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Cone A_b-metric space and some coupled fixed point theorems

K. Anthony Singh[a,*], M. R. Singh[b], M. Bina Devi[a], Th. Chhatrajit Singh[c]

[a]Department of Mathematics, D. M. College of Science, Dhanamanjuri University, Imphal (Manipur), India.

[b]Department of Mathematics, Manipur University, Canchipur (Manipur), India.

[c]Department of Mathematics, Manipur Technical University, Takyelpat (Manipur)-795004, India.

Abstract

In this paper, we extend the definition of coupled fixed point to mappings on cone A_b-metric space and prove some coupled fixed point theorems. Our results extend the coupled fixed point results of Singh and Singh [K. A. Singh, M. R. Singh, J. Math. Comput. Sci., **10** (2020), 891–905] to cone A_b-metric space. An example is also given to illustrate the validity of our result.

Keywords: Coupled fixed point, cone metric space, cone A_b-metric space.

2020 MSC: 47H10, 54H25.

1. Introduction

The classical Banach contraction principle which forms the foundation of metric fixed point theory has been studied and generalized in various directions by many authors. As a generalization of metric space, Huang and Zhang [5] introduced the concept of cone metric space by replacing the set of real numbers by a general Banach space E which is partially ordered with respect to a cone $P \subset E$.

On the other hand, Sedghi et al. [9] generalized metric space to S-metric space and Bakhtin [2] generalized it to b-metric space. Nizar and Nabil [13] introduced the concept of S_b-metric space. The concepts of S-metric space and S_b-metric space are further extended to A-metric space and A_b-metric space respectively by Abbas et al. [1] and Ughade et al. [14]. Dhamodharan and Krishnakumar [3] also again extended S-metric space to cone S-metric space. Singh and Singh [10] then generalized cone S-metric space to cone S_b-metric space and prove some fixed point theorems. Further, Singh et al. [12] generalized cone S_b-metric space to cone A_b-metric space and proved some fixed point theorems. In the meantime, Bhaskar and Lakshmikantham [4] introduced the concept of coupled fixed point of a mapping $F : X \times X \to X$. Lakshmikantham and Ciric [6] investigated some more coupled fixed point theorems in partially ordered sets. Then, Sabetghadam et al. [8] considered the corresponding definition of coupled

*Corresponding author
Email address: anthonykumam@manipuruniv.ac.in (K. Anthony Singh)

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fixed point for mappings on cone metric spaces and proved some coupled fixed point theorems. Singh and Singh [11] further extended these results of Sabetghadam et al. [8] to cone S_b-metric space.

The aim of this paper is to further extend the definition of coupled fixed point to mappings on cone A_b-metric space and prove some coupled fixed point theorems. Our results extend the coupled fixed point results of Singh and Singh [11] to cone A_b-metric space. We also give an example to illustrate the validity of our result.

2. Preliminaries

Following definitions, properties will be needed in the sequel.

Definition 2.1 ([5]). Let E be a Banach space. A subset P of E is called a cone if and only if

1. P is closed, nonempty and $P \neq \{0\}$;
2. $ax + by \in P$ for all $x, y \in P$ and nonnegative real numbers a, b;
3. $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering $\leqslant$ in E with respect to P by $x \leqslant y$ if and only if $y - x \in P$. We will write $x < y$ to indicate that $x \leqslant y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in int.P$, where $int.P$ denotes the interior of P. The cone P is called normal if there is a number $K > 0$ such that $0 \leqslant x \leqslant y$ implies $\| x \| \leqslant K \| y \|$ for all $x, y \in E$. The least positive number satisfying the above is called the normal constant of P.

The cone P is called regular if every increasing sequence which is bounded from above is convergent. That is, if $\{x_n\}$ is a sequence such that $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n \leqslant \cdots \leqslant y$ for some $y \in E$, then there is $x \in E$ such that $\| x_n - x \| \to 0$ as $n \to \infty$. Equivalently, the cone P is regular if and only if every decreasing sequence which is bounded from below is convergent. It is well known that a regular cone is a normal cone.

Definition 2.2 ([11]). Let X be a non-empty set. Suppose the mapping $d : X \times X \to E$ satisfies

1. $d(x, y) \geqslant 0$, and $d(x, y) = 0$ if and only if $x = y$, $\forall x, y \in X$;
2. $d(x, y) = d(y, x)$, $\forall x, y \in X$;
3. $d(x, y) \leqslant d(x, z) + d(z, y)$, $\forall x, y, z \in X$.

Then (X, d) is called a cone metric space or simply CMS.

Example 2.3 ([5]). Let $E = \mathbb{R}^2, P = \{(x, y) : x, y \geqslant 0\}, X = \mathbb{R}$ and $d : X \times X \to E$ such that

$$d(x, y) = (| x - y |, \alpha | x - y |),$$

where $\alpha \geqslant 0$ is a constant. Then (X, d) is a cone metric space.

Definition 2.4 ([7]). Let X be a non-empty set and let $b \geqslant 1$ be a given real number. A function $A : X^n \to [0, \infty)$ is called an A_b-metric on X if for all $x_i,\ a \in X, i = 1, 2, 3, \cdots, n$, the following conditions are satisfied

1. $A(x_1, x_2, \cdots, x_{n-1}, x_n) \geqslant 0$;
2. $A(x_1, x_2, \cdots, x_{n-1}, x_n) = 0$ if and only if $x_1 = x_2 = \cdots = x_{n-1} = x_n$;
3. $$A(x_1, x_2, \cdots, x_{n-1}, x_n) \leqslant b[A(x_1, x_1, \cdots, (x_1)_{n-1}, a) + A(x_2, x_2, \cdots, (x_2)_{n-1}, a) + \cdots + A(x_{n-1}, x_{n-1}, \cdots, (x_{n-1})_{n-1}, a) + A(x_n, x_n, \cdots, (x_n)_{n-1}, a)].$$

The pair (X, A) is called an A_b-metric space.

Example 2.5 ([7]). Let $X = [1, \infty)$. Define $A_b : X^n \to [0, \infty)$ by

$$A_b(x_1, x_2, \cdots, x_n) = \sum_{i=1}^{n} \sum_{i<j} | x_i - x_j |^2,$$

for all $x_i \in X, i = 1, 2, 3, \cdots n$. Then (X, A_b) is an A_b-metric space with $b = 2 > 1$.

Definition 2.6 ([12]). Suppose that E is a real Banach space, P is a cone in E with int.$P \neq \phi$ and $\leqslant$ is partial ordering in E with respect to P. Let X be a non-empty set, and let the function $A : X^n \to E$ satisfy the following conditions

1. $A(x_1, x_2, \cdots, x_{n-1}, x_n) \geqslant 0$;
2. $A(x_1, x_2, \cdots, x_{n-1}, x_n) = 0$ if and only if $x_1 = x_2 = \cdots = x_{n-1} = x_n$;
3. $$\begin{aligned} A(x_1, x_2, \cdots, x_{n-1}, x_n) \leqslant b[&A(x_1, x_1, \cdots, (x_1)_{n-1}, a) + A(x_2, x_2, \cdots, (x_2)_{n-1}, a) + \cdots \\ &+ A(x_{n-1}, x_{n-1}, \cdots, (x_{n-1})_{n-1}, a) \\ &+ A(x_n, x_n, \cdots, (x_n)_{n-1}, a)], \quad \forall x_i, \ a \in X, \ i = 1, 2, 3, \cdots, n, \end{aligned}$$

where $b \geqslant 1$ is a constant. Then, the function A is called a cone A_b-metric on X and the pair (X, A) is called a cone A_b-metric space.

We note that cone A_b-metric spaces are generalizations of cone S_b-metric spaces since every cone S_b-metric is a cone A_b-metric with $n = 3$.

Example 2.7 ([12]). Let $E = \mathbb{R}^2$, the Euclidean plane, and $P = \{(x, y) \in E : x, y \geqslant 0\}$, a normal cone in E. Let $X = \mathbb{R}$ and $A : X^n \to E$ be such that $A(x_1, x_2, \cdots, x_n) = A_*(x_1, x_2, \cdots, x_n)(\alpha, \beta)$, where $\alpha, \beta > 0$ are constants and A_* is an A_b-metric on X. Then A is a cone A_b-metric on X. In particular, we have the function

$$A_*(x_1, x_2, \cdots, x_n) = \sum_{i=1}^{n} \sum_{i<j} | x_i - x_j |^2, \quad x_i \in X, \quad i = 1, 2, 3, \cdots, n,$$

is an A_b-metric on X with $b = 2$. Therefore, the function

$$A(x_1, x_2, \cdots, x_n) = \left(\sum_{i=1}^{n} \sum_{i<j} | x_i - x_j |^2, \frac{1}{4} \sum_{i=1}^{n} \sum_{i<j} | x_i - x_j |^2 \right),$$

is a cone A_b-metric on X with $b = 2$.

Definition 2.8 ([12]). Let (X, A) be a cone A_b-metric space.

1. A sequence $\{x_n\}$ in X is said to converge to x if for each $c \in E$, $0 \ll c$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$, $A(x_n, x_n, \cdots, x_n, x) \ll c$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.
2. A sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $c \in E$, $0 \ll c$ there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geqslant n_0$, $A(x_n, x_n, \cdots, x_n, x_m) \ll c$.
3. The cone A_b-metric space (X, A) is called complete if every Cauchy sequence is convergent.

Lemma 2.9 ([12]). *Let (X, A) be a cone A_b-metric space. Then, for all $x, y, a \in X$,*

(i) $A(x, x, \cdots, x, y) \leqslant bA(y, y, \cdots, y, x)$;

(ii) $A(x, x, \cdots, x, y) \leqslant (n-1)bA(x, x, \cdots, x, a) + bA(y, y, \cdots, y, a)$.

Lemma 2.10 ([12]). *Let (X, A) be a cone A_b-metric space, P be a normal cone with normal constant K. Then a sequence $\{x_n\}$ in X converges to x if and only if $A(x_n, x_n, \cdots, x_n, x) \to 0$ as $n \to \infty$.*

Lemma 2.11 ([12]). *Let (X, A) be a cone A_b-metric space, P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X. If $\{x_n\}$ converges to w_1 and $\{x_n\}$ converges to w_2, then $w_1 = w_2$. That is, the limit of a convergent sequence is unique.*

Lemma 2.12 ([12]). *Let (X, A) be a cone A_b-metric space, P be a normal cone with normal constant K. Then a sequence $\{x_n\}$ in X is a Cauchy sequence if and only if $A(x_n, x_n, \cdots, x_n, x_m) \to 0$ as $n, m \to \infty$.*

Lemma 2.13 ([12]). *Let (X, A) be a cone A_b-metric space, P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X. If $\{x_n\}$ converges to w, then $\{x_n\}$ is a Cauchy sequence. That is, every convergent sequence is Cauchy.*

3. Main results

We now state and prove our main results.

First we give the corresponding definition of coupled fixed point in cone A_b-metric space.

Definition 3.1. Let (X, A) be a cone A_b-metric space. An element $(x, y) \in X \times X$ is said to be a coupled fixed point of $F : X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$.

Theorem 3.2. *Let (X, A) be a complete cone A_b-metric space and P be a normal cone with normal constant K. Suppose the mapping $F : X \times X \to X$ satisfies the following contractive condition*

$$A(F(x, y), \cdots, F(x, y), F(u, v)) \leqslant kA(x, \cdots, x, u) + lA(y, \cdots, y, v), \tag{3.1}$$

for all $x, y, u, v \in X$, where k, l are nonnegative constants with $k + l \in \left[0, \frac{1}{b^2}\right)$. Then, F has a unique coupled fixed point.

Proof. Let us choose $x_0, y_0 \in X$ and set $x_1 = F(x_0, y_0)$, $y_1 = F(y_0, x_0)$, $x_2 = F(x_1, y_1)$, $y_2 = F(y_1, x_1), \cdots,$ $x_{n+1} = F(x_n, y_n)$, $y_{n+1} = F(y_n, x_n)$. Then using condition (3.1), we obtain

$$\begin{aligned} A(x_n, \cdots, x_n, x_{n+1}) &= A(F(x_{n-1}, y_{n-1}), \cdots, F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \\ &\leqslant kA(x_{n-1}, \cdots, x_{n-1}, x_n) + lA(y_{n-1}, \cdots, y_{n-1}, y_n). \end{aligned}$$

Also, we have

$$\begin{aligned} A(y_n, \cdots, y_n, y_{n+1}) &= A(F(y_{n-1}, x_{n-1}), \cdots, F(y_{n-1}, x_{n-1}), F(y_n, x_n)) \\ &\leqslant kA(y_{n-1}, \cdots, y_{n-1}, y_n) + lA(x_{n-1}, \cdots, x_{n-1}, x_n). \end{aligned}$$

If we let $A_n = A(x_n, \cdots, x_n, x_{n+1}) + A(y_n, \cdots, y_n, y_{n+1})$, then we have

$$\begin{aligned} A_n &= A(x_n, \cdots, x_n, x_{n+1}) + A(y_n, \cdots, y_n, y_{n+1}) \\ &\leqslant (k+l)(A(x_{n-1}, \cdots, x_{n-1}, x_n) + A(y_{n-1}, \cdots, y_{n-1}, y_n)) \\ &= (k+l)A_{n-1}. \end{aligned}$$

Thus if we take $\alpha = k + l < 1$, then for each $n \in \mathbb{N}$, we have

$$0 \leqslant A_n \leqslant \alpha A_{n-1} \leqslant \alpha^2 A_{n-2} \leqslant \cdots \leqslant \alpha^n A_0.$$

If $A_0 = 0$, (x_0, y_0) is a coupled fixed point of F. Let us therefore suppose that $A_0 > 0$. Then for $m > n$, we have

$$A(x_n, \cdots, x_n, x_m) \leqslant b\,[(n-1)A(x_n, \cdots, x_n, x_{n+1}) + A(x_m, \cdots, x_m, x_{n+1})]$$

$$\begin{aligned}
&\leqslant (n-1)bA(x_n,\cdots,x_n,x_{n+1}) + b^2A(x_{n+1},\cdots,x_{n+1},x_m)\\
&\leqslant (n-1)bA(x_n,\cdots,x_n,x_{n+1}) + (n-1)b^3A(x_{n+1},\cdots,x_{n+1},x_{n+2})\\
&\quad + b^4A(x_{n+2},\cdots,x_{n+2},x_m)\\
&\leqslant (n-1)bA(x_n,\cdots,x_n,x_{n+1}) + (n-1)b^3A(x_{n+1},\cdots,x_{n+1},x_{n+2})\\
&\quad + (n-1)b^5A(x_{n+2},\cdots,x_{n+2},x_{n+3}) + \cdots + b^{2(m-n-1)}A(x_{m-1},\cdots,x_{m-1},x_m)\\
&\leqslant (n-1)b\{A(x_n,\cdots,x_n,x_{n+1}) + b^2A(x_{n+1},\cdots,x_{n+1},x_{n+2})\\
&\quad + b^4A(x_{n+2},\cdots,x_{n+2},x_{n+3}) + \cdots + b^{2(m-n-1)}A(x_{m-1},\cdots,x_{m-1},x_m)\}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
A(y_n,\cdots,y_n,y_m) \leqslant (n-1)b\{&A(y_n,\cdots,y_n,y_{n+1}) + b^2A(y_{n+1},\cdots,y_{n+1},y_{n+2})\\
&+ b^4A(y_{n+2},\cdots,y_{n+2},y_{n+3}) + \cdots + b^{2(m-n-1)}A(y_{m-1},\cdots,y_{m-1},y_m)\}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
A(x_n,\cdots,x_n,x_m) + A(y_n,\cdots,y_n,y_m) &\leqslant (n-1)b\{A_n + b^2A_{n+1} + b^4A_{n+2} + \cdots\\
&\quad + b^{2(m-n-1)}A_{m-1}\}\\
&\leqslant (n-1)b\{\alpha^n + b^2\alpha^{n+1} + b^4\alpha^{n+2} + \cdots\\
&\quad + b^{2(m-n-1)}\alpha^{m-1}\}A_0\\
&= (n-1)b\alpha^n\{1 + b^2\alpha + b^4\alpha^2 + \cdots\\
&\quad + b^{2(m-n-1)}\alpha^{m-n-1}\}A_0\\
&= (n-1)b\alpha^n\left\{1 + b^2\alpha + (b^2\alpha)^2 + \cdots + (b^2\alpha)^{m-n-1}\right\}A_0\\
&\leqslant \frac{(n-1)b\alpha^n}{1-b^2\alpha}A_0\\
\Rightarrow \| A(x_n,\cdots,x_n,x_m) + A(y_n,\cdots,y_n,y_m) \| &\leqslant \frac{(n-1)b\alpha^nK}{1-b^2\alpha} \| A_0 \| \to 0 \quad \text{as} \quad n,m\to\infty.
\end{aligned}$$

Therefore, $\{x_n\},\{y_n\}$ are Cauchy sequences in X. Using the completeness hypothesis, there exist $x^*,y^*\in X$ such that $\lim_{n\to\infty} x_n = x^*$ and $\lim_{n\to\infty} y_n = y^*$. We show that (x^*,y^*) is a coupled fixed point of F. Using condition 3 of Definition 2.6 and (3.1), we have

$$\begin{aligned}
A(F(x^*,y^*),\cdots,F(x^*,y^*),x^*) &\leqslant (n-1)bA(F(x^*,y^*),\cdots,F(x^*,y^*),x_{n+1})\\
&\quad + bA(x^*,\cdots,x^*,x_{n+1})\\
&\leqslant (n-1)bA(F(x^*,y^*),\cdots,F(x^*,y^*),F(x_n,y_n)) + bA(x^*,\cdots,x^*,x_{n+1})\\
&\leqslant (n-1)bkA(x^*,\cdots,x^*,x_n) + (n-1)blA(y^*,\cdots,y^*,y_n)\\
&\quad + bA(x^*,\cdots,x^*,x_{n+1})\\
\Rightarrow \| A(F(x^*,y^*),\cdots,F(x^*,y^*),x^*) \| &\leqslant K[(n-1)bk \| A(x^*,\cdots,x^*,x_n) \|\\
&\quad + (n-1)bl \| A(y^*,\cdots,y^*,y_n) \|\\
&\quad + b \| A(x^*,\cdots,x^*,x_{n+1}) \|] \to 0 \quad \text{as} \quad n\to\infty\\
&\Rightarrow \| A(F(x^*,y^*),\cdots,F(x^*,y^*),x^*) \| = 0\\
&\Rightarrow A(F(x^*,y^*),\cdots,F(x^*,y^*),x^*) = 0\\
&\Rightarrow F(x^*,y^*) = x^*.
\end{aligned}$$

Similarly, we have $F(y^*,x^*) = y^*$. Thus, (x^*,y^*) is a coupled fixed point of F.

Now if (x',y') is another coupled fixed point of F, then we have

$$A(x',\cdots,x',x^*) = A(F(x',y'),\cdots,F(x',y'),F(x^*,y^*))$$

$$\leqslant kA(x',\cdots,x',x^*)+lA(y',\cdots,y',y^*),$$

and

$$\begin{aligned}A(y',\cdots,y',y^*) &= A(F(y',x'),\cdots,F(y',x'),F(y^*,x^*))\\ &\leqslant kA(y',\cdots,y',y^*)+lA(x',\cdots,x',x^*).\end{aligned}$$

Therefore, we have

$$A(x',\cdots,x',x^*)+A(y',\cdots,y',y^*)\leqslant (k+l)(A(x',\cdots,x',x^*)+A(y',\cdots,y',y^*)). \tag{3.2}$$

Since $k+l<1$, (3.2) implies that $A(x',\cdots,x',x^*)+A(y',\cdots,y',y^*)=0$. This means that

$$A(x',\cdots,x',x^*)=0,$$

and $A(y',\cdots,y',y^*)=0$. Hence we have $(x',y')=(x^*,y^*)$. Therefore, the coupled fixed point of F is unique. □

Theorem 3.3. *Let (X,A) be a complete cone A_b-metric space and P be a normal cone with normal constant K. Suppose the mapping $F:X\times X\to X$ satisfies the following contractive condition*

$$A(F(x,y),\cdots,F(x,y),F(u,v))\leqslant kA(F(x,y),\cdots,F(x,y),x)+lA(F(u,v),\cdots,F(u,v),u), \tag{3.3}$$

for all $x,y,u,v\in X$, where k,l are nonnegative constants with $k+l\in\left[0,\frac{1}{(n-1)b^3}\right)$. Then, F has a unique coupled fixed point.

Proof. Let us choose $x_0,y_0\in X$ and set $x_1=F(x_0,y_0)$, $y_1=F(y_0,x_0)$, $x_2=F(x_1,y_1)$, $y_2=F(y_1,x_1),\cdots$, $x_{n+1}=F(x_n,y_n)$, $y_{n+1}=F(y_n,x_n)$. Using (3.3) and condition 3 of Definition 2.6, we get

$$\begin{aligned}A(x_n,,x_n,x_{n+1}) &= A(F(x_{n-1},y_{n-1}),\cdots,F(x_{n-1},y_{n-1}),F(x_n,y_n))\\ &\leqslant kA(F(x_{n-1},y_{n-1}),\cdots,F(x_{n-1},y_{n-1}),x_{n-1})\\ &\quad +lA(F(x_n,y_n),\cdots,F(x_n,y_n),x_n)\\ &\leqslant kA(x_n,\cdots,x_n,x_{n-1})+lA(x_{n+1},\cdots,x_{n+1},x_n)\\ &\leqslant kbA(x_{n-1},\cdots,x_{n-1},x_n)+lbA(x_n,\cdots,x_n,x_{n+1})\\ &\Rightarrow A(x_n,\cdots,x_n,x_{n+1})\leqslant \frac{kb}{1-lb}A(x_{n-1},\cdots,x_{n-1},x_n)\\ &\Rightarrow A(x_n,\cdots,x_n,x_{n+1})\leqslant \beta A(x_{n-1},\cdots,x_{n-1},x_n),\quad \text{where}\quad \beta=\frac{kb}{1-lb}<1.\end{aligned}$$

Similarly, we have

$$A(y_n,\cdots,y_n,y_{n+1})\leqslant \beta A(y_{n-1},\cdots,y_{n-1},y_n).$$

Therefore for $p>q$, we get

$$A(x_q,\cdots,x_q,x_p)\leqslant \frac{(n-1)b\beta^q}{1-b^2\beta}A(x_0,\cdots,x_0,x_1),$$

and

$$A(y_q,\cdots,y_q,y_p)\leqslant \frac{(n-1)b\beta^q}{1-b^2\beta}A(y_0,\cdots,y_0,y_1).$$

$$\Rightarrow A(x_q,\cdots,x_q,x_p)\to\infty \quad\text{and}\quad A(y_q,\cdots,y_q,y_p)\to\infty \quad\text{as}\quad q,p\to\infty.$$

Here we note that $b^2\beta=\frac{kb^3}{1-lb}<1$.

Therefore $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X. Using the completeness hypothesis, there exist $x^*, y^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$ and $\lim_{n\to\infty} y_n = y^*$. We show that (x^*, y^*) is a coupled fixed point of F.

Using condition 3 of Definition 2.6 and (3.3), we have

$$\begin{aligned}
A(F(x^*,y^*),\cdots,F(x^*,y^*),x^*) &\leqslant (n-1)bA(F(x^*,y^*),\cdots,F(x^*,y^*),x_{n+1}) \\
&\quad + bA(x^*,\cdots,x^*,x_{n+1}) \\
&= (n-1)bA(F(x^*,y^*),\cdots,F(x^*,y^*),F(x_n,y_n)) \\
&\quad + bA(x^*,\cdots,x^*,x_{n+1}) \\
&\leqslant (n-1)bkA(F(x^*,y^*),\cdots,F(x^*,y^*),x^*) \\
&\quad + (n-1)blA(F(x_n,y_n),\cdots,F(x_n,y_n),x_n) + bA(x^*,\cdots,x^*,x_{n+1}) \\
\Rightarrow A(F(x^*,y^*),\cdots,F(x^*,y^*),x^*) &\leqslant \frac{(n-1)bl}{1-(n-1)bk}A(F(x_n,y_n),\cdots,F(x_n,y_n),x_n) \\
&\quad + \frac{b}{1-(n-1)bk}A(x^*,\cdots,x^*,x_{n+1}) \\
\Rightarrow A(F(x^*,y^*),\cdots,F(x^*,y^*),x^*) &\leqslant \frac{(n-1)bl}{1-(n-1)bk}A(x_{n+1},\cdots,x_{n+1},x_n) \\
&\quad + \frac{b^2}{1-(n-1)bk}A(x_{n+1},\cdots,x_{n+1},x^*) \\
\Rightarrow\| A(F(x^*,y^*),\cdots,F(x^*,y^*),x^*) \| &\leqslant K[\frac{(n-1)bl}{1-(n-1)bk} \| A(x_{n+1},\cdots,x_{n+1},x_n) \| \\
&\quad + \frac{b^2}{1-(n-1)bk} \| A(x_{n+1},\cdots,x_{n+1},x^*) \|] \quad \to 0 \quad \text{as} \quad n\to\infty. \\
&\Rightarrow \| A(F(x^*,y^*),\cdots,F(x^*,y^*),x^*) \| = 0 \\
&\Rightarrow A(F(x^*,y^*),\cdots,F(x^*,y^*),x^*) = 0 \\
&\Rightarrow F(x^*,y^*) = x^*.
\end{aligned}$$

Similarly, we can get $F(y^*, x^*) = y^*$. Thus (x^*, y^*) is a coupled fixed point of F.

Now if (x', y') is another coupled fixed point of F, then we have

$$\begin{aligned}
A(x',\cdots,x',x^*) &= A(F(x',y'),\cdots,F(x',y'),F(x^*,y^*)) \\
&\leqslant kA(F(x',y'),\cdots,F(x',y'),x') \\
&\quad + lA(F(x^*,y^*),\cdots,F(x^*,y^*),x^*) \\
&= 0.
\end{aligned}$$

Therefore, $A(x', \cdots, x', x^*) = 0$ and so $x' = x^*$. Similarly, we can get $y' = y^*$. Hence we have $(x', y') = (x^*, y^*)$ which shows that the coupled fixed point of F is unique. □

Theorem 3.4. *Let (X, A) be a complete cone A_b-metric space and P be a normal cone with normal constant K. Suppose the mapping $F : X \times X \to X$ satisfies the following contractive condition*

$$A(F(x,y),\cdots,F(x,y),F(u,v)) \leqslant kA(F(x,y),\cdots,F(x,y),u) + lA(F(u,v),\cdots,F(u,v),x), \tag{3.4}$$

for all $x, y, u, v \in X$, where k, l are nonnegative constants with $k + l \in \left[0, \frac{1}{(n-1)^2b^3}\right)$. Then, F has a unique coupled fixed point.

Proof. Let us choose $x_0, y_0 \in X$ and set $x_1 = F(x_0, y_0)$, $y_1 = F(y_0, x_0)$, $x_2 = F(x_1, y_1)$, $y_2 = F(y_1, x_1), \cdots$, $x_{n+1} = F(x_n, y_n)$, $y_{n+1} = F(y_n, x_n)$. Using (3.4) and Condition 3 of Definition 2.6, we get

$$A(x_n,\cdots,x_n,x_{n+1}) = A(F(x_{n-1},y_{n-1}),\cdots,F(x_{n-1},y_{n-1}),F(x_n,y_n))$$

$$\begin{aligned}
&\leqslant kA(F(x_{n-1},y_{n-1}),\cdots,F(x_{n-1},y_{n-1}),x_n)\\
&\quad + lA(F(x_n,y_n),\cdots,F(x_n,y_n),x_{n-1})\\
&= kA(x_n,\cdots,x_n,x_n) + lA(x_{n+1},\cdots,x_{n+1},x_{n-1})\\
&= lA(x_{n+1},\cdots,x_{n+1},x_{n-1})\\
&\leqslant l((n-1)bA(x_{n+1},\cdots,x_{n+1},x_n) + bA(x_{n-1},\cdots,x_{n-1},x_n))\\
&\leqslant l((n-1)b^2A(x_n,\cdots,x_n,x_{n+1}) + bA(x_{n-1},\cdots,x_{n-1},x_n))
\end{aligned}$$

$$\Rightarrow A(x_n,\cdots,x_n,x_{n+1}) \leqslant \frac{lb}{1-(n-1)lb^2}A(x_{n-1},\cdots,x_{n-1},x_n)$$

$$\Rightarrow A(x_n,\cdots,x_n,x_{n+1}) \leqslant \delta A(x_{n-1},\cdots,x_{n-1},x_n) \quad \text{where} \quad \delta = \frac{lb}{1-(n-1)lb^2} < 1.$$

Similarly, we can get

$$A(y_n,\cdots,y_n,y_{n+1}) \leqslant \delta A(y_{n-1},\cdots,y_{n-1},y_n).$$

Therefore for $p > q$, we get

$$A(x_q,\cdots,x_q,x_p) \leqslant \frac{(n-1)b\delta^q}{1-b^2\delta}A(x_0,\cdots,x_0,x_1),$$

and

$$A(y_q,\cdots,y_q,y_p) \leqslant \frac{(n-1)b\delta^q}{1-b^2\delta}A(y_0,\cdots,y_0,y_1).$$

$$\Rightarrow A(x_q,\cdots,x_q,x_p) \to \infty \quad \text{and} \quad A(y_q,\cdots,y_q,y_p) \to \infty \quad \text{as} \quad q,p \to \infty.$$

Here we note that $b^2\delta = \frac{lb^3}{1-(n-1)lb^2} < 1$.

Therefore $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X. Using the completeness hypothesis, there exist $x^*, y^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$ and $\lim_{n\to\infty} y_n = y^*$. We show that (x^*, y^*) is a coupled fixed point of F.

Using condition 3 of Definition 2.6 and (3.4), we have

$$\begin{aligned}
A(F(x^*,y^*),\cdots,F(x^*,y^*),x^*) &\leqslant (n-1)bA(F(x^*,y^*),\cdots,F(x^*,y^*),x_{n+1})\\
&\quad + bA(x^*,\cdots,x^*,x_{n+1})\\
&= (n-1)bA(F(x^*,y^*),\cdots,F(x^*,y^*),F(x_n,y_n))\\
&\quad + bA(x^*,\cdots,x^*,x_{n+1})\\
&\leqslant (n-1)bkA(F(x^*,y^*),\cdots,F(x^*,y^*),x_n)\\
&\quad + (n-1)blA(F(x_n,y_n),\cdots,F(x_n,y_n),x^*)) + bA(x^*,\cdots,x^*,x_{n+1})\\
&\leqslant (n-1)bkA(F(x^*,y^*),\cdots,F(x^*,y^*),x_n))\\
&\quad + (n-1)blA(x_{n+1},\cdots,x_{n+1},x^*) + b^2A(x_{n+1},\cdots,x_{n+1},x^*)\\
&\leqslant (n-1)^2b^2kA(F(x^*,y^*),\cdots,F(x^*,y^*),x^*)\\
&\quad + (n-1)b^2kA(x_n,\cdots,x_n,x^*)\\
&\quad + ((n-1)bl + b^2)A(x_{n+1},\cdots,x_{n+1},x^*)
\end{aligned}$$

$$\begin{aligned}
\Rightarrow A(F(x^*,y^*),\cdots,F(x^*,y^*),x^*) &\leqslant \frac{(n-1)b^2k}{1-(n-1)^2b^2k}A(x_n,\cdots,x_n,x^*)\\
&\quad + \frac{(n-1)bl + b^2}{1-(n-1)^2b^2k}A(x_{n+1},\cdots,x_{n+1},x^*)
\end{aligned}$$

$$\begin{aligned}\Rightarrow \| A(F(x^*,y^*),\cdots,F(x^*,y^*),x^*)\| \leqslant K[&\frac{(n-1)b^2k}{1-(n-1)^2b^2k}\| A(x_n,\cdots,x_n,x^*)\| \\ &+\frac{(n-1)bl+b^2}{1-(n-1)^2b^2k}\| A(x_{n+1},\cdots,x_{n+1},x^*)\|] \to 0 \text{ as } n\to\infty \\ &\Rightarrow \| A(F(x^*,y^*),\cdots,F(x^*,y^*),x^*)\| = 0 \\ &\Rightarrow A(F(x^*,y^*),\cdots,F(x^*,y^*),x^*) = 0 \\ &\Rightarrow F(x^*,y^*) = x^*.\end{aligned}$$

Similarly, we can get $F(y^*,x^*) = y^*$. Thus (x^*,y^*) is a coupled fixed point of F.

Now if (x',y') is another coupled fixed point of F, then we have

$$\begin{aligned}A(x',\cdots,x',x^*) &= A(F(x',y'),\cdots,F(x',y'),F(x^*,y^*)) \\ &\leqslant kA(F(x',y'),\cdots,F(x',y'),x^*) \\ &\quad + lA(F(x^*,y^*),\cdots,F(x^*,y^*),x') \\ &= kA(x',\cdots,x',x^*) + lA(x^*,\cdots,x^*,x') \\ &\leqslant (k+lb)A(x',\cdots,x',x^*).\end{aligned}$$

Since $k+lb<1$, the above inequality implies $A(x',\cdots,x',x^*) = 0$ and so $x' = x^*$. Similarly, we can get $y' = y^*$. Hence we have $(x',y') = (x^*,y^*)$ which shows that the coupled fixed point of F is unique. □

When $k = l$ in Theorems 3.2, 3.3 and 3.4, we get the following corollaries.

Corollary 3.5. *Let (X, A) be a complete cone A_b-metric space and P be a normal cone with normal constant K. Suppose the mapping $F : X \times X \to X$ satisfies the following contractive condition*

$$A(F(x,y),\cdots,F(x,y),F(u,v)) \leqslant k\left(A(x,\cdots,x,u) + A(y,\cdots,y,v)\right),$$

for all $x,y,u,v \in X$, where $k \in [0,\frac{1}{2b^2})$ is a constant. Then, F has a unique coupled fixed point.

Corollary 3.6. *Let (X, A) be a complete cone A_b-metric space and P be a normal cone with normal constant K. Suppose the mapping $F : X \times X \to X$ satisfies the following contractive condition*

$$A(F(x,y),\cdots,F(x,y),F(u,v)) \leqslant k\left(A(F(x,y),\cdots,F(x,y),x) + A(F(u,v),\cdots,F(u,v),u)\right),$$

for all $x,y,u,v \in X$, where $k \in \left[0,\frac{1}{2(n-1)b^3}\right)$ is a constant. Then, F has a unique coupled fixed point.

Corollary 3.7. *Let (X, A) be a complete cone A_b-metric space and P be a normal cone with normal constant K. Suppose the mapping $F : X \times X \to X$ satisfies the following contractive condition*

$$A(F(x,y),\cdots,F(x,y),F(u,v)) \leqslant k\left(A(F(x,y),\cdots,F(x,y),u) + A(F(u,v),\cdots,F(u,v),x)\right),$$

for all $x,y,u,v \in X$, where $k \in \left[0,\frac{1}{2(n-1)^2b^3}\right)$ is a constant. Then, F has a unique coupled fixed point.

Here we give an example to illustrate Corollary 3.5.

Example 3.8. Let $E = \mathbb{R}$, the Euclidean plane, and $P = \{(x,y) \in E : x,y \geqslant 0\}$, a normal cone in E. Let $X = \mathbb{R}$ and $A : X^n \to E$ such that

$$A(x_1,x_2,\cdots,x_n) = \left(\sum_{i=1}^{n-1} |x_i - x_n|, \sum_{i=1}^{n-1} |x_i - x_n|\right),$$

$x_i \in X,\ i = 1, 2, 3, \cdots, n$. Then A is a cone A_b-metric on X with $b = 1$ and (X, A) is a complete cone A_b-metric space. Let us consider the mapping $F : X \times X \to X$ defined by $F(x, y) = \frac{x+y}{5}$. Then we have

$$\begin{aligned} A(F(x,y),\cdots,F(x,y),F(u,v)) &= A(\frac{x+y}{5},\cdots,\frac{x+y}{5},\frac{u+v}{5}) \\ &= \left((n-1) \mid \frac{x+y}{5} - \frac{u+v}{5} \mid, (n-1) \mid \frac{x+y}{5} - \frac{u+v}{5} \mid\right) \\ &= \frac{(n-1)}{5}(\mid x+y-u-v \mid, \mid x+y-u-v \mid) \\ &\leqslant \frac{(n-1)}{5}((\mid x-u \mid, \mid x-u \mid) + (\mid y-v \mid, \mid y-v \mid)), \end{aligned}$$

and

$$\begin{aligned} A(x,\cdots,x,u) + A(y,\cdots,y,v) &= ((n-1) \mid x-u \mid, (n-1) \mid x-u \mid) \\ &\quad + ((n-1) \mid y-v \mid, (n-1) \mid y-v \mid) \\ &= (n-1)((\mid x-u \mid, \mid x-u \mid) + (\mid y-v \mid, \mid y-v \mid)). \end{aligned}$$

Therefore we have

$$A(F(x,y),\cdots,F(x,y),F(u,v)) \leqslant k(A(x,\cdots,x,u) + A(y,\cdots,y,v)),$$

for $k = \frac{1}{5} \in \left[0, \frac{1}{2b^2}\right) = \left[0, \frac{1}{2}\right)$. Thus the condition of Corollary 3.5 is satisfied and F has a unique coupled fixed point $(0, 0)$.

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