easy to see that $B \subseteq A$. □

The following theorem generalizes [2].

Theorem 4.10. *If A is a proper hyperideal of a right chain ordered semihypergroup S such that $A = \left(A^2\right]$, then $A = (a^n \circ A]$ for any $a \in S - A$ and $n \in \mathbf{Z}^+$.*

Proof. Let $a \in S - A$. Since S is a right chain ordered semihypergroup, we have $A \subseteq (a \circ S]$. Clearly,

$$A = A^2 \subseteq (a \circ S] \circ A \subseteq (a \circ S \circ A] \subseteq (a \circ A] \subseteq (A] = A.$$

Then $A = (a \circ A]$. Suppose that $A = \left(a^k \circ A\right]$ for some $k \in \mathbf{Z}^+$. Consider, $A = \left(a^k \circ A\right] = \left(a^k \circ a \circ A\right] = \left(a^{k+1} \circ A\right]$. Thus the result follows by induction. □

The following two notions are obvious analogues of the concepts defined in [2].

Definition 4.11. A right hyperideal P of a right chain ordered semihypergroup S is called an exceptional prime right hyperideal of S if P is prime but not completely prime.

If $A \subset B$ are right hyperideals of an ordered semihypergroup S such that there are no further ideals properly between A and B, then we say that B is minimal over A. The following theorem is an extension of the Proposition 3.7 [2] to right chain ordered semihypergroups.

Theorem 4.12. *Let S be an ordered semihypergroup with scalar identity. If P is an exceptional prime right hyperideal of a right chain ordered semihypergroup S, then there exists a unique hyperideal A of S such that $P \subset A$ and A is minimal over P.*

Proof. Let B be a hyperideal of S such that $P \subset B$ and let $A = \langle B \rangle$. By Theorem 4.8, $A \subseteq P$ or $P_r(P) \subseteq A$. Now by our hypothesis, $P_r(P) \subseteq A$. By Theorem 3.14, $P_r(P)$ is a completely prime right hyperideal of S containing P. Since P is an exceptional prime right hyperideal of S, we have $P \subseteq P_r(P)$. Thus $P \subset A$. Now since $A = \langle B \rangle$, we have A is minimal over P and obviously A is a unique hyperideal of S. □

Theorem 4.13. *Let S be an ordered semihypergroup with scalar identity. If P is an exceptional prime right hyperideal of a right chain ordered semihypergroup S, then there exists a minimal $A(P \subset A)$ over P such that $A = A^2$.*

Proof. By Theorem 4.12, then there exists a minimal hyperideal A over P such that $P \subset A$. Suppose that $A \neq A^2$. Then $A^2 \subset A$. Since A is minimal over P, we have $A^2 \subseteq P$. By assumption, $A \subseteq P$, a contradiction. Hence $A = A^2$. □

Theorem 4.14. *Let A be a hyperideal of an ordered semihypergroup S with scalar identity. If P is an exceptional prime right hyperideal of a right chain ordered semihypergroup S, then there exists a minimal $A(P \subset A)$ over P and $a \in A - P$ such that $P \subset \bigcap_{n \in \mathbf{Z}^+} (a^n \circ S]$.*

Proof. Set $C = \left\{ s \in S : \bigcap_{n \in \mathbf{Z}^+} (s^n \circ A] \subseteq P \right\}$. Clearly, $P \subseteq C$ and $C \neq \emptyset$. We claim that $C \subseteq A$. By Theorem 4.13, $A = A^2$. If $b \in S - A$, then $A = \bigcap_{n \in \mathbf{Z}^+} (b^n \circ A]$. Since $P \nsubseteq A$, there exists $b \notin C$, which proves our claim.

Let $c^2 \subseteq P$ such that $c \notin P$. If $c \in (C \circ c \circ A]$, then $c \in (d \circ b \circ A]$ for some $d \in C$. Thus $(c \circ A] \subseteq ((d \circ c \circ A] \circ A] = (d \circ c \circ A \circ A] \subseteq (d \circ c \circ A]$. Consider,

$$(c \circ A] \subseteq (d \circ c \circ A] \subseteq (d \circ (c \circ A]] \subseteq (d \circ (d \circ c \circ A]] \subseteq \left(d^2 \circ c \circ A\right] \subseteq \cdots \subseteq (d^n \circ c \circ A].$$

We obtain for any $n \in \mathbf{Z}^+$ such that $(c \circ A] \subseteq (d^n \circ c \circ A] \subseteq (d^n \circ A]$. Now since $d \in C$, we have $(c \circ S] \circ (A] \subseteq (c \circ S \circ A] \subseteq (c \circ A] \subseteq \bigcap_{n \in \mathbf{Z}^+} (d^n \circ A] \subseteq P$. By assumption, $c \in (c \circ S] \subseteq P$ or $A \subseteq P$, a contradiction. It is easy to see that $c \notin (C \circ c \circ A]$. From our hypothesis, $(C \circ c \circ A] \subseteq (c \circ S]$. Now, if $C = A$, then $(A \circ c \circ A] \subseteq (c \circ S]$. Hence

$$\begin{aligned}(c \circ S \circ A]^2 \subseteq (c \circ A]^2 = (c \circ A] \circ (c \circ A] &\subseteq (c \circ (A \circ c \circ A)] \\ &\subseteq (c \circ (A \circ c \circ A]] \subseteq (c \circ (c \circ S]] \subseteq \left(c^2 \circ S\right] \subseteq (P \circ S] \subseteq P.\end{aligned}$$

By the given hypothesis, $(c \circ S] \circ A \subseteq (c \circ S \circ A] \subseteq P$ and so $c \in (c \circ S] \subseteq P$ or $A \subseteq P$, a contradiction. Therefore $C \subseteq A$. To complete the proof, take any $a \in A - C$. Then $\bigcap_{n \in \mathbf{Z}^+} (a^n \circ A] \nsubseteq P$. Now by our hypothesis, $P \subseteq \bigcap_{n \in \mathbf{Z}^+} (a^n \circ A] \subseteq \bigcap_{n \in \mathbf{Z}^+} (a^n \circ S]$. □

5. Prime segments of right chain ordered semihypergroups

A prime segment of a right chain ordered semihypergroup S is a pair $P \subset Q$ of completely prime right hyperideals of S such that no further completely prime right hyperideal of S exists between P and Q. In the following theorem we extend to right chain ordered semigroups the classification of prime segments of right chain ordered semigroups given in [2].

Theorem 5.1. *Let S^0 be a right chain ordered semihypergroup, and let $P \subset Q$ be a prime segment of S^0. Then one of the following possibilities occurs.*

1. *There are no further right hyperideals of S^0 between P and Q; the prime segment is called simple in this case.*
2. *For every $a \in Q - P$ there exists a right hyperideal $A \subseteq Q$ of S such that $a \in A$ and $\bigcap_{n \in \mathbf{Z}^+} (A^n] = P$; the prime segment is called archimedean in this case.*
3. *There exists a prime right hyperideal R of S^0 with $P \subset R \subset Q$; the prime segment is called exceptional in this case.*
4. *There exists a right hyperideal B of S^0 such that $P \subset B \subset Q$ and B is minimal over P; the prime segment is called supplementary in this case.*

Possibilities (1), (2), (3) *are mutually exclusive, and possibilities* (1), (2), (4) *are mutually exclusive.*

Proof. Suppose that a prime segment $P \subset Q$ is not simple case (1) does not hold. Then there exists a right hyperideal A of S^0 such that $P \subset A \subset Q$. If $Q \nsubseteq H_A(S^0)$, then $H_A(S^0) \subset Q$. By Theorem 3.10 and Theorem 4.2, $H_A(S^0)$ is a prime right hyperideal of S^0 lying properly between P and Q. Thus a prime segment $P \subset Q$ is exceptional in this case. Hence, for the remainder of the proof we assume that for every right hyperideal A in this prime segment we have $Q \subseteq H_A(S^0)$. If the prime segment $P \subset Q$ contains a right hyperideal A of S^0 such that $\left(A^{n+1}\right] = (A^n]$ for some $n \in \mathbf{Z}^+$, then $(A^n] = \left(A^{n+1}\right] = (A^n \circ A] = \left(A^{n+1} \circ A\right] = \left(A^{n+2}\right] = \cdots = \left(A^{n+k}\right]$. Hence $(A^n] = \left(A^{n+k}\right]$ for all $k \in \mathbf{Z}^+$. Thus for the right hyperideal $B = A^n$ for all $n \in \mathbf{Z}^+$. It is easy to see that $B = A^n = A^{nk} = (A^n)^k = B^k$ and $B \subseteq A \subset Q$. Next, let $A^n = B \subseteq P$. Since P is completely prime, we have $A \subseteq P \subset A$, a contradiction. Hence, $B \nsubseteq P$. Since S^0 is a right chain ordered semihypergroup, we have $P \subseteq B$. We show that furthermore B is minimal over P. If not, then there exists a right hyperideal C of S^0 such that $P \subset C \subset B$. Then $P \subset C \subset Q$. By assumption, $B \subseteq Q \subseteq H_C(S^0)$. By Theorem 4.5 (1), $\bigcap_{n \in \mathbf{Z}^+} B^n = B$ is a completely prime right hyperideal of S^0. This is a contradiction to the fact that $P \subset Q$ is a prime segment. Hence B is minimal over P and thus the prime segment $P \subset Q$ is supplementary in this case.

We are left with the case where there exists a right hyperideal A of S^0 such that $P \subset A \subset Q$ and for any such a right hyperideal A we have $(A^n] \neq \left(A^{n+1}\right]$ for all $n \in \mathbf{Z}^+$. Let $\mathcal{A}$ be the set of all right hyperideals A of S^0 such that $P \subset A \subset Q$. Since S^0 is a right chain ordered semihypergroup and the right hyperideal P is completely prime, we have $P \subseteq A^n$ for any $A \in \mathcal{A}$ and $n \in \mathbf{Z}^+$. Thus $P \subseteq \bigcap_{n \in \mathbf{Z}^+} (A^n] \subset Q$. By Corollary 4.7, $\bigcap_{n \in \mathbf{Z}^+} (A^n]$ is completely prime. It follows that $\bigcap_{n \in \mathbf{Z}^+} (A^n] = P$ for all $A \in \mathcal{A}$.

Next, let $R = \bigcup\{A : A \in \mathcal{A}\}$. By Lemma 2.6, R is a right hyperideal of S^0. If $R = Q$, then the prime segment $P \subset Q$ is archimedean. Assume that $R \neq Q$. Then $R \subset Q$. We consider two cases.

Case 1. $\left(Q^2\right] \neq Q$. Clearly, $\left(Q^2\right] \subset Q$ it follows that $P \subset \left(Q^2\right] \subset Q$. Thus $\left(Q^2\right] \in \mathcal{A}$. By hypothesis, $\bigcap_{n \in \mathbf{Z}^+} \left(Q^{2n}\right] = \bigcap_{n \in \mathbf{Z}^+} \left((Q^2]^n\right] = P$. Therefore $P \subseteq \bigcap_{n \in \mathbf{Z}^+} (Q^n] \subseteq \bigcap_{n \in \mathbf{Z}^+} \left(Q^{2n}\right] \subseteq P$. Hence $P = \bigcap_{n \in \mathbf{Z}^+} (Q^n]$ and the prime segment $P \subset Q$ is archimedean in this case.

Case 2. $\left(Q^2\right] = Q$. We show that the right hyperideal R is prime in this case. Let A be a right hyperideal A of S^0 such that $A^2 \subseteq R$. Clearly, $A^2 \subseteq Q$ and since Q is completely prime, $A \subseteq Q$ follows. If $A = Q$, then $Q = (Q^2] = (A^2] \subseteq (R] = R \subset Q$, a contradiction. Hence $A \subset Q$. By the definition of R, $A \subseteq R$. Therefore R is semiprime. By Theorem 4.2, R is prime and the prime segment $P \subset Q$ is exceptional in this case.

It is easy to see that possibilities (1), (2), (3) are mutually exclusive. It is also clear that (1) and (4) are mutually exclusive. To complete the proof, assume that the possibility (4) occurs and B is minimal over P. Then $B = (B^2]$. By Lemma 4.10, $P \subset B = \bigcap_{n \in \mathbf{Z}^+} (a^n \circ B] \subseteq \bigcap_{n \in \mathbf{Z}^+} (a^n \circ S^0]$ for any $a \in Q - B$. Hence (2) and (4) cannot happen simultaneously. □

Theorem 5.2. *Let $P \subset Q$ be a prime segment of a right chain ordered semihypergroup S^0. Then the following conditions are equivalent.*

1. *The prime segment $P \subset Q$ is archimedean.*
2. *For any $a \in Q - P$, $\bigcap_{n \in \mathbf{Z}^+} (a^n \circ S^0] = P$.*
3. *For any $a \in Q - P$, $(Q \circ a \circ S^0] \subset (a \circ S^0]$.*

Proof.

$(1 \Rightarrow 2)$ Follows directly from the definition of an archimedean prime segment.

$(2 \Rightarrow 3)$ Let $a \in Q - P$. Suppose that $(a \circ S^0] \subseteq (Q \circ a \circ S^0]$. Then $a \in q \circ a \circ s$ for some $q \in Q$ and $s \in S^0$. If $q \in P$, then $a \in (P \circ a \circ S^0] \subseteq (P] = P$, a contradiction. Hence $q \in Q - P$. Moreover, $q \circ a \circ s \subseteq q \circ (q \circ a \circ s) \circ s = q^2 \circ a \circ s^2 \subseteq q^3 \circ a \circ s^3 \subseteq \cdots \subseteq q^n \circ a \circ s^n$ for any $n \in \mathbf{Z}^+$. Clearly, $a \in q^n \circ a \circ s^n$, and thus $a \in q^n \circ a \circ S^0 \subseteq (q^n \circ a \circ S^0] \subseteq (q^n \circ S^0]$. Hence by (2), $a \in \bigcap_{n \in \mathbf{Z}^+} (q^n \circ S^0] = P$, which is a contradiction. Thus $(a \circ S^0] \nsubseteq (Q \circ a \circ S^0]$. Since S is a right chain ordered semihypergroup, we have $(Q \circ a \circ S^0] \subset (a \circ S^0]$, as desired.

$(3 \Rightarrow 1)$ Let $a \in Q - P$. If $(Q \circ a \circ S^0] \subseteq P$, then $(Q] \circ (a \circ S^0] \subseteq P$. By assumption, $Q \subseteq P$ or $a \in (a \circ S^0] \subseteq P$, a contradiction. Hence $(Q \circ a \circ S^0] \nsubseteq P$. Since S^0 is a right chain ordered semihypergroup, we have $P \subset (Q \circ a \circ S^0]$. By (3), $P \subset (Q \circ a \circ S^0] \subset (a \circ S^0] \subseteq Q$ and thus the prime segment $P \subset Q$ is not simple. Suppose the prime segment $P \subset Q$ is exceptional. Then there exists a prime right hyperideal A of S^0 such that $P \subset A \subset Q$. Then by Theorem 4.12, there exists a hyperideal B of S^0 which is minimal over A. This however is impossible, since (3) implies that for any $a \in B - A$ we have $A \subset (Q \circ a \circ S^0] \subset (a \circ S^0] \subseteq B$. Finally, suppose the prime segment $P \subset Q$ is supplementary. Then there exists a right hyperideal C of S such that $P \subset C \subset Q$ and C is minimal over P. Then by (3), for any $a \in C - P$ we have $P \subset Q \circ a \circ S^0 \subset a \circ S^0 \subseteq C$, a contradiction. Hence the prime segment $P \subset Q$ is neither simple, nor exceptional, nor supplementary, and thus by Theorem 5.1, it must be archimedean. □

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