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Singular value inequalities with applications

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Abstract

Let A_i, B_i, X_i, Y_i be $n \times n$ complex matrices, $i = 1, 2, ..., m$ and let f be a nonnegative increasing convex function on an interval I such that $0 \in I$ and $f(0) \leqslant 0$. Then

$$2s_j\left(f\left(\left|\sum_{i=1}^{m} A_i X_i Y_i^* B_i^*\right|\right)\right) \leqslant (\max\{S,T\})^2 s_j(K)$$

for $j = 1, 2, ..., n$, where

$$S = \left\|\sum_{i=1}^{m} A_i A_i^*\right\|^{1/2}, T = \left\|\sum_{i=1}^{m} B_i B_i^*\right\|^{1/2},$$

$$K = f(|X_1|^2 + |Y_1|^2) \oplus ... \oplus f(|X_m|^2 + |Y_m|^2)$$

and $\max\{S, T\} \leqslant 1$. Several singular value inequalities are also proved.

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1. Introduction

Let $\mathbb{M}_n(\mathbb{C})$ denote the algebra of all $n \times n$ complex matrices. For $A \in \mathbb{M}_n(\mathbb{C})$, the singular values of A are denoted by $s_1(A) \geqslant s_2(A) \geqslant ... \geqslant s_n(A)$, they are precisely the eigenvalues of the positive matrix $|A| = (A^*A)^{1/2}$, Singular values have several properties: Let $A, B \in \mathbb{M}_n(\mathbb{C})$. Then

(a) $s_j(A) = s_j(A^*) = s_j(|A|)$ for $j = 1, 2, ..., n$.

(b) $s_j(AA^*) = s_j(A^*A)$ for $j = 1, 2, ..., n$.

(c) $s_j(A) \leqslant s_j(B)$ if and only if $s_j(A \oplus A) \leqslant s_j(B \oplus B)$ for $j = 1, 2, ..., n$.

(d) $s_j\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = s_j\begin{bmatrix} 0 & B \\ A & 0 \end{bmatrix}$ for $j = 1, 2, ..., n$, where the singular values of $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ consist of these of A together with those of B.

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(Extensive studies on singular values are available in [1, 8, 12]).

Bhatia and Kittaneh in [9] proved the A.G.M.I. for singular values: If $A, B \in \mathbb{M}_n(\mathbb{C})$, then

$$2s_j(AB^*) \leqslant s_j(A^*A + B^*B) \tag{1.1}$$

for $j = 1, 2, ..., n$. Hirzallah in [11] gave a generalization of inequality (1.1): If $A_i \in \mathbb{B}(\mathbb{H})$, $i = 1, 2, 3, 4$, then

$$\sqrt{2}s_j\left(|A_1A_2^* + A_3A_4^*|^{1/2}\right) \leqslant s_j\left(\begin{bmatrix} A_1 & A_3 \\ A_2 & A_4 \end{bmatrix}\right) \tag{1.2}$$

for $j = 1, 2, ..., 2n$. Audeh in [3] created a generalization of inequality (1.2): Let $A_i, B_i, X_i, Y_i \in \mathbb{M}_n(\mathbb{C})$ such that X_i and Y_i are positive, $i = 1, 2, ..., n$. Then

$$2s_j\left(\sum_{i=1}^{n} A_iX_i^{1/2}Y_i^{1/2}B_i^*\right) \leqslant s_j^2(W)$$

for $j = 1, 2, ..., n$, where $W = \begin{bmatrix} A_1X_1^{1/2} & A_2X_2^{1/2} & ... & A_nX_n^{1/2} \\ B_1Y_1^{1/2} & B_2Y_2^{1/2} & ... & B_nY_n^{1/2} \end{bmatrix}$. This generalization contains two attractive special cases. The first one: Let $A, B, X \in \mathbb{B}(\mathbb{H})$ such that X is positive. Then

$$2s_j\left(AX^{1/2}Y^{1/2}B^* + BX^{1/2}Y^{1/2}A^*\right) \leqslant s_j^2\left(\begin{bmatrix} AX^{1/2} & BX^{1/2} \\ BY^{1/2} & AY^{1/2} \end{bmatrix}\right) \tag{1.3}$$

for $j = 1, 2, ..., n$. The second one: Let $A, B, X \in \mathbb{B}(\mathbb{H})$ be positive. Then

$$2s_j(M + N) \leqslant s_j((H + |K^*|) \oplus (L + |K|)) \tag{1.4}$$

for $j = 1, 2, ..., n$, where $M = A^{1/2}X^{1/2}Y^{1/2}A^{1/2}$, $N = B^{1/2}X^{1/2}Y^{1/2}B^{1/2}$, $H = X^{1/2}AX^{1/2} + Y^{1/2}AY^{1/2}$, $K = X^{1/2}A^{1/2}B^{1/2}X^{1/2} + Y^{1/2}A^{1/2}B^{1/2}Y^{1/2}$ and $L = X^{1/2}BX^{1/2} + Y^{1/2}BY^{1/2}$.

In this paper, we provide a new general inequality which is a generalization of several inequalities, one of these inequalities is inequality (1.1). Moreover, we compare our findings with inequalities (1.3) and (1.4). For recent studies on this topic, the reader should return to [2–6].

2. Main results

The following lemmas are essential for supporting our conclusions. The first lemma follows from the min-max principle (see, e.g., [1, p. 75] or [9, p. 27]). The second and third lemmas are shown in [7]. The fourth lemma is proved in [3].

Lemma 2.1. *Let* $A, B, X \in \mathbb{M}_n(\mathbb{C})$. *Then*

$$s_j(AXB) \leqslant \|A\| \, \|B\| \, s_j(X) \tag{2.1}$$

for $j = 1, 2, ..., n$.

Lemma 2.2. *Let* $A \in \mathbb{M}_n(\mathbb{C})$ *and let* $f \geqslant 0$ *and increasing function on an interval* I. *Then,* for $j = 1, 2, ..., n$,

1. $$f(s_j(A)) = s_j(f(|A|)).$$
2. For Hermitian A,
$$f(\lambda_j(A)) = \lambda_j(f(A)).$$

Lemma 2.3. *Let* $A, X \in \mathbb{M}_n(\mathbb{C})$ *such that* A *is Hermitian and* X *is contraction and let* f *be a nonnegative monotone convex function on an interval* I *such that* $0 \in I$ *and* $f(0) \leqslant 0$. *Then*

$$\lambda_j(f(X^*AX)) \leqslant \lambda_j(X^*f(A)X)$$

for $j = 1, 2, ..., n$.

Lemma 2.4. *Let* $A, B, X, Y \in \mathbb{M}_n(\mathbb{C})$. *Then*

$$s_j(AXY^*B^*) \leqslant \frac{1}{2} s_j\left(X^*|A|^2X + Y^*|B|^2Y\right) \tag{2.2}$$

for $j = 1, 2, ..., n$.

Now we are ready to state the first result of this paper, all functions considered in our results are nonnegative increasing convex functions on the interval I such that $0 \in I$ and $f(0) \leqslant 0$.

Theorem 2.5. *Let* $A, B, X, Y \in \mathbb{M}_n(\mathbb{C})$ *such that* $\max\{\|A\|, \|B\|\} \leqslant 1$. *Then*

$$2s_j(f(|AXY^*B^*|)) \leqslant (\max\{\|A\|, \|B\|\})^2 s_j(f(X^*X + Y^*Y)) \tag{2.3}$$

for $j = 1, 2, ..., n$.

Proof. Throughout this proof, let $Z = \begin{bmatrix} X & 0 \\ Y & 0 \end{bmatrix}$, then

$$\begin{aligned}
2s_j(f(|AXY^*B^*|) &= 2f\left(s_j(AXY^*B^*)\right) \ \text{(by Lemma 2.2)} \\
&\leqslant f\left(s_j\left(X^*|A|^2X + Y^*|B|^2Y\right)\right) \\
&\text{(by Lemma 2.4 and since } f \text{ is increasing function)} \\
&= f\left(s_j\left(Z^*\begin{bmatrix} |A|^2 & 0 \\ 0 & |B|^2 \end{bmatrix} Z\right)\right) \\
&= f\left(\lambda_j\left(Z^*\begin{bmatrix} |A|^2 & 0 \\ 0 & |B|^2 \end{bmatrix} Z\right)\right) \\
&= f\left(\lambda_j\left(\begin{bmatrix} |A| & 0 \\ 0 & |B| \end{bmatrix} ZZ^* \begin{bmatrix} |A| & 0 \\ 0 & |B| \end{bmatrix}\right)\right) \\
&= \lambda_j\left(f\left(\begin{bmatrix} |A| & 0 \\ 0 & |B| \end{bmatrix} ZZ^* \begin{bmatrix} |A| & 0 \\ 0 & |B| \end{bmatrix}\right)\right) \\
&\leqslant \lambda_j\left(\begin{bmatrix} |A| & 0 \\ 0 & |B| \end{bmatrix} f(ZZ^*) \begin{bmatrix} |A| & 0 \\ 0 & |B| \end{bmatrix}\right) \\
&\text{(by Lemma 2.3)} \\
&= s_j\left(\begin{bmatrix} |A| & 0 \\ 0 & |B| \end{bmatrix} f(ZZ^*) \begin{bmatrix} |A| & 0 \\ 0 & |B| \end{bmatrix}\right) \\
&\leqslant \left\|\begin{bmatrix} |A| & 0 \\ 0 & |B| \end{bmatrix}\right\|^2 s_j(f(ZZ^*)) \\
&\text{(by Lemma 2.1)} \\
&= \left\|\begin{bmatrix} |A| & 0 \\ 0 & |B| \end{bmatrix}\right\|^2 f(s_j(ZZ^*)) \\
&\text{(by Lemma 2.2)}
\end{aligned}$$

$$\begin{aligned}&= \left\| \begin{bmatrix} |A| & 0 \\ 0 & |B| \end{bmatrix} \right\|^2 f\left(s_j\left(Z^*Z\right)\right) \\ &= \left(\max\{\|A\|, \|B\|\}\right)^2 s_j\left(f\left(X^*X + Y^*Y\right)\right).\end{aligned}$$

□

Corollary 2.6. *Let* $A, B, X, Y \in \mathbb{B}(\mathbb{H})$ *such that* $\max\{\|A\|, \|B\|\} \leqslant 1$, X *and* Y *are positive. Then*

$$2s_j(f(\left|AX^{1/2}Y^{1/2}B^*\right|) \leqslant \left(\max\{\|A\|, \|B\|\}\right)^2 s_j\left(f\left(X+Y\right)\right) \tag{2.4}$$

for $j = 1, 2, ..., n$.

Proof. Substituting X by $X^{1/2}$ and Y by $Y^{1/2}$ in inequality (2.3), we give inequality (2.4). □

Corollary 2.7. *Let* $A, B, X, Y \in \mathbb{M}_n(\mathbb{C})$ *be positive such that* $\max\{\|A\|, \|B\|\} \leqslant 1$. *Then*

$$2s_j(f(\left|A^{1/2}X^{1/2}Y^{1/2}B^{1/2}\right|) \leqslant \left(\max\left\{\|A\|^{1/2}, \|B\|^{1/2}\right\}\right)^2 s_j\left(f\left(X+Y\right)\right) \tag{2.5}$$

for $j = 1, 2, ..., n$.

Proof. The result is deduced from inequality (2.4) by letting $A = A^{1/2}$ and $B = B^{1/2}$. □

The following is our main result.

Theorem 2.8. *Let* $A_i, B_i, X_i, Y_i \in \mathbb{M}_n(\mathbb{C})$, $i = 1, 2, ..., m$. *Then*

$$2s_j\left(f\left(\left|\sum_{i=1}^{m} A_iX_iY_i^*B_i^*\right|\right)\right) \leqslant \left(\max\{S, T\}\right)^2 s_j(K) \tag{2.6}$$

for $j = 1, 2, ..., n$, *where*

$$S = \left\|\sum_{i=1}^{m} A_iA_i^*\right\|^{1/2}, \quad T = \left\|\sum_{i=1}^{m} B_iB_i^*\right\|^{1/2},$$
$$K = f(|X_1|^2 + |Y_1|^2) \oplus ... \oplus f(|X_m|^2 + |Y_m|^2)$$

and $\max\{S, T\} \leqslant 1$.

Proof. On $\oplus_{j=1}^{n} H$, Let

$$A = \begin{bmatrix} A_1 & A_2 & \dots & A_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & B_2 & \dots & B_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

$$X = \begin{bmatrix} X_1 & 0 & \dots & 0 \\ 0 & X_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & X_n \end{bmatrix}, \text{ and } Y = \begin{bmatrix} Y_1 & 0 & \dots & 0 \\ 0 & Y_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & Y_n \end{bmatrix}.$$

Then

$$AXY^*B^* = \begin{bmatrix} \sum\limits_{i=1}^{n} A_iX_iY_i^*B_i^* & 0 & \dots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

$$\|A\| = \left\| \sum_{i=1}^{m} A_i A_i^* \right\|^{1/2}, \|B\| = \left\| \sum_{i=1}^{m} B_i B_i^* \right\|^{1/2}, \text{ and}$$
$$s_j\left(f\left(X^*X + Y^*Y\right)\right) = s_j(K).$$

Apply inequality (2.5) to the operator matrices A, B, X and Y, we give inequality (2.6). ☐

Corollary 2.9. *Let* $A_i, B_i, X_i, Y_i \in \mathbb{M}_n(\mathbb{C})$, $i = 1, 2$ *such that*

$$\max\left\{ \|A_1A_1^* + A_2A_2^*\|^{1/2}, \|B_1B_1^* + B_2B_2^*\|^{1/2} \right\} \leqslant 1.$$

Then

$$2s_j\left(f\left(\left|\sum_{i=1}^{2} A_iX_iY_i^*B_i^*\right|\right)\right) \leqslant (\max\{P, Q\})^2 s_j(M \oplus N) \tag{2.7}$$

for $j = 1, 2, ..., n$, *where*

$$P = \|A_1A_1^* + A_2A_2^*\|^{1/2},$$
$$Q = \|B_1B_1^* + B_2B_2^*\|^{1/2},$$
$$M = f(|X_1|^2 + |Y_1|^2)$$

and

$$N = f(|X_2|^2 + |Y_2|^2).$$

Proof. Letting $A_i = B_i = X_i = Y_i = 0$ for $i = 3, 4, ..., n$ in inequality (2.6), we give inequality (2.7). ☐

Remark 2.10. Let $A_2 = B_2 = X_2 = Y_2 = 0$, $A_1 = B_1 = I$ and let $f(t) = t$ in inequality (2.7), we give inequality (1.1).

We will present some applications.

Corollary 2.11. *Let* $A, B, X, Y \in \mathbb{M}_n(\mathbb{C})$, $i = 1, 2$ *such that* $X_i, Y_i \geqslant 0$ *and* $N = ((X + Y) \oplus (X + Y))$. *Then*

$$2s_j\left(AX^{1/2}Y^{1/2}B^* + BX^{1/2}Y^{1/2}A^*\right) \leqslant \|AA^* + BB^*\|^{1/2} s_j(N) \tag{2.8}$$

for $j = 1, 2, ..., n$.

Proof. Substituting $A_1 = B_2 = A$, $A_2 = B_1 = B$, $X_1 = X_2 = X^{1/2}$, $Y_1 = Y_2 = Y^{1/2}$ and $f(t) = t$ in inequality (2.7), we get our result. ☐

Remark 2.12. You can easily prove that inequality (2.8) is sharper than inequality (1.3) when $A = B = I$. To see this, note that

$$\sqrt{2}s_j\left((X + Y) \oplus (X + Y)\right) \leqslant 2s_j\left((X + Y) \oplus (X + Y)\right).$$

Remark 2.13. Substitute $Y = X$ in inequality (2.8), we obtain

$$s_j\left(AXB^* + BXA^*\right) \leqslant \|AA^* + BB^*\|^{1/2} s_j(X \oplus X) \tag{2.9}$$

for $j = 1, 2, ..., n$.

Remark 2.14. Substitute $X = Y = I$ in inequality (2.9), we obtain

$$s_j\left(AB^* + BA^*\right) \leqslant \|AA^* + BB^*\|^{1/2}$$

for $j = 1, 2, ..., n$. In particular,

$$\|AB^* + BA^*\| \leqslant \|AA^* + BB^*\|^{1/2}.$$

Corollary 2.15. *Let* $A, B, X, Y \in \mathbb{M}_n(\mathbb{C})$ *be positive. Then*

$$2s_j\left(M+N\right) \leqslant \|A+B\|^{1/2}\, s_j\left((X+Y)\oplus(X+Y)\right) \tag{2.10}$$

for $j = 1, 2, ..., 2$, *where* $M = A^{1/2}X^{1/2}Y^{1/2}A^{1/2}$, $N = B^{1/2}X^{1/2}Y^{1/2}B^{1/2}$. *Letting* $Y = X$, *we give*

$$s_j\left(A^{1/2}XA^{1/2} + B^{1/2}XB^{1/2}\right) \leqslant \|A+B\|^{1/2}\, s_j(X \oplus X)$$

for $j = 1, 2, ..., 2$. *Letting* $X = I$, *we give*

$$s_j\left(A+B\right) \leqslant \|A+B\|^{1/2}$$

for $j = 1, 2, ..., n$.

Proof. Letting $A_1 = B_1 = A^{1/2}$, $A_2 = B_2 = B^{1/2}$, $X_1 = X_2 = X^{1/2}$, $Y_1 = Y_2 = Y^{1/2}$ and $f(t) = t$ in inequality (2.7), we give inequality (2.10). ☐

Remark 2.16. It should be noted that Inequality (2.10) is sharper than inequality (1.4) when $A = B = I$. To see this, note that $\sqrt{2}_j\left((X+Y)\oplus(X+Y)\right) \leqslant 2s_j\left((X+Y)\oplus(X+Y)\right)$.

Corollary 2.17. *Let* $A, B, X_1, X_2, Y_1, Y_2 \in \mathbb{M}_n(\mathbb{C})$. *Then*

$$2s_j\left(AX_1Y_1^*A^* - BX_2Y_2^*B^*\right) \leqslant \|AA^* + BB^*\|^{1/2}\, s_j(L) \tag{2.11}$$

for $j = 1, 2, ..., n$, *where*

$$L = \left(X_1^*X_1 + Y_1^*Y_1\right) \oplus \left(X_2^*X_2 + Y_2^*Y_2\right).$$

Proof. In inequality (2.7), letting $A_1 = B_1 = A$, $A_2 = -B_2 = B$, and $f(t) = t$, we give inequality (2.11). ☐

Remark 2.18. Letting $X_2 = Y_2 = B = 0$ in inequality (2.11), we obtain

$$2s_j\left(AX_1Y_1^*A^*\right) \leqslant \|AA^*\|^{1/2}\, s_j\left(X_1^*X_1 + Y_1^*Y_1\right). \tag{2.12}$$

Inequality (1.1) is a special case of inequality (2.12) for $j = 1, 2, ..., n$.

Remark 2.19. Substituting $A = I$ in inequality (2.12), we obtain inequality (1.1).

The following result follows from inequality (2.11).

Corollary 2.20. *Let* $A, B, X \in \mathbb{M}_n(\mathbb{C})$ *where* $\max\{\|A\|, \|B\|\} \leqslant \frac{1}{2}$ *and* X *is positive. Then*

$$s_j(AXB^*) \leqslant \sqrt{2}\left(\max\{\|A\|, \|B\|\}\right) s_j(X) \tag{2.13}$$

for $j = 1, 2, ..., n$.

Proof. In inequality (2.11), let $C = \begin{bmatrix} A \\ B \end{bmatrix}$, $D = \begin{bmatrix} A \\ -B \end{bmatrix}$, and $X_1 = X_2 = Y_1 = Y_2 = X^{1/2}$. Then

$$CX_1Y_1^*C^* - DX_2Y_2^*D^* = \begin{bmatrix} 0 & 2AXB^* \\ 2BXA^* & 0 \end{bmatrix}$$

and

$$CC^* + DD^* = 2\begin{bmatrix} AA^* & 0 \\ 0 & BB^* \end{bmatrix}.$$

Now, applying inequality (2.11), leads to

$$2s_j\left(\begin{bmatrix} 0 & AXB^* \\ BXA^* & 0 \end{bmatrix}\right) \leqslant \sqrt{2}\left\|\begin{bmatrix} AA^* & 0 \\ 0 & BB^* \end{bmatrix}\right\|^{1/2} s_j(2X \oplus 2X)$$

This is equivalent to saying that

$$\begin{aligned} s_j(AXB^*) &\leqslant \sqrt{2}\,(\max\{\|AA^*\|,\|BB^*\|\})^{1/2}\, s_j(X) \\ &= \sqrt{2}\,(\max\{\|A\|,\|B\|\})\, s_j(X) \\ (\text{Since } \|A\|^2 &= \|A^*A\| = \|AA^*\|) \end{aligned}$$

for $j = 1, 2, \ldots, n$. Inequality (2.13) has thus been substantiated. □

Remark 2.21. Letting $B = A$ in inequality (2.13), we give

$$s_j(AXA^*) \leqslant \sqrt{2}\,\|A\|\, s_j(X) \tag{2.14}$$

for $j = 1, 2, \ldots, n$.

Remark 2.22. It should be noted that inequality (2.13) is sharper than inequality (2.1) if $\min\{\|A\|,\|B\|\} > \sqrt{2}$ and inequality (2.14) is sharper than inequality (2.1) if $\|B\| > \sqrt{2}$.

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