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Mersenne Lucas numbers and complete homogeneous symmetric functions

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Abstract

In this paper, we first introduce new definition of Mersenne Lucas numbers sequence as, for $n \geqslant 2$, $m_n = 3m_{n-1} - 2m_{n-2}$ with the initial conditions $m_0 = 2$ and $m_1 = 3$. Considering this sequence, we give Binet's formula, generating function and symmetric function of Mersenne Lucas numbers. By using the Binet's formula we obtain some well-known identities such as Catalan's identity, Cassini's identity and d'Ocagne's identity. After that, we give some new generating functions for products of (p, q)-numbers with Mersenne Lucas numbers at positive and negative indice.

Keywords: Mersenne Lucas numbers, (p, q)-numbers, symmetric functions, Binet's formula, generating functions.

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1. Introduction and some preliminary properties

In recent years, we have seen so many studies on the different (p, q)-numbers sequences such as (p, q)-Fibonacci, (p, q)-Lucas, (p, q)-Pell, (p, q)-Pell Lucas, (p, q)-Jacobsthal and (p, q)-Jacobsthal Lucas sequences, we refer the reader to [6, 13–15]. Now, let we define the generalized (p, q)-Fibonacci sequence $\{f_{p,q,n}(\alpha, \beta, \gamma)\}_{n=0}^{\infty}$, generalized (p, q)-Pell sequence $\{l_{p,q,n}(\alpha, \beta, \gamma)\}_{n=0}^{\infty}$ and generalized (p, q)-Jacobsthal sequence $\{C_{p,q,n}(\alpha, \beta, \gamma)\}_{n=0}^{\infty}$ as follows, for $n \geqslant 2$,

$$f_{p,q,0} = \alpha,\ f_{p,q,1} = \beta + \gamma p \text{ and } f_{p,q,n} = pf_{p,q,n-1} + qf_{p,q,n-2}, \tag{1.1}$$

$$l_{p,q,0} = \alpha,\ l_{p,q,1} = \beta + 2\gamma p \text{ and } l_{p,q,n} = 2pl_{p,q,n-1} + ql_{p,q,n-2}, \tag{1.2}$$

and

$$C_{p,q,0} = \alpha,\ C_{p,q,1} = \beta + \gamma p \text{ and } C_{p,q,n} = pC_{p,q,n-1} + 2qC_{p,q,n-2}, \tag{1.3}$$

respectively. Particular cases of the sequences $\{f_{p,q,n}(\alpha, \beta, \gamma)\}_{n=0}^{\infty}$, $\{l_{p,q,n}(\alpha, \beta, \gamma)\}_{n=0}^{\infty}$, and $\{C_{p,q,n}(\alpha, \beta, \gamma)\}_{n=0}^{\infty}$ are as follows.

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Case 1.1. If we take $\alpha = \gamma = 0$ and $\beta = 1$ in (1.1), (1.2), and (1.3), then we get:

- the $(\mathbf{p}, \mathbf{q})$-Fibonacci sequence

$$F_{p,q,0} = 0, \ F_{p,q,1} = 1 \text{ and } F_{p,q,n} = pF_{p,q,n-1} + qF_{p,q,n-2};$$

- the $(\mathbf{p}, \mathbf{q})$-Pell sequence

$$P_{p,q,0} = 0, \ P_{p,q,1} = 1 \text{ and } P_{p,q,n} = 2pP_{p,q,n-1} + qP_{p,q,n-2};$$

- and the $(\mathbf{p}, \mathbf{q})$-Jacobsthal sequence

$$J_{p,q,0} = 0, \ J_{p,q,1} = 1 \text{ and } J_{p,q,n} = pJ_{p,q,n-1} + 2qJ_{p,q,n-2};$$

respectively.

Case 1.2. If we take $\alpha = 2$, $\gamma = 1$ and $\beta = 0$ in (1.1), (1.2), and (1.3), then we get:

- the $(\mathbf{p}, \mathbf{q})$-Lucas sequence

$$L_{p,q,0} = 2, \ L_{p,q,1} = p \text{ and } L_{p,q,n} = pL_{p,q,n-1} + qL_{p,q,n-2};$$

- the $(\mathbf{p}, \mathbf{q})$-Pell Lucas sequence

$$Q_{p,q,0} = 2, \ Q_{p,q,1} = 2p \text{ and } Q_{p,q,n} = 2pQ_{p,q,n-1} + qQ_{p,q,n-2};$$

- and the $(\mathbf{p}, \mathbf{q})$-Jacobsthal Lucas sequence

$$j_{p,q,0} = 2, \ j_{p,q,1} = p \text{ and } j_{p,q,n} = pj_{p,q,n-1} + 2qj_{p,q,n-2};$$

respectively.

The Mersenne numbers play a key role in an investigations on the prime numbers. In the references [7, 9], certain important studies listed. Note that the Mersenne numbers is defined by the recurrence relation, for $n \geqslant 2$,

$$M_0 = 0, \ M_1 = 1 \text{ and } M_n = 3M_{n-1} - 2M_{n-2}. \tag{1.4}$$

Many important identities such as Catalan's identity, Cassini's identity and d'Ocagne's identity, etc, for Mersenne numbers are shown in [5]. Few properties that the Mersenne numbers satisfy are summarized below.

- Binet's formula for Mersenne numbers:

$$M_n = 2^n - 1.$$

- Negative extension of Mersenne numbers:

$$M_{-n} = \frac{-M_n}{2^n}.$$

- Generating function for Mersenne numbers:

$$g(z) = \sum_{n=0}^{\infty} M_n z^n = \frac{z}{1 - 3z + 2z^2}.$$

- Complete homogeneous symmetric function of Mersenne numbers:

$$M_n = h_{n-1}(2, 1). \tag{1.5}$$

- D'Ocagne's identity for Mersenne numbers:

$$M_r M_{n+1} - M_{r+1} M_n = 2^n M_{r-n}.$$

Next, we recall some properties of the symmetric functions that we will need in the sequel.

Definition 1.3 ([8])**.** Let k and n be two positive integers and $\{a_1, a_2, \dots, a_n\}$ are set of given variables the k-th elementary symmetric function $e_k(a_1, a_2, \dots, a_n)$ is defined by:

$$e_k^{(n)} = e_k(a_1, a_2, \dots, a_n) = \sum_{i_1+i_2+\cdots+i_n=k} a_1^{i_1} a_2^{i_2} \cdots a_n^{i_n} \qquad (0 \leqslant k \leqslant n),$$

with $i_1, i_2, \dots, i_n = 0$ or 1.

Remark 1.4. Set $e_0(a_1, a_2, \dots, a_n) = 1$, by usual convention. For $k < 0$ or $k > n$, we set $e_k(a_1, a_2, \dots, a_n) = 0$.

Example 1.5. The following lists the n elementary symmetric function for the first three positive values of n:

For $n = 1 : e_1(a_1) = a_1$.
For $n = 2 : e_1(a_1, a_2) = a_1 + a_2,\ e_2(a_1, a_2) = a_1 a_2$.
For $n = 3 : e_1(a_1, a_2, a_3) = a_1 + a_2 + a_3,\ e_2(a_1, a_2, a_3) = a_1 a_2 + a_1 a_3 + a_2 a_3,\ e_3(a_1, a_2, a_3) = a_1 a_2 a_3$.

Proposition 1.6. *Given an alphabet* $A = \{a_1, a_2, \dots, a_n\}$*, the generating function for the elementary symmetric function is given as:*

$$\sum_{k=0}^{\infty} e_k(a_1, a_2, \dots, a_n) z^k = \prod_{a \in A} (1 + az).$$

Definition 1.7 ([8])**.** Let k and n be two positive integers and $\{a_1, a_2, \dots, a_n\}$ are set of given variables, the k-th complete homogeneous symmetric function $h_k(a_1, a_2, \dots, a_n)$ is defined by:

$$h_k^{(n)} = h_k(a_1, a_2, \dots, a_n) = \sum_{i_1+i_2+\cdots+i_n=k} a_1^{i_1} a_2^{i_2} \cdots a_n^{i_n}, \qquad (k \geqslant 0), \tag{1.6}$$

with $i_1, i_2, \dots, i_n \geqslant 0$.

Remark 1.8. Set $h_0(a_1, a_2, \dots, a_n) = 1$, by usual convention. For $k < 0$, we set $h_k(a_1, a_2, \dots, a_n) = 0$. If $n = 2$, the k-th complete homogeneous symmetric function (1.6) gives us

$$h_k^{(2)} = h_k(a_1, a_2) = \frac{a_1^{k+1} - a_2^{k+1}}{a_1 - a_2}, \text{ for all } k \in \mathbb{N}_0,$$

with

$$h_0(a_1, a_2) = 1, \quad h_1(a_1, a_2) = a_1 + a_2, \quad h_2(a_1, a_2) = a_1^2 + a_1 a_2 + a_2^2, \quad \dots.$$

Proposition 1.9. *Given an alphabet* $A = \{a_1, a_2, \dots, a_n\}$*, the generating function for the complete homogeneous symmetric function is given as:*

$$\sum_{k=0}^{\infty} h_k(a_1, a_2, \dots, a_n) z^k = \frac{1}{\prod_{a \in A} (1 - az)}.$$

There is a fundamental relation between the elementary symmetric functions and the complete homogeneous ones:

$$\sum_{j=0}^{k} (-1)^j e_j(a_1, a_2, \dots, a_n) h_{k-j}(a_1, a_2, \dots, a_n) = 0,$$

which is valid for all $k > 0$.

Definition 1.10 ([2])**.** Given an alphabet $A = \{a_1, a_2\}$, the symmetrizing operator $\delta^k_{a_1 a_2}$ is defined by:

$$\delta^k_{a_1 a_2} f(a_1) = \frac{a_1^k f(a_1) - a_2^k f(a_2)}{a_1 - a_2}, \text{ for all } k \in \mathbb{N}_0. \tag{1.7}$$

If $f(a_1) = a_1$, the operator (1.7) gives us

$$\delta^k_{a_1 a_2} f(a_1) = \frac{a_1^{k+1} - a_2^{k+1}}{a_1 - a_2} = h_k(a_1, a_2).$$

2. The Mersenne Lucas numbers

In this section, we define the Mersenne Lucas sequence, and then we give Binet's formula and generating function for this sequence. Moreover, we obtain some results related with the Mersenne Lucas numbers. Let us now start at the following definition.

Definition 2.1. Let $n \geqslant 2$ be integer, the recurrence relation of Mersenne Lucas numbers $\{m_n\}_{n=0}^{\infty}$ is given as:

$$m_n = 3m_{n-1} - 2m_{n-2}, \tag{2.1}$$

with the initial conditions $m_0 = 2$ and $m_1 = 3$.

Now, we show that the first few terms of Mersenne Lucas numbers as: $\{2, 3, 5, 9, 17, 33, 65, 129, \dots\}$. Recurrence relationship (2.1) involves the characteristic equation

$$x^2 - 3x + 2 = 0,$$

which has two characteristic real roots:

$$x_1 = 2 \text{ and } x_2 = 1,$$

so that

$$x_1 + x_2 = 3, \quad x_1 x_2 = 2, \quad \text{and } x_1 - x_2 = 1.$$

The next theorem gives the n-th term of the Mersenne Lucas numbers.

Theorem 2.2. *The Binet's formula for Mersenne Lucas numbers is given by:*

$$m_n = 2^n + 1. \tag{2.2}$$

Proof. From the theory of difference equation we know the general term for Mersenne Lucas numbers can be expressed in the following form

$$m_n = C_1 2^n + C_2,$$

where C_1 and C_2 are the coefficients. For $n = 0, 1$, we have $\begin{cases} C_1 + C_2 = 2, \\ 2C_1 + C_2 = 3. \end{cases}$ By these equalities, $C_1 = C_2 = 1$. Therefore $ML_n = 2^n + 1$. This completes the proof. □

The negative extension of Mersenne Lucas numbers m_{-n}, even Mersenne Lucas numbers m_{2n} and odd Mersenne Lucas numbers m_{2n+1} given in the next proposition.

Proposition 2.3. *Let n be any positive integer. Then we have*

$$\begin{aligned} &1)\ m_{-n} = \frac{m_n}{2^n}, \\ &2)\ m_{2n} = m_n^2 - 2^{n+1}, \\ &3)\ m_{2n+1} = 2M_{2n} + 3, \end{aligned} \tag{2.3}$$

where M_n is the n-th Mersenne numbers.

Proof. By the Binet's formula (2.2), we can write

$$m_{-n} = \frac{1}{2^n} + 1 = \frac{2^n + 1}{2^n} = \frac{m_n}{2^n},$$
$$m_{2n} = 2^{2n} + 1 = (2^n + 1)^2 - 2^{n+1} = m_n^2 - 2^{n+1},$$

and

$$m_{2n+1} = 2^{2n+1} + 1 = 2\left(2^{2n} - 1\right) + 3 = 2M_{2n} + 3.$$

Thus, this completes the proof. □

We now investigate some identities and properties of the Mersenne Lucas sequence.

Proposition 2.4. *For $n \geqslant r$, we have*

$$m_{n-r}m_{n+r} - m_n^2 = 2^{n-r}m_r^2 - 2^{n+2}, \tag{2.4}$$

which is Catalan's identity for Mersenne Lucas sequence.

Proof. Using the Binet's formula (2.2), we easily obtain

$$\begin{aligned} m_{n-r}m_{n+r} - m_n^2 &= \left(2^{n-r} + 1\right)\left(2^{n+r} + 1\right) - (2^n + 1)^2 \\ &= 2^{2n} + 2^{n-r} + 2^{n+r} + 1 - \left(2^{2n} + 2^{n+1} + 1\right) \\ &= 2^{n+r} + 2^{n-r} - 2^{n+1} = 2^{n-r}\left(2^{2r} + 1\right) - 2^{n+1} = 2^{n-r}m_r^2 - 2^{n+2}. \end{aligned}$$

Hence, we obtain the desired result. □

By setting $r = 1$ in the relation (2.4), we obtain the following corollary which gives Cassini's identity of the Mersenne Lucas sequence.

Corollary 2.5. *Let n be positive integer, for the sequence $\{m_n\}$, we have*

$$m_{n-1}m_{n+1} - m_n^2 = 2^{n-1}.$$

The following proposition gives d'Ocagne's identity involving the Mersenne Lucas sequence.

Proposition 2.6. *For $n, r \in \mathbb{N}$, the following identity holds true:*

$$m_r m_{n+1} - m_{r+1}m_n = 2^n - 2^r.$$

Proof. Once more, using Binet's formula (2.2), we get

$$\begin{aligned} m_r m_{n+1} - m_{r+1}m_n &= (2^r + 1)\left(2^{n+1} + 1\right) - \left(2^{r+1} + 1\right)(2^n + 1) \\ &= \left(2^{r+n+1} + 2^r + 2^{n+1} + 1\right) - \left(2^{r+n+1} + 2^n + 2^{r+1} + 1\right) = 2^n - 2^r. \end{aligned}$$

As required. □

And we have the following corollary.

Corollary 2.7. *The following identity holds true:*

$$m_r m_{n+1} - m_{r+1}m_n = -\left(M_r M_{n+1} - M_{r+1}M_n\right) = -2^n M_{r-n},$$

where M_n is the n-th Mersenne numbers.

We have the following theorem.

Theorem 2.8. *The sum of the Mersenne Lucas numbers is*

$$\sum_{k=0}^{n} m_k = 2^{n+1} + n.$$

Proof. By using Binet's formula (2.2), we get

$$\sum_{k=0}^{n} m_k = \sum_{k=0}^{n} 2^k + 1 = \sum_{k=0}^{n} 2^k + \sum_{k=0}^{n} 1 = \frac{2^{n+1}-1}{2-1} + n + 1 = 2^{n+1} + n.$$

As required. □

Now, we aim to give the generating function for Mersenne Lucas numbers. For this purpose, we shall prove the following theorem.

Theorem 2.9. *For $n \in \mathbb{N}$, the new generating function of Mersenne Lucas numbers is given by*

$$g(z) = \sum_{n=0}^{\infty} m_n z^n = \frac{2-3z}{1-3z+2z^2}.$$

Proof. The Mersenne Lucas numbers can be considered as the coefficients of the formal power series

$$g(z) = \sum_{n=0}^{\infty} m_n z^n.$$

Using the initial conditions, we get

$$\begin{aligned} g(z) = \sum_{n=0}^{\infty} m_n z^n &= m_0 + m_1 z + \sum_{n=2}^{\infty} m_n z^n \\ &= m_0 + m_1 z + \sum_{n=2}^{\infty} (3m_{n-1} - 2m_{n-2})\, z^n \\ &= m_0 + m_1 z + 3z \sum_{n=1}^{\infty} m_n z^n - 2z^2 \sum_{n=0}^{\infty} m_n z^n \\ &= m_0 + (m_1 - 3m_0)\, z + 3z \sum_{n=0}^{\infty} m_n z^n - 2z^2 \sum_{n=0}^{\infty} m_n z^n \\ &= 2 - 3z + \left(3z - 2z^2\right) g(z). \end{aligned}$$

Hence, we obtain

$$\left(1 - 3z + 2z^2\right) g(z) = 2 - 3z.$$

Therefore

$$g(z) = \sum_{n=0}^{\infty} m_n z^n = \frac{2-3z}{1-3z+2z^2}.$$

Thus, this completes the proof. □

We now investigate some identities of the Mersenne Lucas numbers by using generating function in the following both theorem and proposition.

Theorem 2.10. *Let M_n and m_n be n-th Mersenne and Mersenne Lucas numbers, respectively. Then, we get*

$$m_n = 3M_n - 4M_{n-1}.$$

Proof. Using the generating function of Mersenne Lucas numbers, we get

$$\sum_{n=0}^{\infty} m_n z^n = \frac{2-3z}{1-3z+2z^2} = \frac{2}{1-3z+2z^2} - 3\frac{z}{1-3z+2z^2} = 2\sum_{n=0}^{\infty} M_{n+1} z^n - 3\sum_{n=0}^{\infty} M_n z^n.$$

Therefore

$$\sum_{n=0}^{\infty} m_n z^n = \sum_{n=0}^{\infty} \left(2M_{n+1} - 3M_n\right) z^n. \tag{2.5}$$

From the recurrence relation (1.4), we can write

$$\sum_{n=0}^{\infty} m_n z^n = \sum_{n=0}^{\infty} \left(3M_n - 4M_{n-1}\right) z^n.$$

Comparing of the coefficients of z^n we obtain the desired result. □

Proposition 2.11. *For $n \in \mathbb{N}$, the complete homogeneous symmetric function of Mersenne Lucas numbers is given by*

$$m_n = 2h_n\left(2,1\right) - 3h_{n-1}\left(2,1\right). \tag{2.6}$$

Proof. By relationship (1.5), we have

$$M_n = h_{n-1}\left(2,1\right).$$

Then, from (2.5) we give

$$\sum_{n=0}^{\infty} m_n z^n = \sum_{n=0}^{\infty} \left(2h_n\left(2,1\right) - 3h_{n-1}\left(2,1\right)\right) z^n.$$

Hence, comparing of the coefficients of z^n we obtain the desired result. □

3. Complete homogeneous symmetric functions of binary products of (p,q)-numbers with Mersenne Lucas numbers

The following propositions are key tools of the proof of our main result. They have been proved in [1, 3, 4].

Proposition 3.1. *Given two alphabets $E = \{e_1, e_2\}$ and $A = \{a_1, a_2\}$, then*

$$\sum_{n=0}^{\infty} h_n(a_1,a_2)h_n(e_1,e_2)z^n = \frac{1 - a_1a_2e_1e_2z^2}{\prod_{i=1}^{2}(1-a_ie_1z)\prod_{i=1}^{2}(1-a_ie_2z)}. \tag{3.1}$$

Note that, based on the relationship (3.1), we get

$$\sum_{n=0}^{\infty} h_{n-1}(a_1,a_2)h_{n-1}(e_1,e_2)z^n = \frac{z - a_1a_2e_1e_2z^3}{\prod_{i=1}^{2}(1-a_ie_1z)\prod_{i=1}^{2}(1-a_ie_2z)}. \tag{3.2}$$

Proposition 3.2. *Given two alphabets $E = \{e_1, e_2\}$ and $A = \{a_1, a_2\}$, then*

$$\sum_{n=0}^{\infty} h_{n-1}(a_1,a_2)h_n(e_1,e_2)z^n = \frac{(e_1+e_2)z - e_1e_2(a_1+a_2)z^2}{\prod_{i=1}^{2}(1-a_ie_1z)\prod_{i=1}^{2}(1-a_ie_2z)}. \tag{3.3}$$

Proposition 3.3. *Given two alphabets* $E = \{e_1, e_2\}$ *and* $A = \{a_1, a_2\}$, *then*

$$\sum_{n=0}^{\infty} h_n(a_1, a_2) h_{n-1}(e_1, e_2) z^n = \frac{(a_1 + a_2)z - a_1 a_2 (e_1 + e_2) z^2}{\prod_{i=1}^{2} (1 - a_i e_1 z) \prod_{i=1}^{2} (1 - a_i e_2 z)}. \tag{3.4}$$

In this part, we are now in a position to provide theorems. Also we derive the new generating functions for the products of (p, q)-Fibonacci numbers, (p, q)-Lucas numbers, (p, q)-Pell numbers, (p, q)-Pell Lucas numbers, (p, q)-Jacobsthal numbers and (p, q)-Jacobsthal Lucas numbers with Mersenne Lucas numbers at positive and negative indices.

This part consists of three cases.

Case 1. For $A = \left\{ \frac{p+\sqrt{p^2+4q}}{2}, \frac{p-\sqrt{p^2+4q}}{2} \right\}$ and $E = \{2, 1\}$ in the relationships (3.1)-(3.4), we have

$$\sum_{n=0}^{\infty} h_n \left(\frac{p + \sqrt{p^2 + 4q}}{2}, \frac{p - \sqrt{p^2 + 4q}}{2} \right) h_n (2, 1) z^n = \frac{1 + 2qz^2}{D_1},$$

$$\sum_{n=0}^{\infty} h_{n-1} \left(\frac{p + \sqrt{p^2 + 4q}}{2}, \frac{p - \sqrt{p^2 + 4q}}{2} \right) h_{n-1} (2, 1) z^n = \frac{z + 2qz^3}{D_1}, \tag{3.5}$$

$$\sum_{n=0}^{\infty} h_{n-1} \left(\frac{p + \sqrt{p^2 + 4q}}{2}, \frac{p - \sqrt{p^2 + 4q}}{2} \right) h_n (2, 1) z^n = \frac{3z - 2pz^2}{D_1}, \tag{3.6}$$

$$\sum_{n=0}^{\infty} h_n \left(\frac{p + \sqrt{p^2 + 4q}}{2}, \frac{p - \sqrt{p^2 + 4q}}{2} \right) h_{n-1} (2, 1) z^n = \frac{pz + 3qz^2}{D_1},$$

with

$$D_1 = 1 - 3pz - \left(5q - 2p^2\right) z^2 + 6pqz^3 + 4q^2 z^4,$$

and we deduce the following theorems.

Theorem 3.4. *For* $n \in \mathbb{N}$, *the new generating function of the product of* (p, q)-*Fibonacci numbers with Mersenne Lucas numbers is given by*

$$\sum_{n=0}^{\infty} F_{p,q,n} m_n z^n = \frac{3z - 4pz^2 - 6qz^3}{1 - 3pz - (5q - 2p^2) z^2 + 6pqz^3 + 4q^2 z^4}. \tag{3.7}$$

Proof. In [12], we have $F_{p,q,n} = h_{n-1}\left(\frac{p+\sqrt{p^2+4q}}{2}, \frac{p-\sqrt{p^2+4q}}{2} \right)$, and according to the relationship (2.6) we get

$$\begin{aligned}
\sum_{n=0}^{\infty} F_{p,q,n} m_n z^n &= \sum_{n=0}^{\infty} h_{n-1} \left(\frac{p + \sqrt{p^2 + 4q}}{2}, \frac{p - \sqrt{p^2 + 4q}}{2} \right) \left(2h_n (2, 1) - 3h_{n-1} (2, 1)\right) z^n \\
&= 2 \sum_{n=0}^{\infty} h_{n-1} \left(\frac{p + \sqrt{p^2 + 4q}}{2}, \frac{p - \sqrt{p^2 + 4q}}{2} \right) h_n (2, 1) z^n \\
&\quad - 3 \sum_{n=0}^{\infty} h_{n-1} \left(\frac{p + \sqrt{p^2 + 4q}}{2}, \frac{p - \sqrt{p^2 + 4q}}{2} \right) h_{n-1} (2, 1) z^n,
\end{aligned}$$

by using the relationships (3.5) and (3.6), we obtain

$$\sum_{n=0}^{\infty} F_{p,q,n} m_n z^n = \frac{2\left(3z - 2pz^2\right) - 3\left(z + 2qz^3\right)}{D_1} = \frac{3z - 4pz^2 - 6qz^3}{1 - 3pz - (5q - 2p^2) z^2 + 6pqz^3 + 4q^2 z^4},$$

which completes the proof. □

Theorem 3.5. *For $n \in \mathbb{N}$, the new generating function of the product of (p, q)-Fibonacci numbers with Mersenne Lucas numbers at negative indice is given by*

$$\sum_{n=0}^{\infty} F_{p,q,n} m_{-n} z^n = \frac{6z - 4pz^2 - 3qz^3}{4 - 6pz - (5q - 2p^2) z^2 + 3pqz^3 + q^2 z^4}. \tag{3.8}$$

Proof. We use the change of variable $z = \frac{z}{2}$ in (3.7) and according to relationship (2.3), we get

$$\begin{aligned}\sum_{n=0}^{\infty} F_{p,q,n} m_{-n} z^n = \sum_{n=0}^{\infty} F_{p,q,n} m_n \left(\frac{z}{2}\right)^n &= \frac{3\left(\frac{z}{2}\right) - 4p\left(\frac{z}{2}\right)^2 - 6q\left(\frac{z}{2}\right)^3}{1 - 3p\left(\frac{z}{2}\right) - (5q - 2p^2)\left(\frac{z}{2}\right)^2 + 6pq\left(\frac{z}{2}\right)^3 + 4q^2\left(\frac{z}{2}\right)^4} \\ &= \frac{6z - 4pz^2 - 3qz^3}{4 - 6pz - (5q - 2p^2) z^2 + 3pqz^3 + q^2 z^4}.\end{aligned}$$

Thus, this completes the proof. □

Theorem 3.6. *For $n \in \mathbb{N}$, the new generating function of the product of (p, q)-Lucas numbers with Mersenne Lucas numbers is given by*

$$\sum_{n=0}^{\infty} L_{p,q,n} m_n z^n = \frac{4 - 9pz + 2\left(2p^2 - 5q\right) z^2 + 6pqz^3}{1 - 3pz - (5q - 2p^2) z^2 + 6pqz^3 + 4q^2 z^4}. \tag{3.9}$$

Proof. Recall that, we have $L_{p,q,n} = 2h_n\left(\frac{p+\sqrt{p^2+4q}}{2}, \frac{p-\sqrt{p^2+4q}}{2}\right) - ph_{n-1}\left(\frac{p+\sqrt{p^2+4q}}{2}, \frac{p-\sqrt{p^2+4q}}{2}\right)$, (see [12]).

By the same method given in Theorem 3.4, the proof can be easily made. □

Theorem 3.7. *For $n \in \mathbb{N}$, the new generating function of the product of (p, q)-Lucas numbers with Mersenne Lucas numbers at negative indice is given by*

$$\sum_{n=0}^{\infty} L_{p,q,n} m_{-n} z^n = \frac{16 - 18pz + 2\left(2p^2 - 5q\right) z^2 + 3pqz^3}{4 - 6pz - (5q - 2p^2) z^2 + 3pqz^3 + q^2 z^4}. \tag{3.10}$$

Proof. By the same method given in Theorem 3.5, the proof can be easily made. So we omit the proof. □

By putting $p = q = 1$ in the relationships (3.7), (3.8), (3.9), and (3.10), we obtain Table 1.

Table 1: New generating functions of the products of Fibonacci and Lucas numbers with Mersenne Lucas numbers.

Coefficient of z^n	Generating function
$F_n m_n$	$\frac{3z-4z^2-6z^3}{1-3z-3z^2+6z^3+4z^4}$
$F_n m_{-n}$	$\frac{6z-4z^2-3z^3}{4-6z-3z^2+3z^3+z^4}$
$L_n m_n$	$\frac{4-9z-6z^2+6z^3}{1-3z-3z^2+6z^3+4z^4}$
$L_n m_{-n}$	$\frac{16-18z-6z^2+3z^3}{4-6z-3z^2+3z^3+z^4}$

Case 2. For $A = \left\{p + \sqrt{p^2 + q}, p - \sqrt{p^2 + q}\right\}$ and $E = \{2, 1\}$ in the relationships (3.1)-(3.4), we have

$$\begin{aligned}\sum_{n=0}^{\infty} h_n\left(p + \sqrt{p^2 + q}, p - \sqrt{p^2 + q}\right) h_n(2, 1) z^n &= \frac{1 + 2qz^2}{D_2}, \\ \sum_{n=0}^{\infty} h_{n-1}\left(p + \sqrt{p^2 + q}, p - \sqrt{p^2 + q}\right) h_{n-1}(2, 1) z^n &= \frac{z + 2qz^3}{D_2},\end{aligned} \tag{3.11}$$

$$\sum_{n=0}^{\infty} h_{n-1}\left(p+\sqrt{p^2+q}, p-\sqrt{p^2+q}\right) h_n(2,1) z^n = \frac{3z-4pz^2}{D_2}, \tag{3.12}$$
$$\sum_{n=0}^{\infty} h_n\left(p+\sqrt{p^2+q}, p-\sqrt{p^2+q}\right) h_{n-1}(2,1) z^n = \frac{2pz+3qz^2}{D_2},$$

with

$$D_2 = 1-6pz-\left(5q-8p^2\right)z^2+12pqz^3+4q^2z^4,$$

and we deduce the following theorems.

Theorem 3.8. *For $n \in \mathbb{N}$, the new generating function of the product of (p, q)-Pell numbers with Mersenne Lucas numbers is given by*

$$\sum_{n=0}^{\infty} P_{p,q,n} m_n z^n = \frac{3z-8pz^2-6qz^3}{1-6pz-\left(5q-8p^2\right)z^2+12pqz^3+4q^2z^4}. \tag{3.13}$$

Proof. By referred to [12], we have $P_{p,q,n} = h_{n-1}\left(p+\sqrt{p^2+q}, p-\sqrt{p^2+q}\right)$, and according to the relationship (2.6) we can easily see that

$$\begin{aligned}\sum_{n=0}^{\infty} P_{p,q,n} m_n z^n &= \sum_{n=0}^{\infty} h_{n-1}\left(p+\sqrt{p^2+q}, p-\sqrt{p^2+q}\right)\left(2h_n(2,1)-3h_{n-1}(2,1)\right) z^n \\ &= 2\sum_{n=0}^{\infty} h_{n-1}\left(p+\sqrt{p^2+q}, p-\sqrt{p^2+q}\right) h_n(2,1) z^n \\ &\quad -3\sum_{n=0}^{\infty} h_{n-1}\left(p+\sqrt{p^2+q}, p-\sqrt{p^2+q}\right) h_{n-1}(2,1) z^n,\end{aligned}$$

by using the relationships (3.11) and (3.12), we obtain

$$\sum_{n=0}^{\infty} P_{p,q,n} m_n z^n = \frac{2\left(3z-4pz^2\right)-3\left(z+2qz^3\right)}{D_2} = \frac{3z-8pz^2-6qz^3}{1-6pz-\left(5q-8p^2\right)z^2+12pqz^3+4q^2z^4}.$$

Hence, we obtain the desired result. □

Theorem 3.9. *For $n \in \mathbb{N}$, the new generating function of the product of (p, q)-Pell numbers with Mersenne Lucas numbers at negative indice is given by*

$$\sum_{n=0}^{\infty} P_{p,q,n} m_{-n} z^n = \frac{6z-8pz^2-3qz^3}{4-12pz-\left(5q-8p^2\right)z^2+6pqz^3+q^2z^4}. \tag{3.14}$$

Proof. We use the change of variable $z = \frac{z}{2}$ in (3.13) and according to relationship (2.3), we get

$$\begin{aligned}\sum_{n=0}^{\infty} P_{p,q,n} m_{-n} z^n = \sum_{n=0}^{\infty} P_{p,q,n} m_n \left(\frac{z}{2}\right)^n &= \frac{3\left(\frac{z}{2}\right)-8p\left(\frac{z}{2}\right)^2-6q\left(\frac{z}{2}\right)^3}{1-6p\left(\frac{z}{2}\right)-\left(5q-8p^2\right)\left(\frac{z}{2}\right)^2+12pq\left(\frac{z}{2}\right)^3+4q^2\left(\frac{z}{2}\right)^4} \\ &= \frac{6z-8pz^2-3qz^3}{4-12pz-\left(5q-8p^2\right)z^2+6pqz^3+q^2z^4}.\end{aligned}$$

As required. □

Theorem 3.10. *For $n \in \mathbb{N}$, the new generating function of the product of (p,q)-Pell Lucas numbers with Mersenne Lucas numbers is given by*

$$\sum_{n=0}^{\infty} Q_{p,q,n} m_n z^n = \frac{4 - 18pz + 2\left(8p^2 - 5q\right) z^2 + 12pqz^3}{1 - 6pz - \left(5q - 8p^2\right) z^2 + 12pqz^3 + 4q^2z^4}. \tag{3.15}$$

Proof. We know that

$$Q_{p,q,n} = 2h_n\left(p + \sqrt{p^2+q}, p - \sqrt{p^2+q}\right) - 2ph_{n-1}\left(p + \sqrt{p^2+q}, p - \sqrt{p^2+q}\right), \text{ (see [12]).}$$

By the same method given in Theorem 3.8, the proof can be easily made. □

Theorem 3.11. *For $n \in \mathbb{N}$, the new generating function of the product of (p,q)-Pell Lucas numbers with Mersenne Lucas numbers at negative indice is given by*

$$\sum_{n=0}^{\infty} Q_{p,q,n} m_{-n} z^n = \frac{16 - 36pz + 2\left(8p^2 - 5q\right) z^2 + 6pqz^3}{4 - 12pz - \left(5q - 8p^2\right) z^2 + 6pqz^3 + q^2z^4}. \tag{3.16}$$

Proof. By the same method given in Theorem 3.9, the proof can be easily made. So we omit the proof. □

By setting $p = q = 1$ in the relationships (3.13), (3.14), (3.15), and (3.16), we obtain Table 2.

Table 2: New generating functions of the products of Pell and Pell Lucas numbers with Mersenne Lucas numbers.

Coefficientof z^n	Generating function
$P_n m_n$	$\frac{3z-8z^2-6z^3}{1-6z+3z^2+12z^3+4z^4}$
$P_n m_{-n}$	$\frac{6z-8z^2-3z^3}{4-12z+3z^2+6z^3+z^4}$
$Q_n m_n$	$\frac{4-18z+6z^2+12z^3}{1-6z+3z^2+12z^3+4z^4}$
$Q_n m_{-n}$	$\frac{16-36z+6z^2+6z^3}{4-12z+3z^2+6z^3+z^4}$

Case 3. For $A = \left\{\frac{p+\sqrt{p^2+8q}}{2}, \frac{p-\sqrt{p^2+8q}}{2}\right\}$ and $E = \{2, 1\}$ in the relationships (3.1)-(3.4), we have

$$\sum_{n=0}^{\infty} h_n\left(\frac{p+\sqrt{p^2+8q}}{2}, \frac{p-\sqrt{p^2+8q}}{2}\right) h_n(2,1) z^n = \frac{1+4qz^2}{D_3}, \tag{3.17}$$

$$\sum_{n=0}^{\infty} h_{n-1}\left(\frac{p+\sqrt{p^2+8q}}{2}, \frac{p-\sqrt{p^2+8q}}{2}\right) h_{n-1}(2,1) z^n = \frac{z+4qz^3}{D_3}, \tag{3.18}$$

$$\sum_{n=0}^{\infty} h_{n-1}\left(\frac{p+\sqrt{p^2+8q}}{2}, \frac{p-\sqrt{p^2+8q}}{2}\right) h_n(2,1) z^n = \frac{3z-2pz^2}{D_3}, \tag{3.19}$$

$$\sum_{n=0}^{\infty} h_n\left(\frac{p+\sqrt{p^2+8q}}{2}, \frac{p-\sqrt{p^2+8q}}{2}\right) h_{n-1}(2,1) z^n = \frac{pz+6qz^2}{D_3},$$

with

$$D_3 = 1 - 3pz - 2\left(5q - p^2\right) z^2 + 12pqz^3 + 16q^2z^4,$$

and we deduce the following theorems.

Theorem 3.12. *For $n \in \mathbb{N}$, the new generating function of the product of (p,q)-Jacobsthal numbers with Mersenne Lucas numbers is given by*

$$\sum_{n=0}^{\infty} J_{p,q,n} m_n z^n = \frac{3z - 4pz^2 - 12qz^3}{1 - 3pz - 2\left(5q - p^2\right) z^2 + 12pqz^3 + 16q^2z^4}. \tag{3.20}$$

Proof. Recall that, we have $J_{p,q,n} = h_{n-1}\left(\frac{p+\sqrt{p^2+8q}}{2}, \frac{p-\sqrt{p^2+8q}}{2}\right)$ (see [12]), and according to the relationship (2.6) we see that

$$\begin{aligned}\sum_{n=0}^{\infty} J_{p,q,n} m_n z^n &= \sum_{n=0}^{\infty} h_{n-1}\left(\frac{p+\sqrt{p^2+8q}}{2}, \frac{p-\sqrt{p^2+8q}}{2}\right)\left(2h_n(2,1) - 3h_{n-1}(2,1)\right) z^n \\ &= 2\sum_{n=0}^{\infty} h_{n-1}\left(\frac{p+\sqrt{p^2+8q}}{2}, \frac{p-\sqrt{p^2+8q}}{2}\right) h_n(2,1) z^n \\ &\quad - 3\sum_{n=0}^{\infty} h_{n-1}\left(\frac{p+\sqrt{p^2+8q}}{2}, \frac{p-\sqrt{p^2+8q}}{2}\right) h_{n-1}(2,1) z^n,\end{aligned}$$

by using the relationships (3.18) and (3.19), we obtain

$$\sum_{n=0}^{\infty} J_{p,q,n} m_n z^n = \frac{2\left(3z - 2pz^2\right) - 3\left(z + 4qz^3\right)}{D_3} = \frac{3z - 4pz^2 - 12qz^3}{1 - 3pz - 2\left(5q - p^2\right) z^2 + 12pqz^3 + 16q^2 z^4},$$

which completes the proof. □

Theorem 3.13. *For $n \in \mathbb{N}$, the new generating function of the product of (p, q)-Jacobsthal numbers with Mersenne Lucas numbers at negative indice is given by*

$$\sum_{n=0}^{\infty} J_{p,q,n} m_{-n} z^n = \frac{6z - 4pz^2 - 6qz^3}{4 - 6pz - 2\left(5q - p^2\right) z^2 + 6pqz^3 + 4q^2 z^4}.$$

Proof. We use the change of variable $z = \frac{z}{2}$ in (3.20) and according to relationship (2.3), we get

$$\begin{aligned}\sum_{n=0}^{\infty} J_{p,q,n} m_{-n} z^n = \sum_{n=0}^{\infty} J_{p,q,n} m_n \left(\frac{z}{2}\right)^n &= \frac{3\left(\frac{z}{2}\right) - 4p\left(\frac{z}{2}\right)^2 - 12q\left(\frac{z}{2}\right)^3}{1 - 3p\left(\frac{z}{2}\right) - 2\left(5q - p^2\right)\left(\frac{z}{2}\right)^2 + 12pq\left(\frac{z}{2}\right)^3 + 16q^2\left(\frac{z}{2}\right)^4} \\ &= \frac{6z - 4pz^2 - 6qz^3}{4 - 6pz - 2\left(5q - p^2\right) z^2 + 6pqz^3 + 4q^2 z^4}.\end{aligned}$$

Hence, we obtain the desired result. □

Theorem 3.14. *For $n \in \mathbb{N}$, the new generating function of the product of (p, q)-Jacobsthal Lucas numbers with Mersenne Lucas numbers is given by*

$$\sum_{n=0}^{\infty} j_{p,q,n} m_n z^n = \frac{4 - 9pz + 4\left(p^2 - 5q\right) z^2 + 12pqz^3}{1 - 3pz - 2\left(5q - p^2\right) z^2 + 12pqz^3 + 16q^2 z^4}. \tag{3.21}$$

Proof. Recall that, we have $j_{p,q,n} = 2h_n\left(\frac{p+\sqrt{p^2+8q}}{2}, \frac{p-\sqrt{p^2+8q}}{2}\right) - ph_{n-1}\left(\frac{p+\sqrt{p^2+8q}}{2}, \frac{p-\sqrt{p^2+8q}}{2}\right)$, (see [12]).

By the same method given in Theorem 3.12, the proof can be easily made. □

Theorem 3.15. *For $n \in \mathbb{N}$, the new generating function of the product of (p, q)-Jacobsthal Lucas numbers with Mersenne Lucas numbers at negative indice is given by*

$$\sum_{n=0}^{\infty} j_{p,q,n} m_{-n} z^n = \frac{16 - 18pz + 4\left(p^2 - 5q\right) z^2 + 6pqz^3}{4 - 6pz - 2\left(5q - p^2\right) z^2 + 6pqz^3 + 4q^2 z^4}. \tag{3.22}$$

Proof. By the same method given in Theorem 3.13, the proof can be easily made. So we omit the proof. □

By taking $p = q = 1$ in the relationships (3.20)-(3.22), we obtain Table 3.

Table 3: New generating functions of the products of Jacobsthal and Jacobsthal Lucas numbers with Mersenne Lucas numbers.

Coefficient of z^n	Generating function
$J_n m_n$	$\frac{3z-4z^2-12z^3}{1-3z-8z^2+12z^3+16z^4}$
$J_n m_{-n}$	$\frac{6z-4z^2-6z^3}{4-6z-8z^2+6z^3+4z^4}$
$j_n m_n$	$\frac{4-9z-16z^2+12z^3}{1-3z-8z^2+12z^3+16z^4}$
$j_n m_{-n}$	$\frac{16-18z-16z^2+6z^3}{4-6z-8z^2+6z^3+4z^4}$

4. Conclusion

In this study, we have introduced the concept of the Mersenne Lucas numbers. We also gave some results, such as generating function and Binet's formula of these numbers. Moreover, by making use of the complete homogeneous symmetric function we have obtained the generating functions for the products of (p, q)-numbers and Mersenne Lucas numbers.

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