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r-fuzzy δ-ℓ-open sets and fuzzy upper (lower) δ-ℓ-continuity via fuzzy idealization

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Abstract

In this study, the concepts of r-fuzzy δ-ℓ-open and r-fuzzy strong β-ℓ-open sets are defined in a fuzzy ideal topological space (X, τ, ℓ) based on the sense of Šostak. Some properties of these sets along with their mutual relationships are discussed with the help of examples. Also, the concepts of fuzzy upper and lower δ-ℓ-continuous (resp. strong β-ℓ-continuous) multifunctions are introduced and studied. Moreover, the decomposition of fuzzy upper (resp. lower) semi-ℓ-continuity and the decomposition of fuzzy upper (resp. lower) α-ℓ-continuity are obtained. Finally, we constructed a new form of r-fuzzy connected set called r-fuzzy ℓ-connected and studied some of its properties via fuzzy ideals.

Keywords: Fuzzy ideal topological space, r-fuzzy δ-ℓ-open (resp. strong β-ℓ-open) set, fuzzy upper and lower δ-ℓ-continuity (resp. strong β-ℓ-continuity), connectedness.

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1. Introduction and preliminaries

The theory of fuzzy sets provides a framework for mathematical modeling of those real world situations, which involve an element of uncertainty, imprecision, or vagueness in their description. Since its inception fifty years ago by Zadeh [22], this theory has found wide applications in information sciences, engineering, medicine, etc; for details the reader is referred to [12, 23].

A fuzzy multifunction is a fuzzy set valued function [5, 13, 20, 21]. Fuzzy multifunctions arise in many applications, for instance, the budget multifunction occurs in decision theory, artificial intelligence and economic theory. The biggest difference between fuzzy functions and fuzzy multifunctions has to do with the definition of an inverse image. For a fuzzy multifunction there are two types of inverses. These two definitions of the inverse then lead to two definitions of continuity. Al-shami [6–8], defined and studied new generalization of open set. Taha [17], introduced and studied the concepts of r-fuzzy ℓ-open, r-fuzzy semi-ℓ-open, r-fuzzy α-ℓ-open and r-fuzzy β-ℓ-open sets in a fuzzy ideal topological space (X, τ, ℓ). Also, Taha [18] introduced and studied the concepts of fuzzy upper and lower α-ℓ-continuous (resp. β-ℓ-continuous, semi-ℓ-continuous and pre-ℓ-continuous) multifunctions via fuzzy ideals; for more details the reader is referred to [2–4, 10, 11, 14, 17–19].

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In this article, we introduce the notions of r-fuzzy δ-ℓ-open and r-fuzzy strong β-ℓ-open sets in a fuzzy ideal topological space (X, τ, ℓ) and study their various properties. Also, we introduce the notions of fuzzy upper and lower δ-ℓ-continuous (resp. strong β-ℓ-continuous) multifunctions via fuzzy ideals. Several characterizations of these multifunctions along with their mutual relationships are established. Furthermore, we give the decomposition of fuzzy upper (resp. lower) semi-ℓ-continuity and the decomposition of fuzzy upper (resp. lower) α-ℓ-continuity [18]. In the end, we introduce and explore new form of r-fuzzy connected set [1] called r-fuzzy ℓ-connected set.

Throughout this paper, X refers to an initial universe. The family of all fuzzy sets in X is denoted by I^X and for $\lambda \in I^X$, $\lambda^c(x) = 1 - \lambda(x)$ for all $x \in X$ (where $I = [0, 1]$ and $I_\circ = (0, 1]$). For $t \in I$, $\underline{t}(x) = t$ for all $x \in X$. The fuzzy difference between two fuzzy sets [19] $\lambda, \mu \in I^X$ is defined as follows:

$$\lambda \,\overline{\wedge}\, \mu = \begin{cases} \underline{0}, & \text{if } \lambda \leqslant \mu, \\ \lambda \wedge \mu^c, & \text{otherwise.} \end{cases}$$

All other notations are standard notations of fuzzy set theory.

Now, we recall that a fuzzy idea ℓ on X [15], is a map $\ell : I^X \longrightarrow I$ that satisfies the following conditions:

(i) $\forall\, \lambda, \mu \in I^X$ and $\lambda \leqslant \mu \Rightarrow \ell(\mu) \leqslant \ell(\lambda)$;
(ii) $\forall\, \lambda, \mu \in I^X \Rightarrow \ell(\lambda \vee \mu) \geqslant \ell(\lambda) \wedge \ell(\mu)$.

Also, ℓ is called proper if $\ell(\underline{1}) = 0$ and there exists $\mu \in I^X$ such that $\ell(\mu) > 0$. The simplest fuzzy ideals on X, ℓ_0 and ℓ_1 are defined as follows:

$$\ell_0(\lambda) = \begin{cases} 1, & \text{if } \lambda = \underline{0}, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad \ell_1(\lambda) = 1,\ \forall\ \lambda \in I^X.$$

If ℓ^1 and ℓ^2 are fuzzy ideals on X, we say that ℓ^1 is finer than ℓ^2 (ℓ^2 is coarser than ℓ^1), denoted by $\ell^2 \leqslant \ell^1$, iff $\ell^2(\lambda) \leqslant \ell^1(\lambda)\ \forall\ \lambda \in I^X$.

Let (X, τ) be a fuzzy topological space in Šostak sense [16], the interior and the closure of any fuzzy set $\lambda \in I^X$ is denoted by $I_\tau(\lambda, r)$ and $C_\tau(\lambda, r)$, respectively. Let (X, τ, ℓ) be a fuzzy ideal topological space in Šostak sense, $\lambda \in I^X$ and $r \in I_\circ$, then the r-fuzzy local function [19] λ^*_r of λ is defined as follows:

$$\lambda^*_r = \bigwedge\{\mu \in I^X \ :\ \ell(\lambda \,\overline{\wedge}\, \mu) \geqslant r,\ \tau(\mu^c) \geqslant r\}.$$

If we take $\ell = \ell_0$, for each $\lambda \in I^X$ we have $\lambda^*_r = \bigwedge\{\mu \in I^X \ :\ \lambda \leqslant \mu,\ \tau(\mu^c) \geqslant r\} = C_\tau(\lambda, r)$. Also, if we take $\ell = \ell_1$ (resp. $\ell(\lambda) \geqslant r$), for each $\lambda \in I^X$ we have $\lambda^*_r = \underline{0}$. Moreover, we define an operator $C^*_\tau : I^X \times I_\circ \to I^X$ as follows: $C^*_\tau(\lambda, r) = \lambda \ \vee\ \lambda^*_r$. In (X, τ, ℓ), $\lambda \in I^X$ is said to be r-fuzzy ℓ-open (resp. semi-ℓ-open, pre-ℓ-open, α-ℓ-open and β-ℓ-open) [17] iff $\lambda \leqslant I_\tau(\lambda^*_r, r)$ (resp. $\lambda \leqslant C^*_\tau(I_\tau(\lambda, r), r)$, $\lambda \leqslant I_\tau(C^*_\tau(\lambda, r), r)$, $\lambda \leqslant I_\tau(C^*_\tau(I_\tau(\lambda, r), r), r)$ and $\lambda \leqslant C_\tau(I_\tau(C^*_\tau(\lambda, r), r), r)$). The complement of r-fuzzy ℓ-open set is r-fuzzy ℓ-closed.

A mapping $F : X \multimap Y$ is called a fuzzy multifunction [9] iff $F(x) \in I^Y$ for each $x \in X$. The degree of membership of y in $F(x)$ is denoted by $F(x)(y) = G_F(x, y)$ for any $(x, y) \in X \times Y$. Also, F is a Crisp iff $G_F(x, y) = 1$ for each $x \in X$, $y \in Y$ and F is Normalized iff for each $x \in X$, there exists $y_0 \in Y$ such that $G_F(x, y_0) = 1$. The image $F(\lambda)$ of $\lambda \in I^X$, the lower inverse $F^l(\mu)$ and the upper inverse $F^u(\mu)$ of $\mu \in I^Y$ are defined as follows: $F(\lambda)(y) = \bigvee_{x \in X}[G_F(x, y) \wedge \lambda(x)]$, $F^l(\mu)(x) = \bigvee_{y \in Y}[G_F(x, y) \wedge \mu(y)]$ and $F^u(\mu)(x) = \bigwedge_{y \in Y}[G^c_F(x, y) \vee \mu(y)]$. All definitions and properties of upper and lower are found in [2].

2. On r-fuzzy δ-ℓ-open and r-fuzzy strong β-ℓ-open sets

In this section, the concepts of r-fuzzy δ-ℓ-open and r-fuzzy strong β-ℓ-open sets are defined in a fuzzy ideal topological space (X, τ, ℓ) based on the sense of Šostak. Some properties of these sets along with their mutual relationships are discussed with the help of examples.

Definition 2.1. Let (X, τ, ℓ) be a fuzzy ideal topological space, $\lambda \in I^X$ and $r \in I_\circ$. Then λ is said to be:

(1) r-fuzzy δ-ℓ-open iff $I_\tau(C^*_\tau(\lambda, r), r) \leqslant C^*_\tau(I_\tau(\lambda, r), r)$;
(2) r-fuzzy strong β-ℓ-open iff $\lambda \leqslant C^*_\tau(I_\tau(C^*_\tau(\lambda, r), r), r)$.

The following implications hold:

$$\begin{array}{ccccc} & & \text{r-fuzzy}\alpha\text{-}\ell\text{-open} \Rightarrow & \text{r-fuzzy semi-}\ell\text{-open} \Rightarrow & \text{r-fuzzy}\delta\text{-}\ell\text{-open} \\ & & \Downarrow & \Downarrow & \\ \text{r-fuzzy}\ell\text{-open} \Rightarrow & \text{r-fuzzy pre-}\ell\text{-open} \Rightarrow & \text{r-fuzzy strong}\beta-\ell\text{-open} & & \end{array}$$

In general the converses are not true.

Problem 2.2. Define $\tau, \ell : I^X \longrightarrow I$ as follows:

$$\tau(\lambda) = \begin{cases} 1, & \text{if } \lambda \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \text{if } \lambda = \underline{0.2}, \\ \frac{3}{4}, & \text{if } \lambda = \underline{0.8}, \\ 0, & \text{otherwise}, \end{cases} \qquad \ell(\nu) = \begin{cases} 1, & \text{if } \nu = \underline{0}, \\ \frac{1}{2}, & \text{if } \underline{0} < \nu < \underline{0.4}, \\ 0, & \text{otherwise}. \end{cases}$$

Then, $\underline{0.5}$ is $\frac{1}{2}$-fuzzy strong β-ℓ-open set but it is not $\frac{1}{2}$-fuzzy semi-ℓ-open.

Problem 2.3. Let $X = \{x, y, z\}$ be a set and $\mu_1, \mu_2 \in I^X$ defined as follows: $\mu_1 = \{\frac{x}{0.3}, \frac{y}{0.4}, \frac{z}{0.8}\}$ and $\mu_2 = \{\frac{x}{0.2}, \frac{y}{0.3}, \frac{z}{0.2}\}$. Define $\tau, \ell : I^X \to I$ as follows:

$$\tau(\lambda) = \begin{cases} 1, & \text{if } \lambda \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \text{if } \lambda = \mu_2, \\ 0, & \text{otherwise}, \end{cases} \qquad \ell(\nu) = \begin{cases} 1, & \text{if } \nu = \underline{0}, \\ 0, & \text{otherwise}. \end{cases}$$

Then, μ_1 is $\frac{1}{2}$-fuzzy strong β-ℓ-open set but it is not $\frac{1}{2}$-fuzzy pre-ℓ-open.

Remark 2.4. r-fuzzy δ-ℓ-open and r-fuzzy strong β-ℓ-open are independent notions as shown by the Problems 2.5 and 2.6.

Problem 2.5. Let $X = \{x, y, z, w\}$ be a set and $\mu_1, \mu_2, \mu_3, \mu_4 \in I^X$ defined as follows: $\mu_1 = \{\frac{x}{1.0}, \frac{y}{0.0}, \frac{z}{0.0}, \frac{w}{0.0}\}$, $\mu_2 = \{\frac{x}{0.0}, \frac{y}{1.0}, \frac{z}{0.0}, \frac{w}{0.0}\}$, $\mu_3 = \{\frac{x}{0.0}, \frac{y}{0.0}, \frac{z}{1.0}, \frac{w}{0.0}\}$, and $\mu_4 = \{\frac{x}{0.0}, \frac{y}{0.0}, \frac{z}{0.0}, \frac{w}{1.0}\}$. Define $\tau, \ell : I^X \to I$ as follows:

$$\tau(\lambda) = \begin{cases} 1, & \text{if } \lambda \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \text{if } \lambda = \mu_4, \\ \frac{3}{4}, & \text{if } \lambda = \mu_1 \vee \mu_2 \vee \mu_3, \\ 0, & \text{otherwise}, \end{cases} \qquad \ell(\nu) = \begin{cases} 1, & \text{if } \nu = \underline{0}, \\ \frac{1}{2}, & \text{if } \nu = \mu_3, \\ \frac{3}{4}, & \text{if } \underline{0} < \nu < \mu_3, \\ 0, & \text{otherwise}. \end{cases}$$

Then, $\mu_3 \vee \mu_4$ is $\frac{1}{2}$-fuzzy δ-ℓ-open set but it is neither $\frac{1}{2}$-fuzzy strong β-ℓ-open nor $\frac{1}{2}$-fuzzy semi-ℓ-open.

Problem 2.6. Let $X = \{x, y, z, w\}$ be a set and $\mu_1, \mu_2, \mu_3, \mu_4 \in I^X$ defined as follows: $\mu_1 = \{\frac{x}{1.0}, \frac{y}{0.0}, \frac{z}{0.0}, \frac{w}{0.0}\}$, $\mu_2 = \{\frac{x}{0.0}, \frac{y}{1.0}, \frac{z}{0.0}, \frac{w}{0.0}\}$, $\mu_3 = \{\frac{x}{0.0}, \frac{y}{0.0}, \frac{z}{1.0}, \frac{w}{0.0}\}$, and $\mu_4 = \{\frac{x}{0.0}, \frac{y}{0.0}, \frac{z}{0.0}, \frac{w}{1.0}\}$. Define $\tau, \ell : I^X \to I$ as follows:

$$\tau(\lambda) = \begin{cases} 1, & \text{if } \lambda \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \text{if } \lambda = \mu_1 \vee \mu_2, \\ 0, & \text{otherwise}, \end{cases} \qquad \ell(\nu) = \begin{cases} 1, & \text{if } \nu = \underline{0}, \\ 0, & \text{otherwise}. \end{cases}$$

Then, $\mu_1 \vee \mu_3$ is $\frac{1}{2}$-fuzzy strong β-ℓ-open set but it is not $\frac{1}{2}$-fuzzy δ-ℓ-open.

Corollary 2.7. *Let (X, τ, ℓ) be a fuzzy ideal topological space, $\lambda \in I^X$ and $r \in I_\circ$. If we take $\ell = \ell_0$,*

(1) *r-fuzzy δ-ℓ-open and r-fuzzy δ-open are equivalent;*
(2) *r-fuzzy strong β-ℓ-open and r-fuzzy β-open are equivalent.*

Remark 2.8. The complement of r-fuzzy δ-ℓ-open (resp. r-fuzzy strong β-ℓ-open) set is said to be r-fuzzy δ-ℓ-closed (resp. r-fuzzy strong β-ℓ-closed).

Proposition 2.9. *Let (X, τ, ℓ) be a fuzzy ideal topological space, $\lambda \in I^X$ and $r \in I_\circ$. The following statements are equivalent,*

(1) *λ is r-fuzzy semi-ℓ-open;*
(2) *λ is r-fuzzy δ-ℓ-open and r-fuzzy strong β-ℓ-open.*

Proof.

(1) ⇒ (2) Let λ be r-fuzzy semi-ℓ-open, then

$$\lambda \leqslant C^*_\tau(I_\tau(\lambda, r), r) \leqslant C^*_\tau(I_\tau(C^*_\tau(\lambda, r), r), r).$$

This shows that λ is r-fuzzy strong β-ℓ-open. Moreover,

$$I_\tau(C^*_\tau(\lambda, r), r) \leqslant C^*_\tau(\lambda, r) \leqslant C^*_\tau(C^*_\tau(I_\tau(\lambda, r), r), r) = C^*_\tau(I_\tau(\lambda, r), r).$$

Therefore, λ is r-fuzzy δ-ℓ-open.

(2) ⇒ (1) Let λ be r-fuzzy δ-ℓ-open and r-fuzzy strong β-ℓ-open, $I_\tau(C^*_\tau(\lambda, r), r) \leqslant C^*_\tau(I_\tau(\lambda, r), r)$ and $\lambda \leqslant C^*_\tau(I_\tau(C^*_\tau(\lambda, r), r), r)$. Thus,

$$\lambda \leqslant C^*_\tau(I_\tau(C^*_\tau(\lambda, r), r), r) \leqslant C^*_\tau(C^*_\tau(I_\tau(\lambda, r), r), r) = C^*_\tau(I_\tau(\lambda, r), r).$$

This shows that λ is r-fuzzy semi-ℓ-open. □

Proposition 2.10. *Let (X, τ, ℓ) be a fuzzy ideal topological space, $\lambda \in I^X$ and $r \in I_\circ$. The following statements are equivalent,*

(1) *λ is r-fuzzy α-ℓ-open;*
(2) *λ is r-fuzzy δ-ℓ-open and r-fuzzy pre-ℓ-open.*

Proof.

(1) ⇒ (2) From Proposition 2.9 the proof is straightforward.

(2) ⇒ (1) Let λ be r-fuzzy pre-ℓ-open and r-fuzzy δ-ℓ-open. Then,

$$\lambda \leqslant I_\tau(C^*_\tau(\lambda, r), r) \leqslant I_\tau(C^*_\tau(I_\tau(\lambda, r), r), r).$$

This shows that λ is r-fuzzy α-ℓ-open. □

Theorem 2.11. *Let (X, τ, ℓ) be a fuzzy ideal topological space, $\lambda, \mu \in I^X$ and $r \in I_\circ$. If λ is r-fuzzy δ-ℓ-open set such that $\lambda \leqslant \mu \leqslant C^*_\tau(\lambda, r)$, then μ is also r-fuzzy δ-ℓ-open.*

Proof. Suppose that λ is r-fuzzy δ-ℓ-open and $\lambda \leqslant \mu \leqslant C^*_\tau(\lambda, r)$. Then,

$$I_\tau(C^*_\tau(\lambda, r), r) \leqslant C^*_\tau(I_\tau(\lambda, r), r) \leqslant C^*_\tau(I_\tau(\mu, r), r).$$

Since $\mu \leqslant C^*_\tau(\lambda, r)$, $I_\tau(C^*_\tau(\mu, r), r) \leqslant I_\tau(C^*_\tau(\lambda, r), r) \leqslant C^*_\tau(I_\tau(\mu, r), r)$, this shows that μ is r-fuzzy δ-ℓ-open. □

3. Decomposition of fuzzy semi-ℓ-continuity and fuzzy α-ℓ-continuity

In this section, the concepts of fuzzy upper and lower δ-ℓ-continuous (resp. strong β-ℓ-continuous) multifunctions are introduced and studied. Moreover, the decomposition of fuzzy upper (resp. lower) semi-ℓ-continuity and the decomposition of fuzzy upper (resp. lower) α-ℓ-continuity [18] are obtained.

Definition 3.1. A fuzzy multifunction $F : (X, \tau, \ell) \multimap (Y, \eta)$ is called:

(1) fuzzy upper δ-ℓ-continuous (resp. strong β-ℓ-continuous) iff $F^u(\mu)$ is r-fuzzy δ-ℓ-open (resp. r-fuzzy strong β-ℓ-open) for every $\mu \in I^Y$ with $\eta(\mu) \geqslant r$ and $r \in I_o$;

(2) Fuzzy lower δ-ℓ-continuous (resp. strong β-ℓ-continuous) iff $F^l(\mu)$ is r-fuzzy δ-ℓ-open (resp. r-fuzzy strong β-ℓ-open) for every $\mu \in I^Y$ with $\eta(\mu) \geqslant r$ and $r \in I_o$.

The following implications hold:

$$\begin{array}{ccccc} & & \alpha\text{-}\ell\text{-continuity} & \Rightarrow & \text{semi-}\ell\text{-continuity} \Rightarrow \delta\text{-}\ell\text{-continuity} \\ & & \Downarrow & & \Downarrow \\ \ell\text{-continuity} & \Rightarrow & \text{pre-}\ell\text{-continuity} & \Rightarrow & \text{strong } \beta\text{-}\ell\text{-continuity} \end{array}$$

In general the converses are not true.

Problem 3.2. Let $X = \{x_1, x_2\}$, $Y = \{y_1, y_2, y_3\}$ and $F : X \multimap Y$ be a fuzzy multifunction defined by $G_F(x_1, y_1) = 0.1$, $G_F(x_1, y_2) = 1.0$, $G_F(x_1, y_3) = 0.3$, $G_F(x_2, y_1) = 0.5$, $G_F(x_2, y_2) = 0.1$ and $G_F(x_2, y_3) = 1.0$. Define $\tau, \ell : I^X \longrightarrow I$ and $\eta : I^Y \to I$ as follows:

$$\tau(\lambda) = \begin{cases} 1, & \text{if } \lambda \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \text{if } \lambda = \underline{0.2}, \\ \frac{3}{4}, & \text{if } \lambda = \underline{0.8}, \\ 0, & \text{otherwise,} \end{cases} \qquad \ell(\nu) = \begin{cases} 1, & \text{if } \nu = \underline{0}, \\ \frac{1}{2}, & \text{if } \underline{0} < \nu < \underline{0.4}, \\ 0, & \text{otherwise,} \end{cases} \qquad \eta(\mu) = \begin{cases} 1, & \text{if } \mu \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \text{if } \mu = \underline{0.5}, \\ 0, & \text{otherwise.} \end{cases}$$

Then, $F : (X, \tau, \ell) \multimap (Y, \eta)$ is fuzzy upper (resp. lower) strong β-ℓ-continuous but it is not fuzzy upper (resp. lower) semi-ℓ-continuous.

Problem 3.3. Let $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2, y_3\}$ and $F : X \multimap Y$ be a fuzzy multifunction defined by $G_F(x_1, y_1) = 0.8$, $G_F(x_1, y_2) = 0.3$, $G_F(x_1, y_3) = 0.3$, $G_F(x_2, y_1) = 0.6$, $G_F(x_2, y_2) = 0.1$, $G_F(x_2, y_3) = 0.4$, $G_F(x_3, y_1) = 0.1$, $G_F(x_3, y_2) = 0.2$, $G_F(x_3, y_3) = 1.0$. Define $\mu_1 \in I^X$ and $\mu_2 \in I^Y$ as follows: $\mu_1 = \{\frac{x_1}{0.2}, \frac{x_2}{0.3}, \frac{x_3}{0.2}\}$ and $\mu_2 = \{\frac{y_1}{0.3}, \frac{y_2}{0.4}, \frac{y_3}{0.8}\}$. Define $\tau, \ell : I^X \longrightarrow I$ and $\eta : I^Y \to I$ as follows:

$$\tau(\lambda) = \begin{cases} 1, & \text{if } \lambda \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \text{if } \lambda = \mu_1, \\ 0, & \text{otherwise,} \end{cases} \qquad \ell(\nu) = \begin{cases} 1, & \text{if } \nu = \underline{0}, \\ 0, & \text{otherwise,} \end{cases} \qquad \eta(\mu) = \begin{cases} 1, & \text{if } \mu \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \text{if } \mu = \mu_2, \\ 0, & \text{otherwise.} \end{cases}$$

Then, $F : (X, \tau, \ell) \multimap (Y, \eta)$ is fuzzy upper (resp. lower) strong β-ℓ-continuous but it is not fuzzy upper (resp. lower) pre-ℓ-continuous.

Remark 3.4. Fuzzy upper (resp. lower) δ-ℓ-continuity and fuzzy upper (resp. lower) strong β-ℓ-continuity are independent notions as shown by Problems 3.5 and 3.6.

Problem 3.5. Let $X = \{x_1, x_2, x_3, x_4\}$, $Y = \{y_1, y_2, y_3, y_4\}$ and $F : X \multimap Y$ be a fuzzy multifunction defined by $G_F(x_1, y_1) = 0.8$, $G_F(x_1, y_2) = 1.0$, $G_F(x_1, y_3) = 0.3$, $G_F(x_1, y_4) = 1.0$, $G_F(x_2, y_1) = 1.0$, $G_F(x_2, y_2) = 0.1$, $G_F(x_2, y_3) = 1.0$, $G_F(x_2, y_4) = 0.4$, $G_F(x_3, y_1) = 0.0$, $G_F(x_3, y_2) = 0.0$, $G_F(x_3, y_3) = 0.4$, $G_F(x_3, y_4) = 0.5$, $G_F(x_4, y_1) = 0.0$, $G_F(x_4, y_2) = 0.0$, $G_F(x_4, y_3) = 0.7$, $G_F(x_4, y_4) = 0.4$. Define $\mu_1, \mu_2, \mu_3, \mu_4 \in I^X$ and $\mu_5 \in I^Y$ as follows: $\mu_1 = \{\frac{x_1}{1.0}, \frac{x_2}{0.0}, \frac{x_3}{0.0}, \frac{x_4}{0.0}\}$, $\mu_2 = \{\frac{x_1}{0.0}, \frac{x_2}{1.0}, \frac{x_3}{0.0}, \frac{x_4}{0.0}\}$, $\mu_3 = \{\frac{x_1}{0.0}, \frac{x_2}{0.0}, \frac{x_3}{1.0}, \frac{x_4}{0.0}\}$, $\mu_4 = \{\frac{x_1}{0.0}, \frac{x_2}{0.0}, \frac{x_3}{0.0}, \frac{x_4}{1.0}\}$ and $\mu_5 = \{\frac{y_1}{0.0}, \frac{y_2}{0.0}, \frac{y_3}{1.0}, \frac{y_4}{1.0}\}$. Define $\tau, \ell : I^X \longrightarrow I$ and $\eta : I^Y \to I$ as follows:

$$\tau(\lambda) = \begin{cases} 1, & \text{if } \lambda \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \text{if } \lambda = \mu_4, \\ \frac{3}{4}, & \text{if } \lambda = \mu_1 \vee \mu_2 \vee \mu_3, \\ 0, & \text{otherwise,} \end{cases} \qquad \ell(\nu) = \begin{cases} 1, & \text{if } \nu = \underline{0}, \\ \frac{1}{2}, & \text{if } \nu = \mu_3, \\ \frac{3}{4}, & \text{if } \underline{0} < \nu < \mu_3, \\ 0, & \text{otherwise,} \end{cases} \qquad \eta(\mu) = \begin{cases} 1, & \text{if } \mu \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \text{if } \mu = \mu_5, \\ 0, & \text{otherwise.} \end{cases}$$

Then, $F : (X, \tau, \ell) \multimap (Y, \eta)$ is fuzzy upper δ-ℓ-continuous but it is neither fuzzy upper strong β-ℓ-continuous nor fuzzy upper semi-ℓ-continuous.

Problem 3.6. Let $X = \{x_1, x_2, x_3, x_4\}$, $Y = \{y_1, y_2, y_3, y_4\}$ and $F : X \multimap Y$ be a fuzzy multifunction defined by $G_F(x_1, y_1) = 0.8$, $G_F(x_1, y_2) = 0.0$, $G_F(x_1, y_3) = 0.3$, $G_F(x_1, y_4) = 0.0$, $G_F(x_2, y_1) = 1.0$, $G_F(x_2, y_2) = 1.0$, $G_F(x_2, y_3) = 1.0$, $G_F(x_2, y_4) = 0.0$, $G_F(x_3, y_1) = 0.0$, $G_F(x_3, y_2) = 0.0$, $G_F(x_3, y_3) = 0.4$, $G_F(x_3, y_4) = 0.0$, $G_F(x_4, y_1) = 0.0$, $G_F(x_4, y_2) = 0.0$, $G_F(x_4, y_3) = 0.7$, $G_F(x_4, y_4) = 1.0$. Define $\mu_1 \in I^X$ and $\mu_2, \mu_3 \in I^Y$ as follows: $\mu_1 = \{\frac{x_1}{1.0}, \frac{x_2}{1.0}, \frac{x_3}{0.0}, \frac{x_4}{0.0}\}$, $\mu_2 = \{\frac{y_1}{1.0}, \frac{y_2}{0.0}, \frac{y_3}{1.0}, \frac{y_4}{0.0}\}$ and $\mu_3 = \{\frac{y_1}{1.0}, \frac{y_2}{1.0}, \frac{y_3}{1.0}, \frac{y_4}{0.0}\}$. Define $\tau, \ell : I^X \longrightarrow I$ and $\eta : I^Y \to I$ as follows:

$$\tau(\lambda) = \begin{cases} 1, & \text{if } \lambda \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \text{if } \lambda = \mu_1, \\ 0, & \text{otherwise}, \end{cases} \qquad \ell(\nu) = \begin{cases} 1, & \text{if } \nu = \underline{0}, \\ 0, & \text{otherwise}, \end{cases} \qquad \eta(\mu) = \begin{cases} 1, & \text{if } \mu \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \text{if } \mu = \mu_2, \\ 0, & \text{otherwise}. \end{cases}$$

Then, $F : (X, \tau, \ell) \multimap (Y, \eta)$ is fuzzy upper strong β-ℓ-continuous but it is not fuzzy upper δ-ℓ-continuous.

Corollary 3.7. *Let $F : (X, \tau, \ell) \multimap (Y, \eta)$ be a fuzzy multifunction (resp. normalized fuzzy multifunction). If we take $\ell = \ell_0$, F is fuzzy lower (resp. upper) strong β-ℓ-continuous iff it is fuzzy lower (resp. upper) β-continuous.*

According to Propositions 2.9 and 2.10 we have the following decomposition of fuzzy upper (resp. lower) semi-ℓ-continuity and decomposition of fuzzy upper (resp. lower) α-ℓ-continuity.

Theorem 3.8. *A fuzzy multifunction $F : (X, \tau, \ell) \multimap (Y, \eta)$ is fuzzy upper (resp. lower) semi-ℓ-continuous iff it is both fuzzy upper (resp. lower) δ-ℓ-continuous and fuzzy upper (resp. lower) strong β-ℓ-continuous.*

Proof. The proof is obvious by Proposition 2.9. □

Theorem 3.9. *A fuzzy multifunction $F : (X, \tau, \ell) \multimap (Y, \eta)$ is fuzzy upper (resp. lower) α-ℓ-continuous iff it is both fuzzy upper (resp. lower) δ-ℓ-continuous and fuzzy upper (resp. lower) pre-ℓ-continuous.*

Proof. The proof is obvious by Proposition 2.10. □

Theorem 3.10. *For a fuzzy multifunction $F : (X, \tau, \ell) \multimap (Y, \eta)$, $\lambda \in I^X$, $\mu \in I^Y$ and $r \in I_o$ the following statements are equivalent:*

(1) *F is fuzzy upper strong β-ℓ-continuous;*
(2) $I^*_\tau(C_\tau(I^*_\tau(F^l(\mu), r), r), r) \leqslant F^l(\mu)$, *if* $\eta(\mu^c) \geqslant r$;
(3) $I^*_\tau(C_\tau(I^*_\tau(F^l(\mu), r), r), r) \leqslant F^l(C_\eta(\mu, r))$;
(4) $F^u(I_\eta(\mu, r)) \leqslant C^*_\tau(I_\tau(C^*_\tau(F^u(\mu), r), r), r)$.

Proof.

(1) ⇒ (2) Let $\mu \in I^Y$ with $\eta(\mu^c) \geqslant r$. Then by Definition 3.1,

$$(F^l(\mu))^c = F^u(\mu^c) \leqslant C^*_\tau(I_\tau(C^*_\tau(F^u(\mu^c), r), r), r) = (I^*_\tau(C_\tau(I^*_\tau(F^l(\mu), r), r), r))^c.$$

Thus, $F^l(\mu) \geqslant I^*_\tau(C_\tau(I^*_\tau(F^l(\mu), r), r), r)$.

(2) ⇒ (3) It is obvious.

(3) ⇒ (4) Since $(I^*_\tau(C_\tau(I^*_\tau(F^l(\mu), r), r), r))^c = C^*_\tau(I_\tau(C^*_\tau(F^u(\mu^c), r), r), r)$ and $(F^l(C_\eta(\mu, r)))^c = F^u(I_\eta(\mu^c, r))$, then, $F^u(I_\eta(\mu, r)) \leqslant C^*_\tau(I_\tau(C^*_\tau(F^u(\mu), r), r), r)$ for each $\mu \in I^Y$.

(4) ⇒ (1) Let $\mu \in I^Y$ with $\eta(\mu) \geqslant r$. Then by (4) and $\mu = I_\eta(\mu, r)$, $F^u(\mu) \leqslant C^*_\tau(I_\tau(C^*_\tau(F^u(\mu), r), r), r)$. Thus, F is fuzzy upper strong β-ℓ-continuous. □

The following theorems are similarly proved as in Theorem 3.10.

Theorem 3.11. *For a fuzzy multifunction* $F : (X, \tau, \ell) \multimap (Y, \eta)$, $\lambda \in I^X$, $\mu \in I^Y$ *and* $r \in I_o$ *the following statements are equivalent:*

(1) F *is fuzzy lower strong* β-ℓ*-continuous;*
(2) $I^*_\tau(C_\tau(I^*_\tau(F^u(\mu), r), r), r) \leqslant F^u(\mu)$, *if* $\eta(\mu^c) \geqslant r$*;*
(3) $I^*_\tau(C_\tau(I^*_\tau(F^u(\mu), r), r), r) \leqslant F^u(C_\eta(\mu, r))$*;*
(4) $F^l(I_\eta(\mu, r)) \leqslant C^*_\tau(I_\tau(C^*_\tau(F^l(\mu), r), r), r)$.

Theorem 3.12. *For a fuzzy multifunction* $F : (X, \tau, \ell) \multimap (Y, \eta)$, $\lambda \in I^X$, $\mu \in I^Y$ *and* $r \in I_o$ *the following statements are equivalent:*

(1) F *is fuzzy upper* δ-ℓ*-continuous;*
(2) $I^*_\tau(C_\tau(F^l(\mu), r), r) \leqslant C_\tau(I^*_\tau(F^l(\mu), r), r)$, *if* $\eta(\mu^c) \geqslant r$*;*
(3) $I^*_\tau(C_\tau(F^l(\mu), r), r) \leqslant C_\tau(I^*_\tau(F^l(C_\eta(\mu, r)), r), r)$*;*
(4) $I_\tau(C^*_\tau(F^u(I_\eta(\mu, r)), r), r) \leqslant C^*_\tau(I_\tau(F^u(\mu), r), r)$.

Theorem 3.13. *For a fuzzy multifunction* $F : (X, \tau, \ell) \multimap (Y, \eta)$, $\lambda \in I^X$, $\mu \in I^Y$ *and* $r \in I_o$ *the following statements are equivalent:*

(1) F *is fuzzy lower* δ-ℓ*-continuous;*
(2) $I^*_\tau(C_\tau(F^u(\mu), r), r) \leqslant C_\tau(I^*_\tau(F^u(\mu), r), r)$, *if* $\eta(\mu^c) \geqslant r$*;*
(3) $I^*_\tau(C_\tau(F^u(\mu), r), r) \leqslant C_\tau(I^*_\tau(F^u(C_\eta(\mu, r)), r), r)$*;*
(4) $I_\tau(C^*_\tau(F^l(I_\eta(\mu, r)), r), r) \leqslant C^*_\tau(I_\tau(F^l(\mu), r), r)$.

Theorem 3.14. *Let* $F : (X, \tau, \ell) \multimap (Y, \eta)$ *and* $H : (Y, \eta) \multimap (Z, \gamma)$ *be two fuzzy multifunctions. Then* $H \circ F$ *is fuzzy lower* δ-ℓ*-continuous (resp. strong* β-ℓ*-continuous) if* F *is fuzzy lower* δ-ℓ*-continuous (resp. strong* β-ℓ*-continuous) and* H *is fuzzy lower semi-continuous.*

Proof. It is obvious. □

4. Some applications via fuzzy ideals

In this section, we introduced and studied a new form of r-fuzzy connected set called r-fuzzy ℓ-connected via fuzzy ideals.

Definition 4.1 ([17]). Let (X, τ, ℓ) be a fuzzy ideal topological space. Then for each $\lambda \in I^X$ and $r \in I_\circ$, we define an operator $C^\ell_\tau : I^X \times I_\circ \to I^X$ as follows: $C^\ell_\tau(\lambda, r) = \bigwedge\{\mu \in I^X : \lambda \leqslant \mu,\ \mu \text{ is r-fuzzy } \ell\text{-closed}\}$.

Theorem 4.2 ([17]). *Let* (X, τ, ℓ) *be a fuzzy ideal topological space. Then for any fuzzy sets* $\lambda,\ \nu \in I^X$*, the operator* $C^\ell_\tau : I^X \times I_\circ \to I^X$ *satisfies the following properties:*

(1) $C^\ell_\tau(\underline{1}, r) = \underline{1}$*;*
(2) $\lambda \leqslant C^\ell_\tau(\lambda, r)$*;*
(3) *if* $\lambda \leqslant \nu$*, then* $C^\ell_\tau(\lambda, r) \leqslant C^\ell_\tau(\nu, r)$*;*
(4) *if* $\ell(\lambda) \geqslant r$*, then* $C^\ell_\tau(\lambda, r) = \underline{1}$*;*
(5) $C^\ell_\tau(C^\ell_\tau(\lambda, r), r) = C^\ell_\tau(\lambda, r)$*;*
(6) $C^\ell_\tau(\lambda, r) \vee C^\ell_\tau(\nu, r) \leqslant C^\ell_\tau(\lambda \vee \nu, r)$*;*
(7) $\lambda = C^\ell_\tau(\lambda, r)$ *iff* λ *is r-fuzzy* ℓ*-closed.*

Definition 4.3. Let (X, τ, ℓ) be a fuzzy ideal topological space and $r \in I_\circ$.

(1) Two fuzzy sets λ and μ are said to be r-fuzzy ℓ-separated iff $\lambda \not{q}\, C^\ell_\tau(\mu, r)$ and $\mu \not{q}\, C^\ell_\tau(\lambda, r)$.

(2) A fuzzy set which cannot be expressed as the union of two r-fuzzy ℓ-separated sets is said to be r-fuzzy ℓ-connected set.

Remark 4.4. r-fuzzy ℓ-separated (resp. r-fuzzy ℓ-connected) and r-fuzzy separated [1] (resp. r-fuzzy connected [1]) are independent notions because r-fuzzy ℓ-open and r-fuzzy open are independent notions.

Theorem 4.5. *Let (X, τ, ℓ) be a fuzzy ideal topological space and $r \in I_\circ$.*

(1) *If $\lambda, \mu \in I^X$ are r-fuzzy ℓ-separated and $\nu, \omega \in I^X - \{\underline{0}\}$ such that $\nu \leqslant \lambda$ and $\omega \leqslant \mu$, then ν, ω are also r-fuzzy ℓ-separated.*

(2) *If $\lambda \not{q} \mu$ and either both are r-fuzzy ℓ-open or both r-fuzzy ℓ-closed, then λ and μ are r-fuzzy ℓ-separated.*

(3) *If λ, μ are either both r-fuzzy ℓ-open or both r-fuzzy ℓ-closed, then $\lambda \wedge \mu^c$ and $\mu \wedge \lambda^c$ are r-fuzzy ℓ-separated.*

Proof. (1) and (2) are obvious. For (3) let λ and μ be r-fuzzy ℓ-open. Since $\lambda \wedge \mu^c \leqslant \mu^c$, $C^\ell_\tau(\lambda \wedge \mu^c, r) \leqslant \mu^c$ and hence $C^\ell_\tau(\lambda \wedge \mu^c, r) \not{q} \mu$. Then $C^\ell_\tau(\lambda \wedge \mu^c, r) \not{q} (\mu \wedge \lambda^c)$. Again, since $\mu \wedge \lambda^c \leqslant \lambda^c$, $C^\ell_\tau(\mu \wedge \lambda^c, r) \leqslant \lambda^c$ and hence $C^\ell_\tau(\mu \wedge \lambda^c, r) \not{q} \lambda$. Then $C^\ell_\tau(\mu \wedge \lambda^c, r) \not{q} (\lambda \wedge \mu^c)$. Thus $\lambda \wedge \mu^c$ and $\mu \wedge \lambda^c$ are r-fuzzy ℓ-separated. Similarly we can prove when λ and μ are r-fuzzy ℓ-closed. □

Theorem 4.6. *Let (X, τ, ℓ) be a fuzzy ideal topological space, $\lambda, \mu \in I^X - \{\underline{0}\}$ and $r \in I_\circ$. Then λ and μ are r-fuzzy ℓ-separated iff there exist two r-fuzzy ℓ-open sets ν, ω such that $\lambda \leqslant \nu$, $\mu \leqslant \omega$, $\lambda \not{q} \omega$ and $\mu \not{q} \nu$.*

Proof.

($\Rightarrow$) For two r-fuzzy ℓ-separated sets λ and μ, $\mu \leqslant (C^\ell_\tau(\lambda, r))^c = \omega$ (say) and $\lambda \leqslant (C^\ell_\tau(\mu, r))^c = \nu$ (say), where ω and ν are clearly r-fuzzy ℓ-open, then $\omega \not{q} C^\ell_\tau(\lambda, r)$ and $\nu \not{q} C^\ell_\tau(\mu, r)$. Thus, $\lambda \not{q} \omega$ and $\mu \not{q} \nu$.

($\Leftarrow$) Let ν and ω be r-fuzzy ℓ-open sets such that $\lambda \leqslant \nu$, $\mu \leqslant \omega$, $\lambda \not{q} \omega$ and $\mu \not{q} \nu$. Then $\lambda \leqslant \omega^c$, $\mu \leqslant \nu^c$. Hence $C^\ell_\tau(\lambda, r) \leqslant \omega^c$, $C^\ell_\tau(\mu, r) \leqslant \nu^c$, which in turn imply that $C^\ell_\tau(\lambda, r) \not{q} \mu$ and $C^\ell_\tau(\mu, r) \not{q} \lambda$. Thus λ and μ are r- fuzzy ℓ-separated. □

Theorem 4.7. *Let (X, τ, ℓ) be a fuzzy ideal topological space, $\lambda \in I^X - \{\underline{0}\}$ and $r \in I_\circ$. If λ is a r-fuzzy ℓ-connected set such that $\lambda \leqslant \mu \leqslant C^\ell_\tau(\lambda, r)$, then μ is also r-fuzzy ℓ-connected.*

Proof. Suppose that μ is not r-fuzzy ℓ-connected. Then there exist r-fuzzy ℓ-separated sets ω_1 and ω_2 in X such that $\mu = \omega_1 \vee \omega_2$. Let $\nu = \lambda \wedge \omega_1$ and $\omega = \lambda \wedge \omega_2$, $\lambda = \nu \vee \omega$. Since $\nu \leqslant \omega_1$ and $\omega \leqslant \omega_2$, by Theorem 4.5, ν and ω are r-fuzzy ℓ-separated, contradicting the r-fuzzy ℓ-connectedness of λ. Thus μ is r-fuzzy ℓ-connected. □

5. Conclusion

In this paper, we have continued to study the continuity of fuzzy multifunctions via fuzzy ideals. First, we defined the the concepts of r-fuzzy δ-ℓ-open and r-fuzzy strong β-ℓ-open sets. Some properties and relationships between these sets are discussed with the help of examples. Second, we defined the the concepts of fuzzy upper and lower δ-ℓ-continuous (resp. strong β-ℓ-continuous) multifunctions and some properties of these multifunctions along with their mutual relationships are established. Also, we give the decomposition of fuzzy upper (resp. lower) semi-ℓ-continuity and the decomposition of fuzzy upper (resp. lower) α-ℓ-continuity. In the end, we introduced and studied a new form of r-fuzzy connected set called r-fuzzy ℓ-connected via fuzzy ideals. We hope that the findings in this paper will help researcher enhance and promote the further study on continuity of fuzzy multifunctions to carry out a general framework for their applications in practical life.

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