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A (p,q)-analogue of Qi-type formula for r-Dowling numbers

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Abstract

In this paper, (p,q)-analogues of r-Whitney numbers of the first and second kinds are defined using horizontal generating functions. Several fundamental properties such as orthogonality and inverse relations, an explicit formula, and a kind of exponential generating function are obtained. Moreover, a (p,q)-analogue of r-Whitney-Lah numbers is also defined in terms of a horizontal generating function, where necessary properties are obtained. These properties help develop a (p,q)-analogue of the r-Dowling numbers, particularly, a (p,q)-analogue of a Qi-type formula.

Keywords: r-Whitney numbers, r-Whitney-Lah numbers, r-Dowling numbers, generating function, Qi-type formula.

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1. Introduction

The r-Whitney numbers of the first kind and second kind, denoted by $w_{m,r}(n,k)$ and $W_{m,r}(n,k)$, respectively, were defined by Mező [23] as coefficients in the expansion of the following relations:

$$m^n x^{\underline{n}} = \sum_{k=0}^{n} w_{m,r}(n,k)(mx+r)^k \tag{1.1}$$

and

$$(mx+r)^n = \sum_{k=0}^{n} m^k W_{m,r}(n,k) x^{\underline{k}}, \tag{1.2}$$

where $x^{\underline{k}} = x(x-1)(x-2)\cdots(x-k+1)$, $(k=0,1,2,\dots,n)$ denotes the falling factorial of x of order k. Combinatorial properties of these numbers can be seen in [10, 23]. Now, since

$$m^n t^{\underline{n}} = m^n t(t-1)(t-2)\cdots(t-(n-1))$$

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$$= (mt)(mt-m)(mt-2m)\cdots(mt-(n-1)m) = \prod_{j=0}^{n-1}(mt-jm) = (mt|m)_n,$$

then (1.1) and (1.2) can be re-expressed as follows:

$$(mt|m)_n = \sum_{k=0}^{n} w_{m,r}(n,k)(mt+r)^k$$

and

$$(mt+r)^n = \sum_{k=0}^{n} W_{m,r}(n,k)(mt|m)_k, \tag{1.3}$$

where x is replaced with t. The (r,β)-Stirling numbers, denoted by $\left\langle {n \atop m} \right\rangle_{\beta,r}$, were defined by Corcino [8] as

$$x^n = \sum_{k=0}^{n} \left\langle {n \atop m} \right\rangle_{\beta,r} (x-r|\beta)_k,$$

where

$$(x-r|\beta)_k = \prod_{i=0}^{k-1}(x-r-i\beta)$$

is the generalized factorial of $x-r$ of increment β, with r and β may be real or complex parameters. This can be expressed as

$$(\beta t+r)^n = \sum_{k=0}^{n} \left\langle {n \atop m} \right\rangle_{\beta,r} (\beta t|\beta)_k,$$

when x is replaced by $\beta t+r$. Notice that when $m=\beta$ in (1.3), it is clear to see that

$$W_{\beta,r}(n,k) = \left\langle {n \atop k} \right\rangle_{\beta,r}.$$

Other equivalent numbers may be attributed to Ruciński and Voigt [26] and Mangontarum et al. [21].

Now, the term q-analogue refers to a mathematical expression involving a parameter q which generalizes a known identity and reduces back to its classical form as $q \to 1$. The q-integer n

$$[n]_q = \frac{q^n-1}{q-1},$$

the q-factorial of n

$$[n]_q! = \prod_{j=1}^{n}[j]_q,$$

the q-factorial of n of order k

$$[n]_{k,q} = \prod_{j=0}^{k-1}\frac{q^{n-j}-1}{q-1},$$

and the q-binomial coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \prod_{j=1}^{k}\frac{q^{n-j+1}-1}{q^j-1} = \frac{[n]_q!}{[k]_q![n-k]_q!}$$

are q-analogues of the integer n, factorial $n!$, $n^{\underline{k}}$, and $\binom{n}{k}$, respectively, since the following limits hold:

$$\lim_{q\to 1}[n]_q = n, \quad \lim_{q\to 1}[n]_q! = n!, \quad \lim_{q\to 1}[n]_{k,q} = n^{\underline{k}}, \quad \lim_{q\to 1}\begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k}.$$

On the other hand, a natural extension of q-analogue is called (p,q)-analogue which generalizes a known q-analogue and reduces it to the said q-analogue when $p = 1$. The following are examples of (p,q)-analogues:

$$[n]_{p,q} = \frac{p^n - q^n}{p-q}, \quad [n]_{k,p,q} = \prod_{j=0}^{k-1}\frac{p^{n-j}-q^{n-j}}{p-q}, \quad [n]_{p,q}! = \prod_{j=1}^{n}[j]_{p,q}, \tag{1.4}$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \prod_{j=1}^{k}\frac{p^{n-j+1}-q^{n-j+1}}{p^j-q^j}, \tag{1.5}$$

where $p \neq q$. Notice that when $p = 1$, equations (1.4) and (1.5) coincide with the q-analogues presented earlier. That is,

$$[n]_{1,q} = [n]_q, \quad [n]_{k,1,q} = [n]_{k,q}, \quad [n]_{1,q}! = [n]_q!, \quad \begin{bmatrix} n \\ k \end{bmatrix}_{1,q} = \begin{bmatrix} n \\ k \end{bmatrix}_q.$$

The above-listed (p,q)-analogues are referred to as (p,q)-integer n, (p,q)-falling factorial of n of order k, (p,q)-factorial of n and (p,q)-binomial coefficient, respectively. Also, it is verified that the (p,q)-binomial coefficient satisfies

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{k,p,q}}{[k]_{p,q}!} = \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}.$$

It is known that the transitions of a given sequence of numbers into their q and (p,q)-analogues are not unique. For instance, the q-analogues of the classical Stirling numbers are defined using different motivations by Carlitz [2], Gould [18], Cigler [4], and Ehrenborg [17]. For the q-analogues of r-Whitney-type numbers, some notable works are that of Corcino et al. [16], Corcino and Montero [13], Bent-Usman et al. [1], and Mangontarum and Katriel [22]. On the other hand, the (p,q)-analogues of the generalized Stirling numbers by Hsu and Shiue [19] were done separately by Remmel and Wachs [25] and Corcino and Montero [12].

More precisely, Corcino and Montero [12] defined a pair of (p,q)-analogues of the generalized Stirling numbers $\{\sigma^1[n,k]_{pq}, \sigma^2[n,k]_{pq}\}$ in terms of the (p,q)-exponential-type Stirling number pair

$$\{S^1[n,k]_{pq}, S^2[n,k]_{pq}\} = \{S[n,k:\hat{\alpha},\hat{\beta},\hat{\gamma}]_{pq}, S[n,k;\hat{\beta},\hat{\alpha},-\hat{\gamma}]_{pq}\},$$

as follows:

$$\sigma^1[n,k]_{pq} = \sigma^1[n,k;\alpha,\beta,\gamma]_{pq} := S[n,k;\hat{\alpha},\hat{\beta},\hat{\gamma}]_{pq}(p-q)^{k-n},$$
$$\sigma^2[n,k]_{pq} = \sigma^2[n,k;\alpha,\beta,\gamma]_{pq} := S[n,k;\hat{\beta},\hat{\alpha},-\hat{\gamma}]_{pq}(p-q)^{k-n},$$

where the pair $\{S^1[n,k]_{pq}, S^2[n,k]_{pq}\}$ are defined by the relations:

$$[t|\hat{\alpha}]_n^{pq} = \sum_{k=0}^{n} S^1[n,k]_{pq}[t-\hat{\gamma}|\hat{\beta}]_k^{pq}, \quad \text{and} \quad [t|\hat{\beta}]_n^{pq} = \sum_{k=0}^{n} S^2[n,k]_{pq}[t+\hat{\gamma}|\hat{\alpha}]_k^{pq},$$

with α, β and γ may be real or complex numbers and

$$[t|\hat{\alpha}]_n^{pq} = \prod_{j=0}^{n-1}(t-\hat{\alpha}_j), \hat{\alpha}_j = p^{j\alpha} = q^{j\alpha}, \quad \hat{\alpha}_j = \hat{\alpha}, [t|\hat{\alpha}]_0^{pq} = 1, [t|\hat{\alpha}]_1^{pq} = t,$$

the (p,q)-exponential factorial of t with power α. Several properties and some combinatorial interpretation in the context of A-tableau of this (p,q)-analogue were established in [12].

The results of this paper are organized as follows. In section 2, we will define (p,q)-analogues of the r-Whitney numbers of the first and second kinds in terms of horizontal geenrating functions and obtain combinatorial properties, some of which are important in the succeeding sections. In section 3, we will define a (p,q)-analogue of r-Whitney-Lah numbers which can be expressed as sum of products of the (p,q)-analogues in Section 2, and in section 4, we will define a (p,q)-analogue of the r-Dowling numbers as the sum of the (p,q)-analogue of the r-Whitney numbers of the second kind. Using this notion, we will establish a (p,q)-analogue of a Qi-type formula.

2. A (p,q)-analogue of r-Whitney numbers

Definition 2.1. For real numbers m and r, where $m \neq 0$, the (p,q)-analogues of r-Whitney numbers of the first and second kind, denoted by $W_{m,r}[n,k]_{p,q}$ and $w_{m,r}[n,k]_{p,q}$, respectively, are defined by the following relations:

$$[mt|m]^{n}_{p,q} = \sum_{k=0}^{n} w_{m,r}[n,k]_{p,q}[mt+r]^{k}_{p,q}, \tag{2.1}$$

$$[mt+r]^{n}_{p,q} = \sum_{k=0}^{n} W_{m,r}[n,k]_{p,q}[mt|m]^{k}_{p,q}, \tag{2.2}$$

where

$$[t|m]^{n}_{p,q} = \prod_{j=0}^{n-1}[t-jm]_{p,q}.$$

Substituting (2.1) to (2.2) gives

$$\begin{aligned}[mt+r]^{n}_{p,q} &= \sum_{k=0}^{n} W_{m,r}[n,k]_{p,q} \sum_{j=0}^{k} w_{m,r}[k,j]_{p,q}[mt+r]^{j}_{p,q} \\ &= \sum_{k=0}^{n}\sum_{j=0}^{k} W_{m,r}[n,k]_{p,q} w_{m,r}[k,j]_{p,q}[mt+r]^{j}_{p,q}.\end{aligned}$$

Re-indexing the sums yield

$$[mt+r]^{n}_{p,q} = \sum_{j=0}^{n}\left\{\sum_{k=j}^{n} W_{m,r}[n,k]_{p,q} w_{m,r}[k,j]_{p,q}\right\}[mt+r]^{j}_{p,q}.$$

Comparing the coefficients of $[mt+r]^{j}_{p,q}$,

$$\sum_{k=j}^{n} W_{m,r}[n,k]_{p,q} w_{m,r}[k,j]_{p,q} = 1,$$

when $j=n$ and

$$\sum_{k=j}^{n} W_{m,r}[n,k]_{p,q} w_{m,r}[k,j]_{p,q} = 0,$$

otherwise. On the other hand, substituting (2.2) to (2.1) gives

$$[mt|m]^{n}_{p,q} = \sum_{k=0}^{n} w_{m,r}[n,k]_{p,q} \sum_{j=0}^{k} W_{m,r}[k,j]_{p,q}[mt|m]^{j}_{p,q}$$

$$= \sum_{k=0}^{n}\sum_{j=0}^{k} w_{m,r}[n,k]_{p,q} W_{m,r}[k,j]_{p,q} [mt|m]^{j}_{p,q}.$$

Re-indexing the sums yield

$$[mt|m]^{n}_{p,q} = \sum_{j=0}^{n}\left\{\sum_{k=j}^{n} w_{m,r}[n,k]_{p,q} W_{m,r}[k,j]_{p,q}\right\}[mt|m]^{j}_{p,q}.$$

Comparing the coefficients of $[mt|m]^{j}_{p,q}$,

$$\sum_{k=j}^{n} w_{m,r}[n,k]_{p,q} W_{m,r}[k,j]_{p,q} = 1,$$

when $j = n$ and

$$\sum_{k=j}^{n} w_{m,r}[n,k]_{p,q} W_{m,r}[k,j]_{p,q} = 0,$$

otherwise. Thus, we obtain the following theorem.

Theorem 2.2. *The (p,q)-analogues of the r-Whitney numbers of the first and second kind satisfy the following orthogonality relations:*

$$\sum_{k=j}^{n} W_{m,r}[n,k]_{p,q} w_{m,r}[k,j]_{p,q} = \delta_{j,n} \tag{2.3}$$

and

$$\sum_{k=j}^{n} w_{m,r}[n,k]_{p,q} W_{m,r}[k,j]_{p,q} = \delta_{j,n}. \tag{2.4}$$

The next theorem contains the inverse relations for $w_{m,r}[n,k]_{p.q}$ and $W_{m,r}[n,k]_{p.q}$ which can be proven using the orthogonality relations above.

Theorem 2.3. *The inverse relations of the (p,q)-analogues of the r-Whitney numbers of the first and second kinds are given by the following:*

$$f_n = \sum_{k=0}^{n} w_{m,r}[n,k]_{p,q} g_k \iff g_n = \sum_{k=0}^{n} W_{m,r}[n,k]_{p,q} f_k \tag{2.5}$$

and

$$f_k = \sum_{n=k}^{\infty} w_{m,r}[n,k]_{p,q} g_n \iff g_k = \sum_{n=k}^{\infty} W_{m,r}[n,k]_{p,q} f_n. \tag{2.6}$$

Proof. To prove (2.5), suppose that

$$f_n = \sum_{k=0}^{n} w_{m,r}[n,k]_{p,q} g_k.$$

Then,

$$\sum_{k=0}^{n} W_{m,r}[n,k]_{p,q} f_k = \sum_{k=0}^{n} W_{m,r}[n,k]_{p,q} \sum_{j=0}^{k} w_{m,r}[k,j]_{p,q} g_j = \sum_{j=0}^{n}\left\{\sum_{k=j}^{n} W_{m,r}[n,k]_{p,q} w_{m,r}[k,j]_{p,q}\right\} g_j.$$

Applying the orthogonality relation in (2.4) in this equation,

$$\sum_{k=0}^{n} W_{m,r}[n,k]_{p,q} f_k = \sum_{j=0}^{n} \{\delta_{j,n}\} g_j.$$

Note that $\delta_{j,n} = 0$ for $n = 0, 1, 2, \ldots, n-1$, while $\delta_{n,n} = 1$. Hence,

$$\sum_{k=0}^{n} W_{m,r}[n,k]_{p,q} f_k = \delta_{n,n} g_n = g_n.$$

Conversely, suppose

$$g_n = \sum_{k=0}^{n} W_{m,r}[n,k]_{p,q} f_k.$$

Then,

$$\sum_{k=0}^{n} w_{m,r}[n,k]_{p,q} g_k = \sum_{k=0}^{n} w_{m,r}[n,k]_{p,q} \sum_{j=0}^{k} W_{m,r}[k,j]_{p,q} f_j = \sum_{j=0}^{n} \left\{ \sum_{k=j}^{n} w_{m,r}[n,k]_{p,q} W_{m,r}[k,j]_{p,q} \right\} f_j.$$

By the orthogonality relation in (2.3) and the definition of the Kronecker's delta $\delta_{j,n}$,

$$\sum_{k=0}^{n} w_{m,r}[n,k]_{p,q} g_k = \sum_{j=0}^{n} \{\delta_{j,n}\} f_j = \delta_{n,n} f_n = f_n.$$

Now, to prove (2.6), suppose that

$$f_k = \sum_{n=k}^{\infty} w_{m,r}[n,k]_{p,q} g_n$$

holds. Then combining this with the defining relation in (2.2),

$$\begin{aligned}
\sum_{k=0}^{\infty} \left\{ \sum_{n=k}^{\infty} W_{m,r}[n,k]_{p,q} f_n \right\} [mt|m]^{\underline{k}}_{p,q} &= \sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{n} W_{m,r}[n,k]_{p,q} [mt|m]^{\underline{k}}_{p,q} \right\} f_n \\
&= \sum_{n=0}^{\infty} \left\{ [mt+r]^{n}_{p,q} \right\} f_n = \sum_{n=0}^{\infty} \left\{ \sum_{j=n}^{\infty} w_{m,r}[j,n]_{p,q} g_j \right\} [mt+r]^{n}_{p,q}.
\end{aligned}$$

Re-indexing the sums and again using (2.2) gives

$$\begin{aligned}
\sum_{k=0}^{\infty} \left\{ \sum_{n=k}^{\infty} W_{m,r}[n,k]_{p,q} f_n \right\} [mt|m]^{\underline{k}}_{p,q} &= \sum_{j=0}^{\infty} \left\{ \sum_{n=0}^{j} w_{m,r}[j,n]_{p,q} [mt+r]^{n}_{p,q} \right\} g_j \\
&= \sum_{j=0}^{\infty} \left\{ \sum_{n=0}^{j} w_{m,r}[j,n]_{p,q} \sum_{k=0}^{n} W_{m,r}[n,k]_{p,q} [mt|m]^{\underline{k}}_{p,q} \right\} g_j \\
&= \sum_{j=0}^{\infty} \left\{ \sum_{k=0}^{j} \left\{ \sum_{n=k}^{j} w_{m,r}[j,n]_{p,q} W_{m,r}[n,k]_{p,q} \right\} [mt|m]^{\underline{k}}_{p,q} \right\} g_j.
\end{aligned}$$

Applying the orthogonality relation in (2.3) to this equation yields

$$\begin{aligned}\sum_{k=0}^{\infty}\left\{\sum_{n=k}^{\infty} W_{m,r}[n,k]_{p,q} f_n\right\}[mt|m]^{\underline{k}}_{p,q} &= \sum_{j=0}^{\infty}\left\{\sum_{k=0}^{j}\{\delta_{k,j}\}[mt|m]^{\underline{k}}_{p,q}\right\} g_j\\ &= \sum_{k=0}^{\infty}\left\{\sum_{j=k}^{\infty}\delta_{k,j} g_j\right\}[mt|m]^{\underline{k}}_{p,q}\\ &= \sum_{k=0}^{\infty}\{\delta_{k,k} g_k+\delta_{k,k+1}g_{k+1}+\cdots\}[mt|m]^{\underline{k}}_{p,q}\\ &= \sum_{k=0}^{\infty}\{g_k\}[mt|m]^{\underline{k}}_{p,q}.\end{aligned}$$

By comparing the coefficients of $[mt|m]^{\underline{k}}_{p,q}$,

$$\sum_{n=k}^{\infty} W_{m,r}[n,k]_{p,q} f_n = g_k.$$

Conversely, suppose that

$$g_k = \sum_{n=k}^{\infty} W_{m,r}[n,k]_{p,q} f_n.$$

Then combining this with the defining relation in (2.1),

$$\begin{aligned}\sum_{k=0}^{\infty}\left\{\sum_{n=k}^{\infty} w_{m,r}[n,k]_{p,q} g_n\right\}[mt+r]^{k}_{p,q} &= \sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n} w_{m,r}[n,k]_{p,q}[mt+r]^{k}_{p,q}\right\} g_n\\ &= \sum_{n=0}^{\infty}\left\{[mt|m]^{\underline{n}}_{p,q}\right\} g_n\\ &= \sum_{n=0}^{\infty}\left\{\sum_{j=n}^{\infty} W_{m,r}[j,n]_{p,q} f_j\right\}[mt|m]^{\underline{n}}_{p,q}.\end{aligned}$$

Re-indexing the sums and again using (2.1) gives

$$\begin{aligned}\sum_{k=0}^{\infty}\left\{\sum_{n=k}^{\infty} w_{m,r}[n,k]_{p,q} g_n\right\}[mt+r]^{k}_{p,q} &= \sum_{j=0}^{\infty}\left\{\sum_{n=0}^{j} W_{m,r}[j,n]_{p,q}[mt|m]^{\underline{n}}_{p,q}\right\} f_j\\ &= \sum_{j=0}^{\infty}\left\{\sum_{n=0}^{j} W_{m,r}[j,n]_{p,q}\sum_{k=0}^{n} w_{m,r}[n,k]_{p,q}[mt+r]^{k}_{p,q}\right\} f_j\\ &= \sum_{j=0}^{\infty}\left\{\sum_{k=0}^{j}\left\{\sum_{n=k}^{j} W_{m,r}[j,n]_{p,q} w_{m,r}[n,k]_{p,q}\right\}[mt+r]^{k}_{p,q}\right\} f_j.\end{aligned}$$

By the orthogonality relation in (2.4),

$$\sum_{k=0}^{\infty}\left\{\sum_{n=k}^{\infty} w_{m,r}[n,k]_{p,q} g_n\right\}[mt+r]^{k}_{p,q} = \sum_{j=0}^{\infty}\left\{\sum_{k=0}^{j}\{\delta_{k,j}\}[mt+r]^{k}_{p,q}\right\} f_j$$

$$= \sum_{k=0}^{\infty} \left\{ \sum_{j=k}^{\infty} \delta_{k,j} f_j \right\} [mt+r]_{p,q}^k$$
$$= \sum_{k=0}^{\infty} \{\delta_{k,k} f_k + \delta_{k,k+1} f_{k+1} + \cdots\} [mt+r]_{p,q}^k$$
$$= \sum_{k=0}^{\infty} \{f_k\} [mt+r]_{p,q}^k.$$

By comparing the coefficients of $[mt+r]_{p,q}^k$,

$$\sum_{n=k}^{\infty} w_{m,r}[n,k]_{p,q} g_n = f_k.$$

This completes the proof. □

The next theorem contains an explicit formula for $W_{m,r}[j,n]_{p,q}$ which is analogous to [13, Equation P4]. This can easily be derived using the inverse relation for the (p,q)-binomial coefficients in [9].

Theorem 2.4. *For nonnegative integers* n *and* k*, and real numbers* m*, and* r*, the* (p,q)*-analogue* $W_{m,r}[n,k]_{p,q}$ *is equal to*

$$W_{m,r}[n,k]_{p,q} = \frac{1}{[mk|m]_{p,q}^{\underline{k}}} \sum_{j=0}^{k} (-1)^{k-j} q^{\binom{k-j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_{p,q} [mj+r]_{p,q}^n. \tag{2.7}$$

Proof. By taking $t=k$, Theorem 2.2 gives

$$[mk+r]_{p,q}^n = \sum_{j=0}^{n} W_{m,r}[n,j]_{p,q} [mk|m]_{p,q}^{\underline{j}} = \sum_{j=0}^{k} p^{\binom{k-j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_{q,p} \left\{ \frac{W_{m,r}[n,j]_{p,q} [mk|m]_{p,q}^{\underline{j}}}{p^{\binom{k-j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_{q,p}} \right\}.$$

Applying the inverse relation for (p,q)-binomial coefficients [9] and using the fact that

$$\begin{bmatrix} k \\ j \end{bmatrix}_{q,p} = \begin{bmatrix} k \\ j \end{bmatrix}_{p,q},$$

we obtain the desired result. □

Now, using the explicit formula in Theorem 2.4 and the Cauchy's formula for the product of two power series [6], we get

$$\sum_{n\geqslant 0} W_{m,r}[n,k]_{p,q} \frac{t^n}{n!} = \sum_{n\geqslant 0} \frac{1}{[mk|m]_{p,q}^{\underline{k}}} \sum_{j=0}^{k} (-1)^{k-j} q^{\binom{k-j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_{p,q} [mj+r]_{p,q}^n \frac{t^n}{n!}$$
$$= \frac{1}{[mk|m]_{p,q}^{\underline{k}}} \sum_{j=0}^{k} (-1)^{k-j} q^{\binom{k-j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_{p,q} \sum_{n\geqslant 0} \frac{([mj+r]_{p,q} t)^n}{n!}$$
$$= \frac{1}{[mk|m]_{p,q}^{\underline{k}}} \sum_{j=0}^{k} (-1)^{k-j} q^{\binom{k-j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_{p,q} e^{[mj+r]_{p,q} t}.$$

This is exactly the exponential generating function in the next theorem.

Theorem 2.5. *For nonnegative integers n and k, and real numbers m, and r, the (p,q)-analogue $W_{m,r}[n,k]_{p,q}$ satisfies the following exponential generating function*

$$\sum_{n\geqslant 0} W_{m,r}[n,k]_{p,q}\frac{t^n}{n!} = \frac{1}{[mk|m]^{\underline{k}}_{p,q}}\sum_{j=0}^{k}(-1)^{k-j}q^{\binom{k-j}{2}}\begin{bmatrix}k\\j\end{bmatrix}_{p,q} e^{[mj+r]_{p,q}t}. \tag{2.8}$$

For a more compact form of (2.7) and (2.8), let us consider the following (p,q)-difference operator which is known to be an extension of the q-difference operator of Conrad [7].

The (p,q)-difference operator of degree n, denoted by $\Delta^n_{p,q}$, is a mapping that assigns to every function f the function $\Delta^n_{pq} f$ defined by the rule

$$\Delta^n_{pq} f(x) = \left[\prod_{j=0}^{n-1}(p^jE - q^j)\right] f(x), \qquad n \geqslant 1,$$

where E is the shift operator defined by the rule $Ef(x) = f(x+1)$. As convention, define $\Delta^0_{pq} = 1$ (the identity map, [14, Definition 2.1]).

Note that the q-difference operator of degree n Δ^n_q in [7] can be obtained from Δ^n_{pq} by setting $p = 1$, which further gives the difference operator Δ^n when q tends to 1.

By induction on n and using the triangular recurrence relation of the (p,q)-binomial coefficients in [9], we can easily obtain the following formula

$$\Delta^n_{pq} f(x) = \sum_{k=0}^{n}(-1)^k p^{\binom{n-k}{2}} q^{\binom{k}{2}}\begin{bmatrix}n\\k\end{bmatrix}_{p,q} f(x+n-k). \tag{2.9}$$

Applying (2.9) to the function f_1 and f_2 defined by

$$f_1(x) = \frac{[mx+r]^n_{p,q}}{p^{\binom{x}{2}}[mk|m]^{\underline{k}}_{p,q}} \quad\text{and}\quad f_2(x) = \frac{e^{[mx+r]_{p,q}t}}{p^{\binom{x}{2}}[mk|m]^{\underline{k}}_{p,q}},$$

we can, respectively, express (2.7) and (2.8) in compact form as follows.

Remark 2.6. The (p,q)-analogue of $W_{m,r}[n,k]_{p,q}$ equals to

$$W_{m,r}[n,k]_{p,q} = \left[\Delta^k_{pq}\left(\frac{[mx+r]^n_{p,q}}{p^{\binom{x}{2}}[mk|m]^{\underline{k}}_{p,q}}\right)\right]_{x=0}$$

and has the exponential generating function

$$\sum_{n\geqslant 0} W_{m,r}[n,k]_{p,q}\frac{t^n}{n!} = \left[\Delta^k_{pq}\left(\frac{e^{[mx+r]_{p,q}t}}{p^{\binom{x}{2}}[mk|m]^{\underline{k}}_{p,q}}\right)\right]_{x=0}. \tag{2.10}$$

Remark 2.7. Using the inverse relation in (2.6), the exponential generating function in (2.10) gives

$$\sum_{n\geqslant 0} w_{m,r}[n,k]_{p,q}\frac{k!}{t^k}\left[\Delta^n_{pq}\left(\frac{e^{[mx+r]_{p,q}t}}{p^{\binom{x}{2}}[mn|m]^{\underline{n}}_{p,q}}\right)\right]_{x=0} = 1.$$

3. A (p,q)-analogue of r-Whitney-Lah numbers

In this section, a (p,q)-analogue of r-Whitney-Lah numbers , denoted by $L_{m,r}[n,k]_{p,q}$, will be defined and some necessary properties will be derived, which are useful in establishing a Qi-type formula for the (p,q)-analogue of r-Dowling numbers.

Definition 3.1. The (p,q)-analogue of r-Whitney-Lah numbers is defined by

$$[mt+r|m]^{\underline{n}}_{p,q} = \sum_{k=0}^{n} L_{m,r}[n,k]_{p,q}[mt-r|m]^{\underline{k}}_{p,q}. \tag{3.1}$$

Notice that the horizontal generating function in (3.1) can be written as

$$[mt+2r|m]^{\underline{n}}_{p,q} = \sum_{k=0}^{n} L_{m,r}[n,k]_{p,q}[mt|m]^{\underline{k}}_{p,q}.$$

Theorem 3.2. *The (p,q)-analogue of r-Whitney-Lah numbers satisfies the following explicit formula:*

$$L_{m,r}[n,k]_{p,q} = \frac{1}{[mk-r|m]^{\underline{k}}_{p,q}} \sum_{j=0}^{k} (-1)^{k-j} q^{\binom{k-j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_{p,q} [mj+r|m]^{\underline{k}}_{p,q}. \tag{3.2}$$

Proof. We will rewrite (3.1) as

$$[mk+r|m]^{\underline{n}}_{p,q} = \sum_{j=0}^{k} L_{m,r}[n,k]_{p,q}[mk-r|m]^{\underline{j}}_{p,q} = \sum_{j=0}^{k} p^{\binom{k-j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_{p,q} \left\{ \frac{L_{m,r}[n,k]_{p,q}[mk-r|m]^{\underline{j}}_{p,q}}{p^{\binom{k-j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_{p,q}} \right\}.$$

Applying the (p,q)-binomial inversion and using the fact that

$$\begin{bmatrix} k \\ j \end{bmatrix}_{p,q} = \begin{bmatrix} k \\ j \end{bmatrix}_{q,p},$$

then with $f_k = [mk+r|m]^{\underline{n}}_{p,q}$ and $g_j = \dfrac{L_{m,r}[n,k]_{p,q}[mk-r|m]^{\underline{j}}_{p,q}}{p^{\binom{k-j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_{p,q}}$, we get

$$L_{m,r}[n,k]_{p,q}[mk-r|m]^{\underline{j}}_{p,q} = \sum_{j=0}^{k} (-1)^{k-j} q^{\binom{k-j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_{p,q} [mj+r|m]^{\underline{k}}_{p,q},$$

which gives the desired result. □

Theorem 3.3. *The (p,q)-analogue of r-Whitney-Lah numbers satisfies the following exponential generating function*

$$\sum_{n\geqslant 0} L_{m,r}[n,k]_{p,q} \frac{t^n}{n!} = \frac{1}{[mk-r|m]^{\underline{k}}_{p,q}} \sum_{j=0}^{k} (-1)^{k-j} q^{\binom{k-j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_{p,q} e^{[mj+r|m]_{p,q} t}. \tag{3.3}$$

Proof. Using equation (3.2) and the Cauchy's formula for the product of two power series [6], we get

$$\begin{aligned} \sum_{n\geqslant 0} L_{m,r}[n,k]_{p,q} \frac{t^n}{n!} &= \sum_{n\geqslant 0} \frac{1}{[mk-r|m]^{\underline{k}}_{p,q}} \sum_{j=0}^{k} (-1)^{k-j} q^{\binom{k-j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_{p,q} [mj+r|m]^{\underline{k}}_{p,q} \frac{t^n}{n!} \\ &= \frac{1}{[mk-r|m]^{\underline{k}}_{p,q}} \sum_{j=0}^{k} (-1)^{k-j} q^{\binom{k-j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_{p,q} \sum_{n\geqslant 0} \frac{([mj+r|m]^{\underline{k}}_{p,q} t)^n}{n!} \\ &= \frac{1}{[mk-r|m]^{\underline{k}}_{p,q}} \sum_{j=0}^{k} (-1)^{k-j} q^{\binom{k-j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_{p,q} e^{[mj+r|m]_{p,q} t}. \end{aligned}$$

This is exactly equation (3.3). □

In the next theorem, we will express the (p,q)-analogue of the r-Whitney-Lah numbers as sum of products of the (p,q)-analogues of the r-Whitney numbers of the first and second kinds.

Theorem 3.4. *The* (p,q)*-analogue of* r*-Whitney-Lah numbers satisfies the following generating function*

$$L_{m,r}[n,k]_{p,q} = \sum_{j=k}^{n} w_{m,-r}[n,j]_{p,q} W_{m,r}[j,k]_{p,q}. \tag{3.4}$$

Proof. Using Definitions 3.1 and 2.1, we have

$$\begin{aligned}\sum_{k=0}^{n} L_{m,r}[n,k]_{p,q}[mt-r|m]^{\underline{k}}_{p,q} = [mt+r|m]^{\underline{n}}_{p,q} &= \sum_{j=0}^{n} w_{m,-r}[n,j]_{p,q}[mt]^{j}_{p,q} \\ &= \sum_{j=0}^{n} w_{m,-r}[n,j]_{p,q}\left\{\sum_{k=0}^{j} W_{m,r}[j,k]_{p,q}[mt-r|m]^{\underline{k}}_{p,q}\right\} \\ &= \sum_{k=0}^{n}\left\{\sum_{j=k}^{n} w_{m,-r}[n,j]_{p,q}W_{m,r}[j,k]_{p,q}\right\}[mt-r|m]^{\underline{k}}_{p,q}.\end{aligned}$$

Comparing the coefficients of $[mt-r|m]^{\underline{k}}_{p,q}$ yields (3.4). □

4. A (p,q)-analogue of r-Dowling numbers

The r-Dowling polynomials $D_{m,r}(n,x)$ of Cheon and Jung [3] defined by

$$D_{m,r}(n,x) = \sum_{k=0}^{n} W_{m,r}(n,k)x^k,$$

would consequently yield the r-Dowling numbers, denoted by $D_{m,r}(n)$, when $x=1$. That is,

$$D_{m,r}(n) = D_{m,r}(n,1) = \sum_{k=0}^{n} W_{m,r}(n,k).$$

A q-analogue of r-Dowling numbers has been introduced and investigated in [11, 15] in three forms, namely,

$$D_{m,r}[n]_q := \sum_{k=0}^{n} W_{m,r}[n,k]_q, \qquad D^*_{m,r}[n]_q := \sum_{k=0}^{n} W^*_{m,r}[n,k]_q, \qquad \widetilde{D}_{m,r}[n]_q := \sum_{k=0}^{n} \widetilde{W}_{m,r}[n,k]_q,$$

where $W_{m,r}[n,k]_q$ and

$$\begin{aligned}W^*_{m,r}[n,k]_q &:= q^{-kr-m\binom{k}{2}} W_{m,r}[n,k]_q, \\ \widetilde{W}_{m,r}[n,k]_q &:= q^{kr} W^*_{m,r}[n,k]_q = q^{-m\binom{k}{2}} W_{m,r}[n,k]_q,\end{aligned}$$

denote the first, second and third forms of the q-analogue of r-Whitney numbers of the second kind, respectively, with

$$W_{m,r}[n,k]_q = q^{m(k-1)-r} W_{m,r}[n-1,k-1]_q + [mk-r]_q W_{m,r}[n-1,k]_q,$$

(see [11, 15, 16]). Now, we define a (p,q)-analogue of r-Dowling numbers as follows.

Definition 4.1. A (p, q)-analogue of r-Dowling numbers, denoted by $D_{m,r}[n]_{p,q}$, is defined by

$$D_{m,r}[n]_{p,q} = \sum_{k=0}^{n} W_{m,r}[n,k]_{p,q}. \tag{4.1}$$

Using the explicit formula in Theorem 2.4, we have

$$\begin{aligned}
D_{m,r}[n]_{p,q} = \sum_{k=0}^{n} W_{m,r}[n,k]_{p,q} &= \sum_{k=0}^{\infty} \left(\frac{1}{[mk|m]^{\underline{k}}_{p,q}} \sum_{j=0}^{k} (-1)^{k-j} q^{\binom{k-j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_{p,q} [mj+r]^{n}_{p,q} \right) \\
&= \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{(-1)^{k-j} q^{\binom{k-j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_{p,q} [mj+r]^{n}_{p,q}}{[mk|m]^{\underline{k}}_{p,q}} \\
&= \sum_{j=0}^{\infty} \sum_{k=j}^{\infty} \frac{(-1)^{k-j} q^{\binom{k-j}{2}} [k]_{p,q}! [mj+r]^{n}_{p,q}}{[mk|m]^{\underline{k}}_{p,q} [j]_{p,q}! [k-j]_{p,q}!}.
\end{aligned}$$

Replacing $k-j$ with i gives

$$D_{m,r}[n]_{p,q} = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \frac{(-1)^{i} q^{\binom{i}{2}} [i+j]_{p,q}! [mj+r]^{n}_{p,q}}{[m(i+j)|m]^{\underline{k}}_{p,q} [j]_{p,q}! [i]_{p,q}!},$$

which is exactly equation (4.2) in the next theorem.

Theorem 4.2. *The (p, q)-analogue of r-Dowling numbers $D_{m,r}[n]_{p,q}$ satisfy the following*

$$D_{m,r}[n]_{p,q} = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \frac{(-1)^{i} q^{\binom{i}{2}} [i+j]_{p,q}! [mj+r]^{n}_{p,q}}{[j]_{p,q}! [i]_{p,q}! [m(j+1)|m]^{\underline{n}}_{p,q}}. \tag{4.2}$$

The following theorem contains the exponential generating function of The (p, q)-analogue of r-Dowling number.

Theorem 4.3. *The (p, q)-analogue of r-Dowling numbers $D_{m,r}[n]_{p,q}$ satisfy the following exponential generating function*

$$\sum_{n=0}^{\infty} D_{m,r}[n]_{p,q} \frac{t^n}{n!} = \sum_{k=0}^{\infty} \left\{ \Delta^{k}_{p,q} \left(\frac{e^{[mx+r]_{p,q} t}}{p^{\binom{x}{2}} [mk|m]^{\underline{k}}_{p,q}} \right) \right\}. \tag{4.3}$$

Proof. Multiplying both sides of equation (4.1) with $\frac{t^n}{n!}$ and summing over n gives

$$\sum_{n=0}^{\infty} D_{m,r}[n]_{p,q} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{n} W_{m,r}[n,k]_{p,q} \right\} \frac{t^n}{n!} = \sum_{k=0}^{\infty} \left\{ \sum_{n=0}^{\infty} W_{m,r}[n,k]_{p,q} \frac{t^n}{n!} \right\}.$$

Applying the exponential generating function of the (p, q)-analogue of r-Whitney numbers of the second kind in Remark 2.6 yields the desired exponential generating function in (4.3). □

Now, Qi [24] obtained an explicit formula for the Bell numbers expressed in terms of both the Lah numbers and the Stirling numbers of the second kind by

$$B_n = \sum_{k=0}^{n} (-1)^{n-k} \left\{ \sum_{j=0}^{k} L(k,j) \right\} S(n,k). \tag{4.4}$$

We call (4.4) the Qi formula for the Bell numbers. Qi provided a proof of (4.4) that requires the inverse relation between the Stirling numbers of the first and the second kinds

$$f_n = \sum_{k=0}^{n} S(n,k)g_k \Leftrightarrow g_n = \sum_{k=0}^{n} s(n,k)f_k$$

and the identity for the Lah numbers expressed as the sum of the product of the Stirling numbers of the first and the second kind given by

$$L(n,k) = \sum_{j=k}^{n} (-1)^j s(n,j)S(j,k).$$

Other similar works can be seen in [5, 20].

To obtain the next result, we adopt a process similar with Qi's work.

Theorem 4.4. *The explicit formula for the (p,q)-analogue of r-Dowling numbers $D_{m,r}[n]_{p,q}$ is given by*

$$D_{m,r}[n]_{p,q} = \sum_{k=0}^{n} W_{m,-r}[n,k]_{p,q} \left(\sum_{j=0}^{k} L_{m,r}[k,j]_{p,q} \right).$$

Proof. Using the inverse relation (2.5) with

$$f_n = L_{m,r}[n,j]_{p,q}, \quad g_k = W_{m,r}[k,j]_{p,q},$$

relation in Theorem 3.4 can be transformed as

$$W_{m,r}[n,j]_{p,q} = \sum_{k=j}^{n} W_{m,-r}[n,k]_{p,q} L_{m,r}[k,j]_{p,q}.$$

Then summing up both sides over j yields

$$\begin{aligned} D_{m,r}[n]_{p,q} = \sum_{j=0}^{n} W_{m,r}[n,j]_{p,q} &= \sum_{j=0}^{n} \sum_{k=j}^{n} W_{m,-r}[n,k]_{p,q} L_{m,r}[k,j]_{p,q} \\ &= \sum_{k=0}^{n} W_{m,-r}[n,k]_{p,q} \left(\sum_{j=0}^{k} L_{m,r}[k,j]_{p,q} \right), \end{aligned}$$

which is exactly the desired explicit formula for the (p,q)-analogue of r-Dowling numbers. □

The above explicit formula may also be called a Qi-type formula for the (p,q)-analogue of r-Dowling numbers, which is analogous to explicit formula obtained by Qi [24].

5. Conclusion

This paper have established (p,q)-analogues of r-Whitney and r-Whitney-Lah numbers and obtained some necessary properties which are useful in developing a (p,q)-analogue of r-Dowling numbers, particularly, a (p,q)-analogue of the explicit formula in [24] known as Qi-type formula. Parallel to the Hankel transform of (q,r)-Dowling numbers, it would also be interesting to establish the Hankel transform of (p,q)-analogue of r-Dowling numbers.

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