J. Math. Computer Sci., 24 (2022), 173–185

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

Characteristics of solutions of nonlinear neutral integro-differential equation via Chandrasekhar integral

H. H. G. Hashem[a,b], Hessah O. Alrashidi[b,*]

[a]Faculty of Science, Alexandria University, Alexandria, Egypt.

[b]Department of mathematics, College of Science, Qassim University, P.O. Box 6644, Buraidah 51452, Saudi Arabia.

Abstract

In this paper, we shall study the existence of at least one continuous solution for a nonlinear neutral differential equation via Chandrasekhar integral. Next, continuous dependence of the solution of that equation on the delay functions will be studied. Also, we use Kransnoselskii theorem to prove the existence of solutions and estimate upper and lower bounds for solutions defined in unbounded interval. Some particular cases and remarks are presented to illustrate our results.

Keywords: Neutral differential equation, Kransnoselskii theorem, continuous dependence, estimate upper and lower bounds for solutions.

2020 MSC: 34A40, 34A12, 34A99, 45D05.

1. Introduction

Neutral differential equations arise in many areas of applied mathematics and for this reason these equations have been investigated extensively in the last decades.

The dynamical systems which depend on present and past states are often described by neutral delay differential equations. Practical examples of neutral delay differential systems include biological models of single species growth [14], processes including steam or water pipes, heat exchanges [11], population ecology [12] and other engineering systems [11].

The study of neutral differential equations has grown rapidly. This is largely due to the fact that often the qualitative behavior of solutions of neutral differential equations is very different from those of nonneutral equations. For example, [10, 15, 16, 19]. It has been shown in the literature that even when all the characteristic roots of a neutral differential equation have negative real parts, it is still possible for the equation to have unbounded solutions. Such a behavior is impossible for nonneutral equations. Banaś et al. [1] proved the existence and asymptotic behaviour of solutions of the differential equation with a deviating argument of neutral type

$$\mathfrak{x}'(\mathfrak{t}) = \mathfrak{f}(\mathfrak{t}, \mathfrak{x}(\mathfrak{H}(\mathfrak{t})), \mathfrak{x}'(\mathfrak{h}(\mathfrak{t}))), \mathfrak{t} \in \mathfrak{R}_+ = [0, \infty) \tag{1.1}$$

*Corresponding author

Email addresses: 3922@qu.edu.sa (H. H. G. Hashem), H.alrashidi@qu.edu.sa (Hessah O. Alrashidi)

doi: 10.22436/jmcs.024.02.08

Received: 2020-08-16 *Revised:* 2021-01-01 *Accepted:* 2021-01-18

together with the initial condition of the form $\mathfrak{x}(0) = 0$. The considered equation contains both delayed and advanced arguments. The method used in the proof of the main result depends on conjunction of the classical Schauder fixed point theorem with the technique of measures of noncompactness [2]. The problem (1.1) was considered in many research papers and monographs under various assumptions, for example [7]. In general, if both types of deviation of argument are admitted, i.e., both delay and advance, the theory of the problem (1.1) is rather difficult and requires strong assumptions.

Chang et al. [4] considered the existence of mild solutions for a class of first-order impulsive neutral integro-differential equations with state-dependent delay such as

$$\frac{d}{dt}\left[\mathfrak{x}(t) - F(t, \mathfrak{x}_t)\right] = A\left[\mathfrak{x}(t) + \int_0^t \mathfrak{f}(t-\mathfrak{s})\mathfrak{x}(\mathfrak{s})\,d\mathfrak{s}\right] + G\left[t, \mathfrak{x}_{\rho(t,\mathfrak{x}_t)}\right],$$
$$t \in J = [0, \mathfrak{a}], t \neq t_i, i = 1, 2, \dots, n,$$
$$\mathfrak{x}_0 = \phi \in \mathcal{B}, \Delta\mathfrak{x}(t_i) = I_i(\mathfrak{x}_{t_i}), i = 1, 2, \dots, n,$$

where A is the infinitesimal generator of a compact, analytic resolvent operator $\mathfrak{R}(t), t > 0$ in a Banach space $(X, \|.\|), \mathfrak{f}(t), t \in J$ is a bounded linear operator.

Hussain et al. [9] presented some fixed point and coupled fixed point results in the generalized setting. Moreover, their purpose in this paper is to concern with the solution of nonlinear neutral differential equation

$$\mathfrak{x}'(t) = -\mathfrak{a}(t)\mathfrak{x}(t) + \mathfrak{b}(t)\mathfrak{g}(\mathfrak{x}(t - r(t))) + \mathfrak{c}(t)\mathfrak{x}'(t - r(t))$$

with unbounded delay using fixed point theory in $\mathcal{F}$-metric space, where $\mathfrak{a}(t), \mathfrak{b}(t)$ are continuous, $\mathfrak{c}(t)$ is continuously differentiable and $r(t) > 0$ for all $t \in R$ and is twice continuously differentiable. The paper [6] is concerned with the existence of a positive solution of the neutral differential equation of the form

$$\frac{d}{dt}\left[\mathfrak{x}(t) - \mathfrak{a}(t)\mathfrak{x}(t-\tau)\right] = \mathfrak{p}(t)\mathfrak{f}(\mathfrak{x}(t-\sigma)), \quad t \geqslant t_0$$

where $\tau > 0, \sigma \geqslant 0, \mathfrak{a} \in C([t_0, \infty), (0, \infty)), \mathfrak{p} \in C(R, (0, \infty)), \mathfrak{f} \in C(R, R), f$ is nondecreasing function, and $\mathfrak{x}\mathfrak{f}(\mathfrak{x}) > 0, \mathfrak{x} \neq 0$. [6] contains some sufficient conditions for the existence of positive solutions which are bounded below and above by positive functions for the first-order nonlinear neutral differential equations. These equations can also support the existence of positive solutions approaching zero at infinity.

Our paper is concerned with the existence of the nonlinear neutral integro-differential equation of the form

$$\begin{cases} \frac{d}{dt}\left[\mathfrak{x}(t) - \mathfrak{a}_2\mathfrak{f}_2\left(t, \mathfrak{x}(\phi_2(t)), \int_{t_0}^1 \frac{t}{t+\mathfrak{s}}\mathfrak{g}(\mathfrak{s}, \mathfrak{x}(\phi_3(\mathfrak{s})))d\mathfrak{s}\right)\right] = \mathfrak{a}_1\mathfrak{f}_1(t, \mathfrak{x}(\phi_1(t))), \\ \left[\mathfrak{x}(t) - \mathfrak{a}_2\mathfrak{f}_2\left(t, \mathfrak{x}(\phi_2(t)), \int_{t_0}^1 \frac{t}{t+\mathfrak{s}}\mathfrak{g}(\mathfrak{s}, \mathfrak{x}(\phi_3(\mathfrak{s})))d\mathfrak{s}\right)\right]_{t=t_0} = 0, \end{cases} \tag{1.2}$$

where $t \in I = [t_0, T], \mathfrak{a}_1, \mathfrak{a}_2 \in R$.

Motivated by this works, we shall prove the existence of at least one positive solution for the nonlinear neutral integro-differential equation via Chandrasekhar integral (1.2) by applying Schauder fixed point theorem, prove the existence of unique solution and prove that this solution is continuously depending on the delay functions. Our next result is concerned with (1.2) on the real half-axis R_+ by applying Kransnoselskii theorem to prove an existence result which give sufficient conditions for the existence of a positive solution which is bounded by two positive functions.

By a solution of the nonlinear neutral integro-differential equation (1.2) we mean a function $\mathfrak{x} \in C[I, R]$ such that

(i) the function $t \to \mathfrak{x}(t) - \mathfrak{a}_2\mathfrak{f}_2\left(t, \mathfrak{x}(\phi_2(t)), \int_{t_0}^1 \frac{t}{t+\mathfrak{s}}\mathfrak{g}(\mathfrak{s}, \mathfrak{x}(\phi_3(\mathfrak{s})))d\mathfrak{s}\right)$ is continuously differentiable on I;

(ii) $\mathfrak{x}$ satisfies the equations in (1.2).

Theorem 1.1 (Schauder fixed point theorem [5]). *Let Q be a nonempty, convex, compact subset of a Banach space X, and $T : Q \to Q$ be a continuous map. Then T has at least one fixed point in Q.*

Theorem 1.2 (Kransnoselskii's fixed point theorem [6]). *Let X be a Banach space, let Ω be a bounded closed convex subset of X and S_1, S_2 be maps of Ω into X such that $S_1x + S_2y \in \Omega$ for every pair $x, y \in \Omega$. If S_1 is contractive and S_2 is completely continuous, then the equation $S_1x + S_2x = x$ has a solution in Ω.*

2. Existence results on bounded interval

(i) The functions $\phi_i : I \to I, i = 1, 2, 3$ are continuous.

(ii) $\mathfrak{f}_1, \mathfrak{g} : I \times R_+ \to R_+$ satisfy Carathéodory condition, i.e., $\mathfrak{f}_1, \mathfrak{g}$ are measurable functions in t for any $\mathfrak{x} \in R_+$ and continuous in $\mathfrak{x}$ for almost all $t \in I$. There exist two nonnegative constants $\mathfrak{b}, \mathfrak{b}_1$ and two functions $t \to \alpha(t)$, $t \to m(t)$, such that

$$|\mathfrak{g}(t,\mathfrak{x})| \leqslant \alpha(t) + \mathfrak{b}_1|\mathfrak{x}|, \forall (t,\mathfrak{x}) \in I \times R_+, \quad \text{and} \quad |\,\mathfrak{f}_1(t,\mathfrak{x})| \leqslant m(t) + \mathfrak{b}\,|\,\mathfrak{x}\,|, \forall (t,\mathfrak{x}) \in I \times R_+,$$

where $\alpha(.), m(.) \in L^1(I)$ and $\sup\limits_{t\in I} |\alpha(t)| = k_1$.

(iii) $\mathfrak{f}_2 : I \times R_+ \times R_+ \to R_+$ is continuous and there exists three positive constants $\mathfrak{l}_0, \mathfrak{l}_1$ and $\mathfrak{l}_2$ satisfying:

$$|\,\mathfrak{f}_2(t_1,\mathfrak{x}_1,\mathfrak{y}_1) - \mathfrak{f}_2(t_2,\mathfrak{x}_2,\mathfrak{y}_2)| \leqslant \mathfrak{l}_0|t_1 - t_2| + \mathfrak{l}_1\,|\,\mathfrak{x}_1 - \mathfrak{x}_2\,| + \mathfrak{l}_2\,|\,\mathfrak{y}_1 - \mathfrak{y}_2\,|,$$

$\forall (t_1,\mathfrak{x}_1,\mathfrak{y}_1), (t_2,\mathfrak{x}_2,\mathfrak{y}_2) \in I \times R_+ \times R_+$.

(iv) $k = \sup\limits_{t\in I}\{\mathfrak{f}_2(t,0,0)\}, \int_{t_0}^{T} m(\mathfrak{s})d\mathfrak{s} \leqslant M$.

Theorem 2.1. *Let assumptions (i)-(iv) be satisfied. If $1 > |\mathfrak{a}_1|\mathfrak{b}T + |\mathfrak{a}_2|\mathfrak{b}_1\mathfrak{l}_2 + |\mathfrak{a}_2|\mathfrak{l}_1$, then there exists at least one continuous positive solution for the nonlinear neutral integro-differential equation via Chandraseker integral (1.2).*

Proof. From assumption (iii), we obtain

$$|\mathfrak{f}_2(t,\mathfrak{x},\mathfrak{y}) - \mathfrak{f}_2(t,0,0)| \leqslant \mathfrak{l}_1\,|\,\mathfrak{x}\,| + \mathfrak{l}_2\,|\,\mathfrak{y}\,|,$$
$$|\mathfrak{f}_2(t,\mathfrak{x},\mathfrak{y})| \leqslant |\mathfrak{f}_2(t,0,0)| + \mathfrak{l}_1\,|\,\mathfrak{x}\,| + \mathfrak{l}_2\,|\,\mathfrak{y}\,| \leqslant k + \mathfrak{l}_1\,|\,\mathfrak{x}\,| + \mathfrak{l}_2\,|\,\mathfrak{y}\,|.$$

Let $C[I, R_+]$ be the set of all continuous function with the norm $\|\mathfrak{x}\| = \sup\limits_{I} |\mathfrak{x}(t)|$. Now define a closed, bounded, and convex subset Ω of $C[I, R_+]$ as follows:

$$\Omega = \{\mathfrak{x} : \mathfrak{x} \in C[I, R_+], \|\mathfrak{x}\| \leqslant r, r \geqslant 0\}.$$

Let $\mathbb{A}$ be an operator defined on Ω by the formula

$$\mathbb{A}\mathfrak{x}(t) = \mathfrak{a}_1 \int_{t_0}^{t} \mathfrak{f}_1(\mathfrak{s},\mathfrak{x}(\phi_1(\mathfrak{s})))d\mathfrak{s} + \mathfrak{a}_2\mathfrak{f}_2\left(t, \mathfrak{x}(\phi_2(t)), \int_{t_0}^{1} \frac{t}{t+\mathfrak{s}}\mathfrak{g}(\mathfrak{s},\mathfrak{x}(\phi_3(\mathfrak{s})))d\mathfrak{s}\right).$$

We shall prove that for each $\mathfrak{x} \in \Omega \Rightarrow \mathbb{A}\mathfrak{x} \in \Omega$. First we make an estimate for r:

$$\begin{aligned}|\,\mathbb{A}\mathfrak{x}(t)\,| &= \left|\mathfrak{a}_1 \int_{t_0}^{t} \mathfrak{f}_1(\mathfrak{s},\mathfrak{x}(\phi_1(\mathfrak{s})))d\mathfrak{s} + \mathfrak{a}_2\mathfrak{f}_2\left(t, \mathfrak{x}(\phi_2(t))), \int_{t_0}^{1} \frac{t}{t+\mathfrak{s}}\mathfrak{g}(\mathfrak{s},\mathfrak{x}(\phi_3(\mathfrak{s})))d\mathfrak{s}\right)\right| \\ &\leqslant |\,\mathfrak{a}_1\,| \int_{t_0}^{T} [m(\mathfrak{s}) + \mathfrak{b}\,|\,\mathfrak{x}(\phi_1(\mathfrak{s}))\,|]d\mathfrak{s} + |\,\mathfrak{a}_2\,|\,k + |\,\mathfrak{a}_2\,||\,\mathfrak{l}_1\,|\,\mathfrak{x}(\phi_2(t))\,| + |\mathfrak{a}_2|\mathfrak{l}_2\left|\int_{t_0}^{1} \frac{t}{t+\mathfrak{s}}\mathfrak{g}(\mathfrak{s},\mathfrak{x}(\phi_3(\mathfrak{s})))d\mathfrak{s}\right|\end{aligned}$$

$$
\begin{aligned}
&\leqslant |\mathfrak{a}_1| \int_{t_0}^{T} \mathfrak{m}(\mathfrak{s})d\mathfrak{s} + |\mathfrak{a}_1|\, \mathfrak{b} \int_{t_0}^{T} |\mathfrak{x}(\phi_1(\mathfrak{s}))|\, d\mathfrak{s} + |\mathfrak{a}_2|\, k + |\mathfrak{a}_2|| \mathfrak{l}_1 | \mathfrak{x}(t)| \\
&\quad + |\mathfrak{a}_2| \mathfrak{l}_2 \alpha(t) + \mathfrak{a}_2 \mathfrak{l}_2 \mathfrak{b}_1 \int_{t_0}^{1} |\mathfrak{x}(\mathfrak{s})| d\mathfrak{s} \\
&\leqslant |\mathfrak{a}_1| M + |\mathfrak{a}_1|\, \mathfrak{b} \int_{t_0}^{T} |\mathfrak{x}(\mathfrak{s})|\, d\mathfrak{s} + |\mathfrak{a}_2| k + |\mathfrak{a}_2|\, \mathfrak{l}_1 r + |\mathfrak{a}_2| \mathfrak{l}_2 k_1 + |\mathfrak{a}_2| \mathfrak{b}_1 \mathfrak{l}_2 r \int_{t_0}^{1} d\mathfrak{s} \\
&\leqslant |\mathfrak{a}_1| M + |\mathfrak{a}_1|\, \mathfrak{b} r \int_{t_0}^{T} d\mathfrak{s} + |\mathfrak{a}_2| k + |\mathfrak{a}_2|\, \mathfrak{l}_1 r + |\mathfrak{a}_2| \mathfrak{l}_2 k_1 + |\mathfrak{a}_2| \mathfrak{b}_1 \mathfrak{l}_2 r \\
&\leqslant |\mathfrak{a}_1| M + |\mathfrak{a}_1|\, \mathfrak{b} r T + |\mathfrak{a}_2| k + |\mathfrak{a}_2|\, \mathfrak{l}_1 r + |\mathfrak{a}_2| \mathfrak{l}_2 k_1 + |\mathfrak{a}_2| \mathfrak{b}_1 \mathfrak{l}_2 r \\
&\leqslant r.
\end{aligned}
$$

From the last estimate we can deduce that

$$
r = \frac{|\mathfrak{a}_1| M + |\mathfrak{a}_2| k + |\mathfrak{a}_2| \mathfrak{l}_2 k_1}{1 - |\mathfrak{a}_1|\, \mathfrak{b} T - |\mathfrak{a}_2|\, \mathfrak{b}_1 \mathfrak{l}_2 - |\mathfrak{a}_2| \mathfrak{l}_1}.
$$

This means that $\mathfrak{x} \in \Omega \Rightarrow \mathbb{A}\mathfrak{x} \in \Omega$. Now, we shall show that $\mathbb{A}\Omega$ is relatively compact. It is sufficient to show that $\mathbb{A}\Omega$ is uniformly bounded and equicontinuous by the Arzela Ascoli theorem. The uniform boundedness follows from the definition of Ω. For $\mathfrak{x} \in \Omega$ and $t_1, t_2 \in I, t_2 > t_1$ (without loss of generality) we get

$$
\begin{aligned}
|\mathbb{A}\mathfrak{x}(t_2) - \mathbb{A}\mathfrak{x}(t_1)| &= \Bigg| \mathfrak{a}_1 \int_{t_0}^{t_2} \mathfrak{f}_1(\mathfrak{s}, \mathfrak{x}(\phi_1(\mathfrak{s})))d\mathfrak{s} + \mathfrak{a}_2 \mathfrak{f}_2 \left(t_2, \mathfrak{x}(\phi_2(t_2)), \int_{t_0}^{1} \frac{t_2}{t_2 + \mathfrak{s}} \mathfrak{g}(\mathfrak{s}, \mathfrak{x}(\phi_3(\mathfrak{s})))d\mathfrak{s} \right) \\
&\quad - \mathfrak{a}_1 \int_{t_0}^{t_1} \mathfrak{f}_1(\mathfrak{s}, \mathfrak{x}(\phi_1(\mathfrak{s})))d\mathfrak{s} - \mathfrak{a}_2 \mathfrak{f}_2 \left(t_1, \mathfrak{x}(\phi_2(t_1)), \int_{t_0}^{1} \frac{t_1}{t_1 + \mathfrak{s}} \mathfrak{g}(\mathfrak{s}, \mathfrak{x}(\phi_3(\mathfrak{s})))d\mathfrak{s} \right) \Bigg| \\
&\leqslant |\mathfrak{a}_1| \int_{t_1}^{t_2} |\mathfrak{f}_1(\mathfrak{s}, \mathfrak{x}(\phi_1(\mathfrak{s})))| d\mathfrak{s} + |\mathfrak{a}_2| \Bigg| \mathfrak{f}_2 \left(t_2, \mathfrak{x}(\phi_2(t_2)), \int_{t_0}^{1} \frac{t_2}{t_2 + \mathfrak{s}} \mathfrak{g}(\mathfrak{s}, \mathfrak{x}(\phi_3(\mathfrak{s})))d\mathfrak{s} \right) \\
&\quad - \mathfrak{f}_2 \left(t_1, \mathfrak{x}(\phi_2(t_1)), \int_{t_0}^{1} \frac{t_1}{t_1 + \mathfrak{s}} \mathfrak{g}(\mathfrak{s}, \mathfrak{x}(\phi_3(\mathfrak{s})))d\mathfrak{s} \right) \Bigg| \\
&\leqslant |\mathfrak{a}_1| \int_{t_1}^{t_2} [\mathfrak{m}(\mathfrak{s}) + \mathfrak{b} |\mathfrak{x}(\phi_1(\mathfrak{s}))|] d\mathfrak{s} + |\mathfrak{a}_2| \Bigg[\mathfrak{l}_o |t_2 - t_1| + \mathfrak{l}_1 |\mathfrak{x}(\phi_2(t_2)) - \mathfrak{x}(\phi_2(t_1))| \\
&\quad + \mathfrak{l}_2 \left| \int_{t_0}^{1} \frac{t_2}{t_2 + \mathfrak{s}} \mathfrak{g}(\mathfrak{s}, \mathfrak{x}(\phi_3(\mathfrak{s})))d\mathfrak{s} - \int_{t_0}^{1} \frac{t_1}{t_1 + \mathfrak{s}} \mathfrak{g}(\mathfrak{s}, \mathfrak{x}(\phi_3(\mathfrak{s})))d\mathfrak{s} \right| \Bigg] \\
&\leqslant |\mathfrak{a}_1| \int_{t_1}^{t_2} \mathfrak{m}(\mathfrak{s})d\mathfrak{s} + |\mathfrak{a}_1| \mathfrak{b} r |t_2 - t_1| + |\mathfrak{a}_2| \Bigg[\mathfrak{l}_o |t_2 - t_1| + \mathfrak{l}_1 |\mathfrak{x}(t_2) - \mathfrak{x}(t_1)| \\
&\quad + \mathfrak{l}_2 \left| \int_{t_0}^{1} \frac{t_2}{t_1 + \mathfrak{s}} \mathfrak{g}(\mathfrak{s}, \mathfrak{x}(\phi_3(\mathfrak{s})))d\mathfrak{s} - \int_{t_0}^{1} \frac{t_1}{t_1 + \mathfrak{s}} \mathfrak{g}(\mathfrak{s}, \mathfrak{x}(\phi_3(\mathfrak{s})))d\mathfrak{s} \right| \Bigg] \\
&\leqslant |\mathfrak{a}_1| \int_{t_1}^{t_2} \mathfrak{m}(\mathfrak{s})d\mathfrak{s} + |\mathfrak{a}_1| \mathfrak{b} r |t_2 - t_1| + |\mathfrak{a}_2| \Bigg[\mathfrak{l}_o |t_2 - t_1| + \mathfrak{l}_1 |\mathfrak{x}(t_2) - \mathfrak{x}(t_1)| \\
&\quad + \mathfrak{l}_2 |t_2 - t_1| \int_{t_0}^{1} \frac{1}{t_1 + \mathfrak{s}} |\mathfrak{g}(\mathfrak{s}, \mathfrak{x}(\phi_3(\mathfrak{s})))| d\mathfrak{s} \Bigg] \\
&\leqslant |\mathfrak{a}_1| \int_{t_1}^{t_2} \mathfrak{m}(\mathfrak{s})d\mathfrak{s} + |\mathfrak{a}_1| \mathfrak{b} r |t_2 - t_1| + |\mathfrak{a}_2| \Bigg[\mathfrak{l}_o |t_2 - t_1| + \mathfrak{l}_1 |\mathfrak{x}(t_2) - \mathfrak{x}(t_1)| \\
&\quad + \mathfrak{l}_2 |t_2 - t_1| \int_{t_0}^{1} \frac{1}{t_1 + \mathfrak{s}} \left[\alpha(\mathfrak{s}) + \mathfrak{b} |\mathfrak{x}(\phi_3(\mathfrak{s}))| \right] d\mathfrak{s} \Bigg]
\end{aligned}
$$

$$\leqslant |\ \mathfrak{a}_1| \int_{t_1}^{t_2} \mathfrak{m}(\mathfrak{s}) d\mathfrak{s} + |\ \mathfrak{a}_1|\mathfrak{b}\mathfrak{r}|t_2 - t_1| + |\mathfrak{a}_2|\mathfrak{l}_o|t_2 - t_1| + |\mathfrak{a}_2|\mathfrak{l}_1|\mathfrak{x}(t_2) - \mathfrak{x}(t_1)|$$
$$+ |\mathfrak{a}_2|\mathfrak{l}_2 \Big[\frac{k_1}{t_0} + \mathfrak{b}\mathfrak{r}\Big]|t_2 - t_1|.$$

Then

$$|(\mathbb{A}\mathfrak{x})(t_2) - (\mathbb{A}\mathfrak{x})(t_1)| \to 0 \text{ as } t_2 \to t_1.$$

This means that the functions from $\mathbb{A}\Omega$ are equi-continuous on I. Then by Arzela-Ascoli Theorem [5] the closure of $\mathbb{A}\Omega$ is compact. It is clear that the set Ω is nonempty, bounded, closed and convex. Assumptions (ii) and (iii) imply that $\mathbb{A} : \Omega \to C[I, R_+]$ is a continuous operator. Since all conditions of the Schauder fixed-point theorem hold, then $\mathbb{A}$ has a fixed point in Ω. □

Remark 2.2. In general, we can study the existence of the nonlinear neutral integro-differential equation of the form

$$\begin{cases} \frac{d}{dt}\left[\mathfrak{x}(t) - \mathfrak{a}_2\mathfrak{f}_2\left(t, \mathfrak{x}(\phi_2(t)), \int_{t_0}^1 \kappa(t,\mathfrak{s})\mathfrak{g}(\mathfrak{s}, \mathfrak{x}(\phi_3(\mathfrak{s})))d\mathfrak{s}\right)\right] = \mathfrak{a}_1\mathfrak{f}_1(t, \mathfrak{x}(\phi_1(t))), \\ \left[\mathfrak{x}(t) - \mathfrak{a}_2\mathfrak{f}_2\left(t, \mathfrak{x}(\phi_2(t)), \int_{t_0}^1 \kappa(t,\mathfrak{s})\mathfrak{g}(\mathfrak{s}, \mathfrak{x}(\phi_3(\mathfrak{s})))d\mathfrak{s}\right)\right]_{t=t_0} = 0, \end{cases} \tag{2.1}$$

where $t \in I = [t_0, T], \mathfrak{a}_1, \mathfrak{a}_2 \in R$. Assume that

(V) $\kappa : I \times I \to R_+$ is a continuous function and the operator $\mathbb{K}$ defined by

$$(\mathbb{K}\mathfrak{y})(t) = \int_{t_0}^t \kappa(t,\mathfrak{s})\mathfrak{y}(\mathfrak{s})d\mathfrak{s}, t \in I$$

and

$$\int_{t_0}^1 |\kappa(t,\mathfrak{s})|\alpha(\mathfrak{s})d\mathfrak{s} < \Lambda.$$

By a similar way, we can deduce that

$$r = \frac{|\ \mathfrak{a}_1\ | M + |\mathfrak{a}_2|k + |\mathfrak{a}_2|\mathfrak{l}_2\Lambda}{1 - |\ \mathfrak{a}_1\ | \mathfrak{b}T - |\ \mathfrak{a}_2\ | \mathfrak{b}_1\mathfrak{l}_2\|\mathbb{K}\| - |\mathfrak{a}_2|\mathfrak{l}_1}.$$

Moreover, we easily can prove the following result.

Theorem 2.3. *Let assumptions (i)-(iv) and* (**V**) *be satisfied. If* $1 > |\mathfrak{a}_1|\mathfrak{b}T + |\mathfrak{a}_2||\mathfrak{b}_1\mathfrak{l}_2\|\mathbb{K}\| + |\mathfrak{a}_2|\mathfrak{l}_1$, *then there exists at least one continuous positive solution for the nonlinear neutral integro-differential equation* (2.1).

2.1. *Existence of unique solution*

Let the functions $\mathfrak{f}_1$ and $\mathfrak{g}$ satisfy the following assumption

(v)

$$|\ \mathfrak{f}_1(t,\mathfrak{x}) - \mathfrak{f}_1(t,\mathfrak{y})| \leqslant L|\mathfrak{x} - \mathfrak{y}|, \quad |\ \mathfrak{g}(t,\mathfrak{x}) - \mathfrak{g}(t,\mathfrak{y})| \leqslant L'|\mathfrak{x} - \mathfrak{y}|, \ \ \forall (t,\mathfrak{x}), (t,\mathfrak{y}) \in I \times R_+.$$

Let $\mathfrak{x}(t)$ and $\tilde{\mathfrak{x}}(t)$ be two solutions of the nonlinear neutral integro-differential equations

$$|\ \mathfrak{x}(t) - \tilde{\mathfrak{x}}(t)\ | = \Bigg|\mathfrak{a}_1 \int_{t_0}^t \mathfrak{f}_1(\mathfrak{s}, \mathfrak{x}(\phi_1(\mathfrak{s})))d\mathfrak{s} + \mathfrak{a}_2\mathfrak{f}_2\left(t, \mathfrak{x}(\phi_2(t)), \int_{t_0}^1 \frac{t}{t+\mathfrak{s}}\mathfrak{g}(\mathfrak{s}, \mathfrak{x}(\phi_3(\mathfrak{s})))d\mathfrak{s}\right)$$
$$- \mathfrak{a}_1 \int_{t_0}^t \mathfrak{f}_1(\mathfrak{s}, \tilde{\mathfrak{x}}(\phi_1(\mathfrak{s})))d\mathfrak{s} - \mathfrak{a}_2\mathfrak{f}_2\left(t, \tilde{\mathfrak{x}}(\phi_2(t), \int_{t_0}^1 \frac{t}{t+\mathfrak{s}}\mathfrak{g}(\mathfrak{s}, \tilde{\mathfrak{x}}(\phi_3(\mathfrak{s})))d\mathfrak{s}\right)\Bigg|$$

$$
\begin{aligned}
&\leqslant |a_1| \int_{t_0}^{t} |f_1(s, x(\phi_1(s))) - f_1(s, \widetilde{x}(\phi_1(s)))| ds + |a_2| \Big| f_2\left(t, x(\phi_2(t)), \int_{t_0}^{1} \frac{t}{t+s} g(s, x(\phi_3(s))) ds\right) \\
&\quad - f_2\left(t, \widetilde{x}(\phi_2(t)), \int_{t_0}^{1} \frac{t}{t+s} g(s, \widetilde{x}(\phi_3(s))) ds\right) \Big| \\
&\leqslant |a_1| TL \sup_{t\in I} |x(\phi_1(t)) - \widetilde{x}(\phi_1(t))| + |a_2| \Big[l_1 |x(\phi_2(t)) - \widetilde{x}(\phi_2(t))| \\
&\quad + l_2 \Big| \int_{t_0}^{1} \frac{t}{t+s} g(s, x(\phi_3(s))) ds - \int_{t_0}^{1} \frac{t}{t+s} g(s, \widetilde{x}(\phi_3(s))) ds \Big| \Big] \\
&\leqslant |a_1| TL \|x(t) - \widetilde{x}(t)\| + |a_2| \Big[l_1 |x(t) - \widetilde{x}(t)| + l_2 \Big| \int_{t_0}^{1} \frac{t}{t+s} g(s, x(\phi_3(s))) ds - \int_{t_0}^{1} \frac{t}{t+s} g(s, \widetilde{x}(\phi_3(s))) ds \Big| \Big] \\
&\leqslant |a_1| TL \|x(t) - \widetilde{x}(t)\| + |a_2| \Big[l_1 \|x(t) - \widetilde{x}(t)\| + l_2 \int_{t_0}^{1} \frac{t}{t+s} |g(s, x(\phi_3(s))) - g(s, \widetilde{x}(\phi_3(s)))| ds \Big] \\
&\leqslant |a_1| TL \|x(t) - \widetilde{x}(t)\| + |a_2| \Big[l_1 \|x(t) - \widetilde{x}(t)\| + l_2 L' \sup_{t\in I} |x(\phi_3(t)) - \widetilde{x}(\phi_3(t))| \Big] \\
&\leqslant |a_1| TL \|x(t) - \widetilde{x}(t)\| + |a_2| [l_1 \|x(t) - \widetilde{x}(t)\| + l_2 L' \|x(t) - \widetilde{x}(t)\|] \\
&\leqslant |a_1| TL \|x(t) - \widetilde{x}(t)\| + |a_2| [l_1 \|x(t) - \widetilde{x}(t)\| + l_2 L' \|x(t) - \widetilde{x}(t)\|].
\end{aligned}
$$

Then

$$
\begin{aligned}
&\|x(t) - \widetilde{x}(t)\| \leqslant |a_1| TL \|x(t) - \widetilde{x}(t)\| + |a_2| [l_1 \|x(t) - \widetilde{x}(t)\| + l_2 L' \|x(t) - \widetilde{x}(t)\|], \\
&\left[1 - |a_1| TL - |a_2| l_1 - |a_2| l_2 L'\right] \|x(t) - \widetilde{x}(t)\| \leqslant 0.
\end{aligned}
$$

Since, $|a_1|TL - |a_2|l_1 - |a_2|l_2L' < 1$, then

$$\|x(t) - \widetilde{x}(t)\| = 0 \Rightarrow x(t) = \widetilde{x}(t).$$

Then the following result is proved

Theorem 2.4. *Let assumptions (i)-(v) be satisfied. If $|a_1|TL - |a_2|l_1 - |a_2|l_2L' < 1$, then there exists unique continuous solution for the nonlinear neutral differential equation via Chandrasekhar integral (1.2).*

2.2. Continuous dependence of solutions on delay functions

Here, we study the continuous dependence of solutions of the nonlinear neutral integro-differential equation via Chandrasekhar integral (1.2) on the functions $\phi_i, i = 1, 2, 3$.

Definition 2.5. The solutions $x \in C(I, R_+)$ of (1.2) are continuously dependent on the function ϕ_1, if $\forall \epsilon > 0, \exists \delta > 0$ such that $\|\phi_1 - \tilde{\phi_1}\| \leqslant \epsilon$ implies that $\|x - \tilde{x}\| \leqslant \delta$, where

$$\tilde{x}(t) = a_1 \int_{t_0}^{t} f_1(s, \tilde{x}(\tilde{\phi}_1(s))) ds + a_2 f_2\left(t, \tilde{x}(\phi_2(t)), \int_{t_0}^{1} \frac{t}{t+s} g(s, \tilde{x}(\phi_3(s))) ds\right).$$

Theorem 2.6. *Let assumptions of Theorem 2.4 be satisfied. If $|a_1|LT + |a_2|l_1 + |a_2|l_2L' < 1$, then the solution $x \in C(I, R_+)$ of the nonlinear neutral integro-differential equation via Chandrasekhar integral (1.2) depends continuously on ϕ_1.*

Proof. Let x and $\tilde{x}$ be two solutions of the nonlinear neutral integro-differential equation via Chandrasekhar integral (1.2),

$$|x(t) - \tilde{x}(t)| = \Big| a_1 \int_{t_0}^{t} f_1(s, x(\phi_1(s))) ds + a_2 f_2\left(t, x(\phi_2(t)), \int_{t_0}^{1} \frac{t}{t+s} g(s, x(\phi_3(s))) ds\right)$$

$$
\begin{aligned}
&- a_1 \int_{t_0}^{t} \mathfrak{f}_1(\mathfrak{s}, \tilde{\mathfrak{x}}(\tilde{\phi}_1(\mathfrak{s})))d\mathfrak{s} - a_2\mathfrak{f}_2\left(t, \tilde{\mathfrak{x}}(\phi_2(t)), \int_{t_0}^{1} \frac{t}{t+\mathfrak{s}}\mathfrak{g}(\mathfrak{s}, \tilde{\mathfrak{x}}(\phi_3(\mathfrak{s})))d\mathfrak{s}\right)\Bigg| \\
\leqslant\ & |a_1| \int_{t_0}^{t} |\mathfrak{f}_1(\mathfrak{s}, \mathfrak{x}(\phi_1(\mathfrak{s}))) - \mathfrak{f}_1(\mathfrak{s}, \tilde{\mathfrak{x}}(\tilde{\phi}_1(\mathfrak{s})))|d\mathfrak{s} \\
&+ |a_2|\Bigg|\mathfrak{f}_2\left(t, \mathfrak{x}(\phi_2(t)), \int_{t_0}^{1} \frac{t}{t+\mathfrak{s}}\mathfrak{g}(\mathfrak{s}, \mathfrak{x}(\phi_3(\mathfrak{s})))d\mathfrak{s}\right) - \mathfrak{f}_2\left(t, \tilde{\mathfrak{x}}(\phi_2(t)), \int_{t_0}^{1} \frac{t}{t+\mathfrak{s}}\mathfrak{g}(\mathfrak{s}, \tilde{\mathfrak{x}}(\phi_3(\mathfrak{s})))d\mathfrak{s}\right)\Bigg| \\
\leqslant\ & |a_1|L \int_{t_0}^{t} |\mathfrak{x}(\phi_1(\mathfrak{s})) - \tilde{\mathfrak{x}}(\tilde{\phi}_1(\mathfrak{s}))|d\mathfrak{s} \\
&+ |a_2|\mathfrak{l}_1|\mathfrak{x}(\phi_2(t)) - \tilde{\mathfrak{x}}(\phi_2(t))| + |a_2|\mathfrak{l}_2\left|\int_{t_0}^{1} \frac{t}{t+\mathfrak{s}}\mathfrak{g}(\mathfrak{s}, \mathfrak{x}(\phi_3(\mathfrak{s})))d\mathfrak{s} - \int_{t_0}^{1} \frac{t}{t+\mathfrak{s}}\mathfrak{g}(\mathfrak{s}, \tilde{\mathfrak{x}}(\phi_3(\mathfrak{s})))d\mathfrak{s}\right| \\
\leqslant\ & |a_1|L \int_{t_0}^{t} |\mathfrak{x}(\phi_1(\mathfrak{s})) - \mathfrak{x}(\tilde{\phi}_1(\mathfrak{s})) + \mathfrak{x}(\tilde{\phi}_1(\mathfrak{s})) - \tilde{\mathfrak{x}}(\tilde{\phi}_1(\mathfrak{s}))|d\mathfrak{s} \\
&+ |a_2|\mathfrak{l}_1|\mathfrak{x}(\phi_2(t)) - \tilde{\mathfrak{x}}(\phi_2(t))| + |a_2|\mathfrak{l}_2L' \int_{t_0}^{1} |\mathfrak{x}(\phi_3(\mathfrak{s})) - \tilde{\mathfrak{x}}(\phi_3(\mathfrak{s}))|d\mathfrak{s} \\
\leqslant\ & |a_1|L \int_{t_0}^{t} |\mathfrak{x}(\phi_1(\mathfrak{s})) - \mathfrak{x}(\tilde{\phi}_1(\mathfrak{s}))|d\mathfrak{s} + |a_1|L \int_{t_0}^{t} |\mathfrak{x}(\tilde{\phi}_1(\mathfrak{s})) - \tilde{\mathfrak{x}}(\tilde{\phi}_1(\mathfrak{s}))|d\mathfrak{s} \\
&+ |a_2|\mathfrak{l}_1|\mathfrak{x}(t) - \tilde{\mathfrak{x}}(t)| + |a_2|\mathfrak{l}_2L' \int_{t_0}^{1} |\mathfrak{x}(\mathfrak{s}) - \tilde{\mathfrak{x}}(\mathfrak{s})|d\mathfrak{s} \\
\leqslant\ & |a_1|L\mathbb{L} \int_{t_0}^{t} |\phi_1(\mathfrak{s}) - \tilde{\phi}_1(\mathfrak{s})|d\mathfrak{s} + |a_1|LT|\mathfrak{x}(t) - \tilde{\mathfrak{x}}(t)|, \mathbb{L} = \sup_{\forall t \in I}\{\mathfrak{x}'(t)\} \\
&+ |a_2|\mathfrak{l}_1|\mathfrak{x}(t) - \tilde{\mathfrak{x}}(t)| + |a_2|\mathfrak{l}_2L'|\mathfrak{x}(t) - \tilde{\mathfrak{x}}(t)| \\
\leqslant\ & |a_1|L\mathbb{L}T\epsilon + |a_1|LT|\mathfrak{x}(t) - \tilde{\mathfrak{x}}(t)| + \left[|a_2|\mathfrak{l}_1 + |a_2|\mathfrak{l}_2L'\right]|\mathfrak{x}(t) - \tilde{\mathfrak{x}}(t)|.
\end{aligned}
$$

Then

$$(1 - (|a_1|LT + |a_2|\mathfrak{l}_1 + |a_2|\mathfrak{l}_2L'))\|\mathfrak{x} - \tilde{\mathfrak{x}}\| \leqslant |a_1|L\mathbb{L}T\epsilon,$$

$$\|\mathfrak{x} - \tilde{\mathfrak{x}}\| \leqslant \frac{|a_1|L\mathbb{L}T\epsilon}{1 - (|a_1|LT + |a_2|\mathfrak{l}_1 + |a_2|\mathfrak{l}_2L')} = \delta.$$

Since $|a_1|LT + |a_2|\mathfrak{l}_1 + |a_2|\mathfrak{l}_2L' < 1$, then the solution $\mathfrak{x} \in C(I, R_+)$ of (1.2) is continuously dependent on the function ϕ_1. □

As done above we can prove the continuous dependence of solutions of (1.2) on the functions ϕ_2 and ϕ_3.

Theorem 2.7. *Let assumptions of Theorem 2.4 be satisfied. If* $|a_2|\mathfrak{l}_2L' + |a_2|\mathfrak{l}_1 + |a_1|LT < 1$, *then the solution* $\mathfrak{x} \in C(I, R_+)$ *of the nonlinear neutral integro-differential equation via Chandrasekhar integral* (1.2) *depends continuously on* ϕ_2.

Proof. Let $\mathfrak{x}$ and $\tilde{\mathfrak{x}}$ be two solutions of the nonlinear neutral differential equation via Chandrasekhar integral (1.2), where

$$\tilde{\mathfrak{x}}(t) = a_1 \int_{t_0}^{t} \mathfrak{f}_1(\mathfrak{s}, \tilde{\mathfrak{x}}(\phi_1(\mathfrak{s})))d\mathfrak{s} + a_2\mathfrak{f}_2\left(t, \tilde{\mathfrak{x}}(\tilde{\phi}_2(t)), \int_{t_0}^{1} \frac{t}{t+\mathfrak{s}}\mathfrak{g}(\mathfrak{s}, \tilde{\mathfrak{x}}(\phi_3(\mathfrak{s})))d\mathfrak{s}\right),$$

$$|\mathfrak{x}(t) - \tilde{\mathfrak{x}}(t)| = \Bigg|a_1 \int_{t_0}^{t} \mathfrak{f}_1(\mathfrak{s}, \mathfrak{x}(\phi_1(\mathfrak{s})))d\mathfrak{s} + a_2\mathfrak{f}_2\left(t, \mathfrak{x}(\phi_2(t)), \int_{t_0}^{1} \frac{t}{t+\mathfrak{s}}\mathfrak{g}(\mathfrak{s}, \mathfrak{x}(\phi_3(\mathfrak{s})))d\mathfrak{s}\right)$$

$$
\begin{aligned}
&- a_1 \int_{t_0}^{t} \mathfrak{f}_1(\mathfrak{s}, \tilde{\mathfrak{x}}(\phi_1(\mathfrak{s})))d\mathfrak{s} - a_2 \mathfrak{f}_2\left(t, \tilde{\mathfrak{x}}(\tilde{\phi}_2(t)), \int_{t_0}^{1} \frac{t}{t+\mathfrak{s}} \mathfrak{g}(\mathfrak{s}, \tilde{\mathfrak{x}}(\phi_3(\mathfrak{s})))d\mathfrak{s}\right)\Bigg| \\
\leqslant\ & |a_1| \int_{t_0}^{t} |\mathfrak{f}_1(\mathfrak{s}, \mathfrak{x}(\phi_1(\mathfrak{s}))) - \mathfrak{f}_1(\mathfrak{s}, \tilde{\mathfrak{x}}(\phi_1(\mathfrak{s})))|d\mathfrak{s} \\
&+ |a_2| \Bigg| \mathfrak{f}_2\left(t, \mathfrak{x}(\phi_2(t)), \int_{t_0}^{1} \frac{t}{t+\mathfrak{s}} \mathfrak{g}(\mathfrak{s}, \mathfrak{x}(\phi_3(\mathfrak{s})))d\mathfrak{s}\right) - \mathfrak{f}_2\left(t, \tilde{\mathfrak{x}}(\tilde{\phi}_2(t)), \int_{t_0}^{1} \frac{t}{t+\mathfrak{s}} \mathfrak{g}(\mathfrak{s}, \tilde{\mathfrak{x}}(\phi_3(\mathfrak{s})))d\mathfrak{s}\right)\Bigg| \\
\leqslant\ & |a_1| L \int_{t_0}^{t} |\mathfrak{x}(\phi_1(\mathfrak{s})) - \tilde{\mathfrak{x}}(\phi_1(\mathfrak{s}))|d\mathfrak{s} \\
&+ |a_2| \mathfrak{l}_1 |\mathfrak{x}(\phi_2(t)) - \tilde{\mathfrak{x}}(\tilde{\phi}_2(t))| + |a_2| \mathfrak{l}_2 \left| \int_{t_0}^{1} \frac{t}{t+\mathfrak{s}} \mathfrak{g}(\mathfrak{s}, \mathfrak{x}(\phi_3(\mathfrak{s})))d\mathfrak{s} - \int_{t_0}^{1} \frac{t}{t+\mathfrak{s}} \mathfrak{g}(\mathfrak{s}, \tilde{\mathfrak{x}}(\phi_3(\mathfrak{s})))d\mathfrak{s} \right| \\
\leqslant\ & |a_1| L \int_{t_0}^{t} |\mathfrak{x}(\phi_1(\mathfrak{s})) - \tilde{\mathfrak{x}}(\phi_1(\mathfrak{s}))|d\mathfrak{s} + |a_2| \mathfrak{l}_1 |\mathfrak{x}(\phi_2(t)) - \mathfrak{x}(\tilde{\phi}_2(t))| \\
&+ |a_2| \mathfrak{l}_1 |\mathfrak{x}(\tilde{\phi}_2(t)) - \tilde{\mathfrak{x}}(\tilde{\phi}_2(t))| + |a_2| \mathfrak{l}_2 L' \int_{t_0}^{1} |\mathfrak{x}(\phi_3(\mathfrak{s})) - \tilde{\mathfrak{x}}(\phi_3(\mathfrak{s}))|d\mathfrak{s} \\
\leqslant\ & |a_1| L \int_{t_0}^{t} |\mathfrak{x}(\phi_1(\mathfrak{s})) - \tilde{\mathfrak{x}}(\phi_1(\mathfrak{s}))|d\mathfrak{s} + |a_2| \mathfrak{l}_1 \mathbb{L} |\phi_2(t) - \tilde{\phi}_2(t)| \\
&+ |a_2| \mathfrak{l}_1 |\mathfrak{x}(t) - \tilde{\mathfrak{x}}(t)| + |a_2| \mathfrak{l}_2 L' \int_{t_0}^{1} |\mathfrak{x}(\mathfrak{s}) - \tilde{\mathfrak{x}}(\mathfrak{s})|d\mathfrak{s} \\
\leqslant\ & |a_1| LT \|\mathfrak{x}(t) - \tilde{\mathfrak{x}}(t)\| + |a_2| \mathfrak{l}_1 \mathbb{L} \epsilon + |a_2| \mathfrak{l}_1 |\mathfrak{x}(t) - \tilde{\mathfrak{x}}(t)| + |a_2| \mathfrak{l}_2 L' \|\mathfrak{x}(t) - \tilde{\mathfrak{x}}(t)\| \\
\leqslant\ & |a_2| \mathfrak{l}_1 \mathbb{L} \epsilon + \left[|a_1| LT + |a_2| \mathfrak{l}_1 + |a_2| \mathfrak{l}_2 L'\right] \|\mathfrak{x}(t) - \tilde{\mathfrak{x}}(t)\|.
\end{aligned}
$$

Then

$$
\begin{aligned}
(1 - (|a_2| \mathfrak{l}_2 L' + |a_2| \mathfrak{l}_1 + |a_1| LT)) \|\mathfrak{x} - \tilde{\mathfrak{x}}\| &\leqslant |a_2| \mathfrak{l}_1 \mathbb{L} \epsilon, \\
\|\mathfrak{x} - \tilde{\mathfrak{x}}\| &\leqslant \frac{|a_2| \mathfrak{l}_1 \mathbb{L} \epsilon}{1 - (|a_2| \mathfrak{l}_2 L' + |a_2| \mathfrak{l}_1 + |a_1| LT)} = \delta.
\end{aligned}
$$

Since $|a_1| LT + |a_2| \mathfrak{l}_1 + |a_2| \mathfrak{l}_2 L' < 1$, then the solution $\mathfrak{x} \in C(I, R_+)$ of (1.2) is continuously dependent on the function ϕ_2. □

Similarly, we have the following result.

Theorem 2.8. *Let assumptions of Theorem 2.4 be satisfied. If* $|a_2| \mathfrak{l}_2 L' + |a_2| \mathfrak{l}_1 + |a_1| TL < 1$, *then the solution* $\mathfrak{x} \in C(I, R_+)$ *of the nonlinear neutral integro-differential equation via Chandrasekhar integral* (1.2) *depends continuously on* ϕ_3.

3. Estimating upper and lower bounds for solutions

In most literature, the existence of solutions which are bounded by constants is handled. In this section, we shall study the existence of positive solutions of the neutral differential equation (1.2) on the interval $I = [t_0, \infty)$ and estimate upper and lower bounds for solutions of the neutral integro-differential equation (1.2) by applying Kransnoselskii, under the following assumptions.

(i:) The functions $\phi_i : R_+ \to R_+, \quad i = 1, 2, 3$ are continuous.

(ii:) $\mathfrak{f}_1, \mathfrak{g} : I \times R_+ \to R_+$ satisfy Carathéodory condition, i.e., $\mathfrak{f}_1, \mathfrak{g}$ are measurable functions in t for any $\mathfrak{x} \in R_+$ and continuous in $\mathfrak{x}$ for almost all $t \in I$. There exist a nonnegative constant $\mathfrak{b}$ and a function $t \to m(t)$, such that

$$
|\mathfrak{g}(t, \mathfrak{x}) - \mathfrak{g}(t, \mathfrak{y})| \leqslant L' |\mathfrak{x} - \mathfrak{y}|, \quad \forall (t, \mathfrak{x}), (t, \mathfrak{y}) \in I \times R_+,
$$

and

$$|\mathfrak{f}_1(t,\mathfrak{x})| \leqslant \mathfrak{m}(t) + \mathfrak{b} \mid \mathfrak{x} \mid, \forall (t,\mathfrak{x}) \in I \times R_+,$$

where $\mathfrak{m}(.) \in L^1(t)$.

(iii:) $\mathfrak{f}_2 : I \times R_+ \times R_+ \to R_+$ is continuous and there exists three positive constants $\mathfrak{l}_0, \mathfrak{l}_1$, and $\mathfrak{l}_2$ satisfying:

$$|\mathfrak{f}_2(t_1,\mathfrak{x}_1,\mathfrak{y}_1) - \mathfrak{f}_2(t_2,\mathfrak{x}_2,\mathfrak{y}_2)| \leqslant \mathfrak{l}_0|t_1 - t_2| + \mathfrak{l}_1 \mid \mathfrak{x}_1 - \mathfrak{x}_2 \mid + \mathfrak{l}_2 \mid \mathfrak{y}_1 - \mathfrak{y}_2 \mid,$$

$\forall (t_1,\mathfrak{x}_1,\mathfrak{y}_1), (t_2,\mathfrak{x}_2,\mathfrak{y}_2) \in I \times R_+ \times R_+$.

(iv:) $\mathfrak{f}_2$ is monotonic nondecreasing function in second and third argument.

(v:) $\mathfrak{f}_1, \mathfrak{g}$ is monotonic nondecreasing function in second argument.

By integrating both sides of (1.2) on R_+, we get

$$\mathfrak{x}(t) = \mathfrak{a}_1 \int_t^\infty \mathfrak{f}_1(\mathfrak{s},\mathfrak{x}(\phi_1(\mathfrak{s})))d\mathfrak{s} + \mathfrak{a}_2\mathfrak{f}_2\left(t,\mathfrak{x}(\phi_2(t)), \int_{t_0}^1 \frac{t}{t+\mathfrak{s}}\mathfrak{g}(\mathfrak{s},\mathfrak{x}(\phi_3(\mathfrak{s})))d\mathfrak{s}\right). \tag{3.1}$$

Now, we define a subset Ω of $C([t_0,\infty), R_+)$ as follows:

$$\Omega = \{\mathfrak{x} = \mathfrak{x}(t) \in C([t_0,\infty), R_+) : \mathfrak{u}(t) \leqslant \mathfrak{x}(t) \leqslant \mathfrak{v}(t), t \geqslant t_0\},$$

and two maps S_1 and $S_2 : \Omega \longrightarrow C[[t_0,\infty), R_+]$ as follows:

$$(S_1\mathfrak{x})(t) = \begin{cases} \mathfrak{a}_2\mathfrak{f}_2\left(t,\mathfrak{x}(\phi_2(t)), \int_{t_0}^1 \frac{t}{t+\mathfrak{s}}\mathfrak{g}(\mathfrak{s},\mathfrak{x}(\phi_3(\mathfrak{s})))d\mathfrak{s}\right), & \text{for } t > t_1, \\ (S_1\mathfrak{x})(t_1), & \text{for } t_0 \leqslant t \leqslant t_1, \end{cases}$$

$$(S_2\mathfrak{x})(t) = \begin{cases} \mathfrak{a}_1 \int_t^\infty \mathfrak{f}_1(\mathfrak{s},\mathfrak{x}(\phi_1(\mathfrak{s})))d\mathfrak{s}, & \text{for } t > t_1, \\ (S_2\mathfrak{x})(t_1) + \mathfrak{v}(t) - \mathfrak{v}(t_1), & \text{for } t_0 \leqslant t \leqslant t_1. \end{cases}$$

Then the functional integral equation (3.1) can be written as:

$$\mathfrak{x}(t) = (S_1\mathfrak{x})(t) + (S_2\mathfrak{y})(t).$$

In this section, we shall consider the existence of a positive solution for the equation (1.2). The next theorem gives us the sufficient conditions for existence of a positive solution which is bounded by two positive functions.

Theorem 3.1. *Let assumptions (i:)-(v:) be satisfied and suppose that there exist bounded functions* $\mathfrak{u}, \mathfrak{v} \in C^1([t_0,\infty), (0,\infty))$ *and* $t_1 \geqslant t_0$ *such that*

$$\mathfrak{u}(t) \leqslant \mathfrak{v}(t), \quad t \geqslant t_0,$$
$$\mathfrak{v}(t) - \mathfrak{v}(t_1) - \mathfrak{u}(t) + \mathfrak{u}(t_1) \geqslant 0, \quad t_0 \leqslant t \leqslant t_1.$$

If $|\mathfrak{a}_2|[\mathfrak{l}_1 + \mathfrak{l}_2 L'] < 1$, *then* (1.2) *has a positive solution which is bounded by functions* $\mathfrak{u}, \mathfrak{v}$.

Proof. Let $C([t_0,\infty), R_+)$ be the set of all continuous functions with the norm $\|\mathfrak{x}\| = \sup_{t \geqslant t_0} |\mathfrak{x}(t)|$. Clearly, the subset Ω of $C([t_0,\infty), R_+)$is a closed, bounded, and convex.

We shall show that for any $\mathfrak{x}, \mathfrak{y} \in \Omega$ we have $S_1\mathfrak{x} + S_2\mathfrak{y} \in \Omega$. For every $\mathfrak{x}, \mathfrak{y} \in \Omega$ and $t \geqslant t_1$, we obtain

$$\begin{aligned}(S_1\mathfrak{x})(t) + (S_2\mathfrak{y})(t) &= \mathfrak{a}_1 \int_t^\infty \mathfrak{f}_1(\mathfrak{s},\mathfrak{y}(\phi_1(\mathfrak{s})))d\mathfrak{s} + \mathfrak{a}_2\mathfrak{f}_2\left(t,\mathfrak{x}(\phi_2(t)), \int_{t_0}^1 \frac{t}{t+\mathfrak{s}}\mathfrak{g}(\mathfrak{s},\mathfrak{x}(\phi_3(\mathfrak{s})))d\mathfrak{s}\right) \\ &\leqslant \mathfrak{a}_1 \int_t^\infty \mathfrak{f}_1(\mathfrak{s},\mathfrak{v}(\phi_1(\mathfrak{s})))d\mathfrak{s} + \mathfrak{a}_2\mathfrak{f}_2\left(t,\mathfrak{v}(\phi_2(t)), \int_{t_0}^1 \frac{t}{t+\mathfrak{s}}\mathfrak{g}(\mathfrak{s},\mathfrak{v}(\phi_3(\mathfrak{s})))d\mathfrak{s}\right) = \mathfrak{v}(t).\end{aligned}$$

For $t \in [t_0, t_1]$ we have:

$$(S_1\mathfrak{x})(t) + (S_2\mathfrak{y})(t) = (S_1\mathfrak{x})(t_1) + (S_2\mathfrak{y})(t_1) + \mathfrak{v}(t) - \mathfrak{v}(t_1) \leqslant \mathfrak{v}(t_1) + \mathfrak{v}(t) - \mathfrak{v}(t_1) = \mathfrak{v}(t).$$

Furthermore for $t \geqslant t_1$ we get:

$$\begin{aligned}(S_1\mathfrak{x})(t) + (S_2\mathfrak{y})(t) &= a_1 \int_t^\infty \mathfrak{f}_1(\mathfrak{s}, \mathfrak{y}(\phi_1(\mathfrak{s})))d\mathfrak{s} + a_2\mathfrak{f}_2\left(t, \mathfrak{x}(\phi_2(t)), \int_{t_0}^1 \frac{t}{t+\mathfrak{s}}\mathfrak{g}(\mathfrak{s}, \mathfrak{x}(\phi_3(\mathfrak{s})))d\mathfrak{s}\right)\\ &\geqslant a_1 \int_t^\infty \mathfrak{f}_1(\mathfrak{s}, u(\phi_1(\mathfrak{s})))d\mathfrak{s} + a_2\mathfrak{f}_2\left(t, u(\phi_2(t)), \int_{t_0}^1 \frac{t}{t+\mathfrak{s}}\mathfrak{g}(\mathfrak{s}, u(\phi_3(\mathfrak{s})))d\mathfrak{s}\right) = u(t).\end{aligned}$$

Let $t \in [t_0, t_1]$, we get:

$$\mathfrak{v}(t) - \mathfrak{v}(t_1) + u(t_1) \geqslant u(t), t_0 \leqslant t \leqslant t_1.$$

Then for $t \in [t_0, t_1]$ and any $\mathfrak{x}, \mathfrak{y} \in \Omega$ we obtain

$$(S_1\mathfrak{x})(t) + (S_2\mathfrak{y})(t) = (S_1\mathfrak{x})(t_1) + (S_2\mathfrak{y})(t_1) + \mathfrak{v}(t) - \mathfrak{v}(t_1) \geqslant u(t_1) + \mathfrak{v}(t) - \mathfrak{v}(t_1) \geqslant u(t).$$

Thus we have proved that $S_1\mathfrak{x} + S_2\mathfrak{y} \in \Omega$ for any $\mathfrak{x},y \in \Omega$. We will show that S_1 is contraction mapping on Ω for $\mathfrak{x}, \mathfrak{y} \in \Omega$ and $t \geqslant t_1$, we have:

$$\begin{aligned}|(S_1\mathfrak{x})(t) - (S_1\mathfrak{y})(t)| &= \Bigg|a_2\mathfrak{f}_2\left(t, \mathfrak{x}(\phi_2(t)), \int_{t_0}^1 \frac{t}{t+\mathfrak{s}}\mathfrak{g}(\mathfrak{s}, \mathfrak{x}(\phi_3(\mathfrak{s})))d\mathfrak{s}\right)\\ &\quad - a_2\mathfrak{f}_2\left(t, \mathfrak{y}(\phi_2(t)), \int_{t_0}^1 \frac{t}{t+\mathfrak{s}}\mathfrak{g}(\mathfrak{s}, \mathfrak{y}(\phi_3(\mathfrak{s})))d\mathfrak{s}\right)\Bigg|\\ &\leqslant |a_2|\mathfrak{l}_1|\mathfrak{x}(\phi_2(t)) - \mathfrak{y}(\phi_2(t))| + |a_2|\mathfrak{l}_2\left|\int_{t_0}^1 \frac{t}{t+\mathfrak{s}}\mathfrak{g}(\mathfrak{s}, \mathfrak{x}(\phi_3(\mathfrak{s})))d\mathfrak{s} - \int_{t_0}^1 \frac{t}{t+\mathfrak{s}}\mathfrak{g}(\mathfrak{s},\mathfrak{y}(\phi_3(\mathfrak{s})))d\mathfrak{s}\right|\\ &\leqslant |a_2|\mathfrak{l}_1|\mathfrak{x}(t) - \mathfrak{y}(t)| + |a_2|\mathfrak{l}_2\int_{t_0}^1 |\mathfrak{g}(\mathfrak{s}, \mathfrak{x}(\phi_3(\mathfrak{s})))d\mathfrak{s} - \mathfrak{g}(\mathfrak{s}, \mathfrak{y}(\phi_3(\mathfrak{s})))|d\mathfrak{s}\\ &\leqslant |a_2|\mathfrak{l}_1|\mathfrak{x}(t) - \mathfrak{y}(t)| + |a_2|\mathfrak{l}_2L'|\mathfrak{x}(\phi_3(t)) - \mathfrak{y}(\phi_3(t))|\\ &\leqslant |a_2|\mathfrak{l}_1|\mathfrak{x}(t) - \mathfrak{y}(t)| + |a_2|\mathfrak{l}_2L'|\mathfrak{x}(t) - \mathfrak{y}(t)|\\ &\leqslant |a_2|[\mathfrak{l}_1 + \mathfrak{l}_2L']|\mathfrak{x}(t) - \mathfrak{y}(t)|.\end{aligned}$$

Then

$$\|(S_1\mathfrak{x})(t) - (S_1\mathfrak{y})(t)\| \leqslant |a_2|[\mathfrak{l}_1 + \mathfrak{l}_2L']\|\mathfrak{x} - \mathfrak{y}\|.$$

Also, for $t \in [t_0, t_1]$ the previous inequality is valid and $|a_2|[\mathfrak{l}_1 + \mathfrak{l}_2L'] < 1$, we conclude that S_1 is a contraction mapping on Ω. We now show that S_2 is completely continuous. First we will show that S_2 is continuous. Let $\mathfrak{x}_k = \mathfrak{x}_k(t) \in \Omega$ be such that $\mathfrak{x}_k(t) \longrightarrow \mathfrak{x}(t)$ as $k \longrightarrow \infty$. Because Ω is closed, $\mathfrak{x} = \mathfrak{x}(t) \in \Omega$. For $t \geqslant t_1$ we have

$$\begin{aligned}|(S_2\mathfrak{x}_k)(t) - (S_2\mathfrak{x}(t))\,| &\leqslant |a_1|\int_t^\infty |\mathfrak{f}_1(\mathfrak{s}, \mathfrak{x}_k(\phi_1(\mathfrak{s}))) - \mathfrak{f}_1(\mathfrak{s}, \mathfrak{x}(\phi_1(\mathfrak{s})))|d\mathfrak{s}\\ &\leqslant |a_1|\int_{t_1}^\infty |\mathfrak{f}_1(\mathfrak{s}, \mathfrak{x}_k(\phi_1(\mathfrak{s}))) - \mathfrak{f}_1(\mathfrak{s}, x(\phi_1(\mathfrak{s})))|d\mathfrak{s}.\end{aligned}$$

Also, we have

$$a_1\int_{t_1}^\infty \mathfrak{f}_1(\mathfrak{s}, \mathfrak{v}(\phi_1(\mathfrak{s})))d\mathfrak{s} < \infty.$$

From assumptions (ii:) and (iii:) we have

$$|\mathfrak{f}_1(\mathfrak{s}, \mathfrak{x}_k(\phi_1(\mathfrak{s})))| \leqslant \mathfrak{m}(\mathfrak{s}) + \mathfrak{b}|\mathfrak{x}_k(\phi_1(\mathfrak{s}))| \leqslant \mathfrak{m}(\mathfrak{s}) + \mathfrak{b}|\mathfrak{v}(\mathfrak{s})| \in L^1(I)$$

and the function $\mathfrak{f}_1(\mathfrak{s}, \mathfrak{x}_k(\phi_1(\mathfrak{s})))$ is continuous in the second argument, i.e.,

$$\mathfrak{f}_1(\mathfrak{s}, \mathfrak{x}_k(\phi_1(\mathfrak{s}))) \to \mathfrak{f}_1(\mathfrak{s}, \mathfrak{x}(\phi_1(\mathfrak{s}))) \text{ as } k \to \infty.$$

Therefore the sequence $\{\mathfrak{f}_1(\mathfrak{s}, \mathfrak{x}_k(\phi_1(\mathfrak{s})))\}$ satisfies Lebesgue dominated convergence theorem [5]. This means that S_2 is continuous.

We now show that $S_2\Omega$ is relatively compact. It is sufficient to show by the Arzela-Ascoli theorem that the family of functions $\{S_2\mathfrak{x} : \mathfrak{x} \in \Omega\}$ is uniformly bounded and equicontinuous on $[t_0, \infty)$. The uniform boundedness follows from the definition of Ω. For the the equicontinuity we only need to show, according to Levitan's result [13], that for any given $\epsilon > 0$ the interval $[t_0, \infty)$ can be decomposed into finite subintervals in such a way that on each subinterval all functions of the family have a change of amplitude less than ϵ, for $\mathfrak{x} \in \Omega$ and any $\epsilon > 0$, we take $t^* \geqslant t_1$ large enough so that

$$\mathfrak{a}_1 \int_{t^*}^{\infty} \mathfrak{f}_1(\mathfrak{s}, \mathfrak{x}(\phi_1(\mathfrak{s})))d\mathfrak{s} < \frac{\epsilon}{2}.$$

Then, for $\mathfrak{x} \in \Omega, T_2 > T_1 \geqslant t^*$, we have

$$|(S_2\mathfrak{x})(T_2) - (S_2\mathfrak{x})(T_1)| \leqslant |\mathfrak{a}_1| \int_{T_2}^{\infty} |\mathfrak{f}_1(\mathfrak{s}, \mathfrak{x}(\phi_1(\mathfrak{s})))|d\mathfrak{s} + |\mathfrak{a}_1| \int_{T_1}^{\infty} |\mathfrak{f}_1(\mathfrak{s}, \mathfrak{x}(\phi_1(\mathfrak{s})))|d\mathfrak{s} \leqslant \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

For $\mathfrak{x} \in \Omega, t_1 \leqslant T_1 \leqslant T_2 \leqslant t^*$, we have

$$\begin{aligned}
|(S_2\mathfrak{x})(T_2) - (S_2\mathfrak{x})(T_1)| &\leqslant |\mathfrak{a}_1 \int_{T_2}^{\infty} \mathfrak{f}_1(\mathfrak{s}, \mathfrak{x}(\phi_1(\mathfrak{s})))d\mathfrak{s} - \mathfrak{a}_1 \int_{T_1}^{\infty} \mathfrak{f}_1(\mathfrak{s}, \mathfrak{x}(\phi_1(\mathfrak{s})))d\mathfrak{s}| \\
&\leqslant |\mathfrak{a}_1| \int_{T_1}^{T_2} |\mathfrak{f}_1(\mathfrak{s}, \mathfrak{x}(\phi_1(\mathfrak{s})))|d\mathfrak{s} \\
&\leqslant |\mathfrak{a}_1| \int_{T_1}^{T_2} \mathfrak{m}(\mathfrak{s})d\mathfrak{s} + |\mathfrak{a}_1|\mathfrak{b} \int_{T_1}^{T_2} |\mathfrak{x}(\phi_1(\mathfrak{s}))|d\mathfrak{s} \\
&\leqslant |\mathfrak{a}_1| \int_{T_1}^{T_2} \mathfrak{m}(\mathfrak{s})d\mathfrak{s} + |\mathfrak{a}_1|\mathfrak{b} \int_{T_1}^{T_2} \mathfrak{v}(\mathfrak{s})d\mathfrak{s} \\
&\leqslant |\mathfrak{a}_1| \int_{T_1}^{T_2} \mathfrak{m}(\mathfrak{s})d\mathfrak{s} + \mathfrak{b}|\mathfrak{a}_1| \max_{t_0 \leqslant \mathfrak{s} \leqslant t^*} |\mathfrak{v}(\mathfrak{s})||T_2 - T_1|.
\end{aligned}$$

Thus there exists $\delta_1 > 0$ such that

$$|(S_2\mathfrak{x})(T_2) - (S_2\mathfrak{x})(T_1)| < \epsilon, \text{ if } 0 < T_2 - T_1 < \delta_1.$$

Finally for any $\mathfrak{x} \in \Omega, t_0 \leqslant T_1 < T_2 \leqslant t_1$, there exists a $\delta_2 > 0$ such that

$$|(S_2\mathfrak{x})(T_2) - (S_2\mathfrak{x})(T_1)| = |\mathfrak{v}(T_2) - \mathfrak{v}(T_1)| \leqslant |\int_{T_1}^{T_2} \mathfrak{v}'(\mathfrak{s})d\mathfrak{s}| \leqslant \max_{t_0 \leqslant \mathfrak{s} \leqslant t^*} |\mathfrak{v}'(\mathfrak{s})||T_2 - T_1| < \epsilon, \text{ if } 0 < T_2 - T_1 < \delta_2.$$

Then $\{S_2 x : \mathfrak{x} \in \Omega\}$ is uniformly bounded and equicontinuous on $[t_0, \infty)$, and hence $S_2\Omega$ is relatively compact subset of $C([t_0, \infty), R)$. By Theorem 1.2 there is an $\mathfrak{x}_0 \in \Omega$ such that $S_1\mathfrak{x}_0 + S_2\mathfrak{x}_0 = \mathfrak{x}_0$. We conclude that $\mathfrak{x}_0(t)$ is a positive solution of (3.1). Thus the proof is complete. □

4. Some applications and remarks

Observe that the equation (1.2) includes several classes of functional, integral, and functional integral equations considered in many literature. As particular cases of equation (1.2), we obtain the following.

• When $\mathfrak{a}_1 = 0, \mathfrak{f}_2(\mathfrak{t}, \mathfrak{x}, \mathfrak{y}) = \mathfrak{q}(\mathfrak{t}) + \mathfrak{x}\mathfrak{y}\psi(\mathfrak{t})$, then we get a quadratic integral equation of Chandrasekhar type

$$\mathfrak{x}(\mathfrak{t}) = \mathfrak{c} + \mathfrak{q}(\mathfrak{t}) + \mathfrak{x}(\phi_2(\mathfrak{t})) \int_0^1 \frac{\mathfrak{t}}{\mathfrak{t}+\mathfrak{s}} \psi(\mathfrak{s})\mathfrak{x}(\phi_3(\mathfrak{s}))d\mathfrak{s}, \quad \mathfrak{t} \in [0,1], \tag{4.1}$$

where $\mathfrak{c}$ is a real constant and $\mathfrak{q}(\mathfrak{t})$ is continuous function and ψ is the characteristic function. Moreover, the quadratic functional integral (4.1) reduces to the well-known Chandrasekhar integral equation in radiative transfer [3],

$$\mathfrak{x}(\mathfrak{t}) = 1 + \mathfrak{x}(\mathfrak{t}) \int_0^1 \frac{\mathfrak{t}}{\mathfrak{t}+\mathfrak{s}} \psi(\mathfrak{s})\mathfrak{x}(\mathfrak{s})d\mathfrak{s}, \quad \mathfrak{t} \in [0,1].$$

It describes a scattering through a homogeneous semi-infinite plane atmosphere. In particular, solutions for this equations need not to be continuous.

• When $\mathfrak{a}_1 = 0$ and $\mathfrak{f}_2(\mathfrak{t}, \mathfrak{x}, \mathfrak{y}) = -\mathfrak{p}(\mathfrak{t})\mathfrak{f}(\mathfrak{x}(\mathfrak{t})) + \mathfrak{r}(\mathfrak{t})$, then we obtain the delay differential equation with a forcing term

$$\mathfrak{x}'(\mathfrak{t}) = -\mathfrak{p}(\mathfrak{t})\mathfrak{f}(\mathfrak{x}(\phi_2(\mathfrak{t}))) + \mathfrak{r}(\mathfrak{t}), \quad \mathfrak{t} \geqslant 0,$$

where $\mathfrak{p} : R_+ \to R_+$ and $\mathfrak{f} : R \to R$ are continuous functions with $\mathfrak{x}\mathfrak{f}(\mathfrak{x}) > 0$ for $\mathfrak{x} \neq 0$, and $\mathfrak{r} : R_+ \to R$ is a continuous function. Which is more general than the delay differential equation by Qian et al. [18] with $\phi_2(\mathfrak{t}) = \mathfrak{t} - \tau,\ \tau \geqslant 0$.

• When $\mathfrak{a}_1 = 0$ and $\mathfrak{f}_2(\mathfrak{t}, N, M) = \left(\mathfrak{p}\frac{K-N(\phi_2(\mathfrak{t}))}{K+\mathfrak{c}\mathfrak{p}N(\phi_2(\mathfrak{t}))} + \mathfrak{r}(\mathfrak{t})\right)N(\mathfrak{t})$, then we obtain

$$N'(\mathfrak{t}) = \left(\mathfrak{p}\frac{K - N(\phi_2(\mathfrak{t}))}{K + \mathfrak{c}\mathfrak{p}N(\phi_2(\mathfrak{t}))} + \mathfrak{r}(\mathfrak{t})\right)N(\mathfrak{t}),$$

which is more general than the well-known delay-logistic equation. For more details about this model, see [8],

$$N'(\mathfrak{t}) = \left(\mathfrak{p}\frac{K - N(\mathfrak{t}-\tau)}{K + \mathfrak{c}\mathfrak{p}N(\mathfrak{t}-\tau)} + \mathfrak{r}(\mathfrak{t})\right)N(\mathfrak{t}), \quad \mathfrak{t} > 0, \tag{4.2}$$

and with initial conditions of the form

$$N(\mathfrak{t}) = \phi(\mathfrak{t}) \text{ for } -\tau \leqslant \mathfrak{t} \leqslant 0,$$

where $\phi(\mathfrak{t}) \in C[[-\tau, 0], [0,\infty)]$ with $\phi(0) > 0$. So, the solution on $[0, \tau]$ is given by

$$N(\mathfrak{t}) = \phi(0)e^{\int_0^{\mathfrak{t}}\left(\mathfrak{p}\frac{K-N(\mathfrak{t}-\tau)}{K+\mathfrak{c}\mathfrak{p}N(\mathfrak{t}-\tau)} + \mathfrak{r}(\mathfrak{t})\right)}. \tag{4.3}$$

Remark 4.1. Also, we can consider the nonlinear neutral retarded differential equations of the form

$$\begin{cases} \frac{d}{d\mathfrak{t}}\left[\mathfrak{x}(\mathfrak{t}) - \mathfrak{a}_2\mathfrak{f}_2\left(\mathfrak{t}, \mathfrak{x}(\mathfrak{t}-\tau_2), \int_0^1 \frac{\mathfrak{t}}{\mathfrak{t}+\mathfrak{s}}\mathfrak{g}(\mathfrak{s}, \mathfrak{x}(\mathfrak{s}-\tau_3))d\mathfrak{s}\right)\right] = \mathfrak{a}_1\mathfrak{f}_1(\mathfrak{t}, \mathfrak{x}(\mathfrak{t}-\tau_1)), \quad \mathfrak{t} > 0, \\ \mathfrak{x}(\mathfrak{t}) = \phi(\mathfrak{t}) \text{ for } -\tau \leqslant \mathfrak{t} \leqslant 0, \quad \tau = \max\{\tau_i, i = 1,2,3\}, \end{cases} \tag{4.4}$$

where $\mathfrak{a}_1, \mathfrak{a}_2 \in R, \tau_i \geqslant 0, i = 1, 2, 3$.

By similar way as done before, we can proof the following theorem.

Theorem 4.2. *Let assumptions (ii:)-(vi:) be satisfied and suppose that there exist bounded functions* $\mathfrak{u}, \mathfrak{v} \in C^1([\mathfrak{t}_0, \infty), (0,\infty))$ *and* $\mathfrak{t}_1 \geqslant \mathfrak{t}_0 + \tau$ *such that*

$$\mathfrak{u}(\mathfrak{t}) \leqslant \mathfrak{v}(\mathfrak{t}),\ \mathfrak{t} \geqslant \mathfrak{t}_0, \qquad \mathfrak{v}(\mathfrak{t}) - \mathfrak{v}(\mathfrak{t}_1) - \mathfrak{u}(\mathfrak{t}) + \mathfrak{u}(\mathfrak{t}_1) \geqslant 0,\ \mathfrak{t}_0 \leqslant \mathfrak{t} \leqslant \mathfrak{t}_1.$$

If $|\mathfrak{a}_2|[\mathfrak{l}_1 + \mathfrak{l}_2 L'] < 1$, *then* (4.4) *has a positive solution which is bounded by functions* $\mathfrak{u}, \mathfrak{v}$.

Conclusion 4.3. We have given some existence theorem for a class of neutral integro-differential equations (1.2) which involved many key integral and functional differential equations that appear in applications of nonlinear analysis. The existence of solutions which are bounded by constants has been treated and received much attention in many papers. The existence of positive solutions which are bounded below and above by positive functions for the nonlinear neutral differential equations (1.2) is proved. In aim of estimating lower bound and upper bound of the solutions, we have made a use of Kransnoselskii fixed point theorem and monotonicity conditions.

Acknowledgment

The authors express their thanks to the anonymous referees for their valuable remarks.

References

[1] J. Banaś, I. J. Cabrera, *On solutions of a neutral differential equation with deviating argument*, Math. Comput. Modelling, **44** (2006), 1080–1088. 1
[2] J. Banaś, K. Goebel, *Measures of Noncomapctness in Banach Spaces*, Lecture Notes in Pure and Applied Math., Dekker, New York, (1980). 1
[3] S. Chandrasekhar, *Radiative Transfer*, Dover Publications, New York, (1960). 4
[4] Y. K. Chang, W. S. Li, *Solvability for Impulsive Neutral Integro-Differential Equations with State-Dependent Delay via Fractional Operators*, J. Optim. Theory Appl., **144** (2010), 445–459. 1
[5] R. F. Curtain, A. J. Pritchard, *Functional Analysis in Modern Applied Mathematics*, Academic Press, (1977). 1.1, 2, 3
[6] B. Dorociaková, A. Najmanová, R. Olach, *Existence of nonoscillatory solutions of first-order neutral differential equatuions*, Abstr. Appl. Anal., **2011** (2011), 9 pages. 1, 1.2
[7] W. G. El-Sayed, *Solvability of a neutral differential equation with deviated argument*, J. Math. Anal. Appl., **327** (2007), 342–350. 1
[8] K. Gopalsmy, M. R. S. Kulenović, G. Ladas, *Time lags in a "food-limited" population model*, Appl. Anal., **31** (1988), 225–237. 4
[9] A. Hussain, T. Kanwal, *Existence and uniqueness for a neutral differential problem with unbounded delay via fixed point results*, Trans. A. Razmadze Math. Inst., **172** (2018), 481–490. 1
[10] P. Kalamani, D. Baleanu, M. M. Arjunan, *Local existence for an impulsive fractional neutral integro-differential system with Riemann-Liouville fractional derivatives in a Banach space*, Adv. Difference Equ., **2018** (2018), 1–26. 1
[11] V. Kolmanovskiĭ, A. Myshkis, *Applied Theory of Functional Differential Equations*, Kluwer Academic, Dordrecht, (1992). 1
[12] Y. Kuang, *Delay Differential Equations with Applications in Population Dynamics*, Academic Press, Boston, (1993). 1
[13] B. M. Levitan, *Some questions of the theory of almost periodic functions. I*, Uspehi Matem. Nauk, **2** (1947), 133–192. 3
[14] B. Lisena, *Global attractivity in nonautonomous logistic equations with delay*, Nonlinear Anal. Real World Appl., **9** (2008), 53–63. 1
[15] D. Mallika, D. Baleanu, S. Suganya, M. M. Arjunan, *Existence results for fractional neutral integro-differential systems with nonlocal condition through resolvent operators*, An. Ştiinţ. Univ. "Ovidius" Constanţa Ser. Mat., **27** (2019), 107–124. 1
[16] O. Moaaz, M. Anis, D. Baleanu, A. Muhib, *More effective criteria for Oscillation of Second-Order Differential Equations with Neutral Arguments*, Mathematics, **8** (2020), 1–13. 1
[17] O. Moaaz, D. Baleanu, A. Muhib, *New Aspects for Non-Existence of Kneser Solutions of Neutral Differential Equations with Odd-Order*, Mathematics, **8** (2020), 1–11.
[18] C. Qian, Y. Sun, *Global attractivity of solutions of nonlinear delay differential equations with a forcing term*, Nonlinear Anal., **66** (2007), 689–703. 4
[19] V. Usha, D. Baleanu, M. M. Arjunan, *Existence results for an impulsive neutral integro-differential equations in Banach spaces*, An. Ştiinţ. Univ. "Ovidius" Constanţa Ser. Mat., **27** (2019), 231–257. 1