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Solving fuzzy differential equations by Runge-Kutta method

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Abstract
In this paper, we interpret a fuzzy differential equation (FDE) by using the strongly generalized differentiability concept. Then we show that by this concept any FDE can be transformed to a system of ordinary differential equations (ODEs). Next by solving the associate ODEs we will find two solutions for FDE. Here we express the generalized Runge-Kutta approximation method of order two and analyze its error. Finally one example in the nuclear decay equation show the rich behavior of the method.

Keywords: fuzzy differential equation, generalized differentiability, generalized Runge-Kutta method.

1. Introduction

Knowledge about the behavior of differential equation is often incomplete or vague. For example, values of parameter, functional relationship or initial conditions, may not be known

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precisely.The concept of fuzzy derivative was first introduced by Chang and Zadeh in [9]. It was followed up by Dubois and Prade in [11], who defined and used the extension principle. Other methods have been discussed by Puri and Ralescu in [19] and Goetshel and Voxman in [12].

The initial value problem for fuzzy differential equation (FIVP)has been studied by Kaleva in [14,15] and by Seikkala in [20].

There are different approaches to the study of fuzzy differential equations. First approach uses H-derivative or its generalization, the Hukuhara derivative. This approach has the disadvantage that it leads to solutions with increasing support, fact which is solved by interpreting a FDE as a system of differential inclusions (see e.g. [13,10]).

The strongly generalized differentiability was introduced in [5] and studied in [6,8]. This concept allows us to resolve the above mentioned shortcoming.Indeed, the strongly generalized derivative is defined for a larger class of fuzzy number valued function than the H-derivative. So, we use this differentiability concept in the present paper.

Under appropriate conditions, the fuzzy initial value problem considered under this interpretation has locally two solutions [6].

The topics of numerical methods for solving FDE have been rapidly growing in recent years.

Hüllermeier in [13] obtained numerical solution of an FDE, by extending the existing classical methods to the fuzzy case. Some numerical methods for FDE under Hukuhara differentiability concept such as the fuzzy Euler method, predictor-corrector method, Taylor method and Nyström method are presented in [1-3,16]. The local existence of two solutions of an FDE under generalized differentiability implies that we present new numerical methods.

In [4], Bede proved a characterization theorem which states that under certain conditions a FDE under Hukuhara differentiability is equivalent to a system of ODEs. Bede also remarked that this theorem can help to solve FDEs numerically by converting them to a system of ODEs, which can then be solved by any numerical method suitable for ODEs.

In [18], the authors extended Bede's characterization theorem to generalized derivatives and then used that result to solve FDE numerically by Euler method for ODEs under strongly generalized differentiability.

In this paper, after preliminary section, we study FDE under this concept of differentiability and present the generalized characterization theorem. In section 4, we extend Runge-Kutta method expressed on [7] for solving ODEs under strongly generalized differentiability and then use for solving FDE numerically.

2. Preliminaries

In this section, we give some definitions and introduce the necessary notation which will be used throughout the paper.

Definition 2.1. Let X be a nonempty set. A fuzzy set u in $X$is characterized by its membership function$u: X \to [0,1]$.Then $u(x)$ is interpreted as the degree of membership of a element x in the fuzzy set u for each $x \in X$. Let us denote by $\mathbb{R}_F$ the class of fuzzy subsets of the real axes (i.e. $u: \mathbb{R} \to [0,1]$) satisfying the following properties:

(i) $\forall u \in \mathbb{R}_F$, u is normal, i.e. $\exists x_0 \in \mathbb{R}$ with $u(x_0) = 1$;
(ii) $\forall u \in \mathbb{R}_F$, u is convex fuzzy set(i.e. $u(tx + (1-t)y) \geq \min\{u(x), u(y)\}, \forall t \in [0,1], x, y \in \mathbb{R}$);
(iii) $\forall u \in \mathbb{R}_F$, u is upper semicontinuous on $\mathbb{R}$;
(iv) $cl\{x \in \mathbb{R}; u(x) > 0\}$ is compact, where $cl(A)$ denotes the closure of subset A.

Then $\mathbb{R}_F$ is called the space of fuzzy numbers. Obviously $\mathbb{R} \subset \mathbb{R}_F$. For $0 < \alpha \leq 1$ denote $[u]^{\propto} = \{x \in \mathbb{R}; u(x) \geq \propto\}$ and $[u]^0 = cl\{x \in \mathbb{R}; u(x) > 0\}$. Then it is well-known that for each $\propto \in [0,1]$, $[u]^{\propto}$ is a bounded closed interval.

For $u, v \in \mathbb{R}_F$, $\lambda \in \mathbb{R}$, the sum $u + v$ and $\lambda . u$ are defined by $[u+v]^{\propto} = [u]^{\propto} + [v]^{\propto}$, $[\lambda . u]^{\propto} = \lambda [u]^{\propto}$, $\forall \propto \in [0,1]$, where $[u]^{\propto} + [v]^{\propto}$ means the usual addition of two intervals of $\mathbb{R}$ and $\lambda [u]^{\propto}$ means the usual product between a scalar and a subset of $\mathbb{R}$.

The metric structure is given by the Hausdorff distance

$$D: \mathbb{R}_F \times \mathbb{R}_F \to \mathbb{R}_+ \cup \{0\}$$

$$D(u, v) = sup_{\propto \in [0,1]} \max\{|\underline{u}^{\propto} - \underline{v}^{\propto}|, |\overline{u}^{\propto} - \overline{v}^{\propto}|\}$$

where $[u]^{\propto} = [\underline{u}^{\propto}, \overline{u}^{\propto}]$, $[v]^{\propto} = [\underline{v}^{\propto}, \overline{v}^{\propto}]$, $(\mathbb{R}_F, D)$ is a complete space and the following properties are well-known:

$$D(u + w, v + w) = D(u, v), \forall u, v, w \in \mathbb{R}_F,$$
$$D(k.u, k.v) = |k| D(u, v), \forall k \in \mathbb{R}, u, v \in \mathbb{R}_F,$$
$$D(u + v, w + e) \leq D(u, w) + D(v, e), \forall u, v, w \in \mathbb{R}_F.$$

Definition 2.2. Let $x, y \in \mathbb{R}_F$. If there exists $z \in \mathbb{R}_F$ such that $x = y + z$, then z is called the H-difference of x, y and it is denoted by $x \ominus y$.

Note that $x \ominus y \neq x + (-1)y = x - y$. In what follows,we fix $I = (a, b)$, for $a, b \in \mathbb{R}$.

Bede in [6] introduced a more general definition of a derivative for a fuzzy-number-valued function. In this paper we consider the following definition [8]:

Definition 2.3. Let $f: I \to \mathbb{R}_F$ be given. Fix $t_0 \in I$. We say f is (1)-differentiable at t_0 and its derivative denoted by $D_1 f$, if there exists an element $f'(t_0) \in \mathbb{R}_F$ such that for all $h > 0$ sufficiently small, there exist $f(t_0 + h) \ominus f(t_0)$, $f(t_0) \ominus f(t_0 - h)$ and the following limits (in metric Hausdorff):

$$\lim_{h \to 0^+} \frac{f(t_0 + h) \ominus f(t_0)}{h} = \lim_{h \to 0^+} \frac{f(t_0) \ominus f(t_0 - h)}{h} = f'(t_0).$$

Similarly a function f is (2)-differentiable at t_0 and its derivative denoted by $D_2 f$, if there exists an element $f'(t_0) \in \mathbb{R}_F$ such that for all $h > 0$ sufficiently small, there exist $f(t_0 + h) \ominus f(t_0)$, $f(t_0) \ominus f(t_0 - h)$ and the following limits:

$$\lim_{h \to 0^-} \frac{f(t_0 + h) \ominus f(t_0)}{h} = \lim_{h \to 0^-} \frac{f(t_0) \ominus f(t_0 - h)}{h} = f'(t_0).$$

Theorem 2.4. Let $F: I \to \mathbb{R}_F$ and put $[F(t)]^{\propto} = [f_{\propto}(t), g_{\propto}(t)]$ for each $\propto \in [0,1]$.

(i) If F is (1)-differentiable then $f_{\propto}$ and $g_{\propto}$ are differentiable functions and $[D_1 F(t)]^{\propto} = [f'_{\propto}(t), g'_{\propto}(t)]$.
(ii) If F is (2)-differentiable then $f_{\propto}$ and $g_{\propto}$ are differentiable functions and $[D_2 F(t)]^{\propto} = [g'_{\propto}(t), f'_{\propto}(t)]$.

Proof. See [8].

3. Generalized characterization theorem

Let us consider the FDE with initial value condition:

$$x^{'}(t) = f(t,x), \quad x(t_0) = x_0 \tag{1}$$

where $f:[t_0,T] \times \mathbb{R}_F \to \mathbb{R}_F$ is a continuous fuzzy mapping and$x_0 \in \mathbb{R}_F$ and T is positive number or infinity.

Theorem 3.1. Let $f:[t_0,T] \times \mathbb{R}_F \to \mathbb{R}_F$ is a continuous fuzzy function. If there exists $k > 0$ such that $D\big(f(t,x), f(t,z)\big) \leq kD(x,z), \forall t \in I,\ x,y \in \mathbb{R}_F$. Then the problem (1) has two solutions on I. One is (1)-differentiable solution and the other one is (2)-differentiable solution.

Proof. See[8].

Definition 3.2. Let $y: I \to \mathbb{R}_F$ be a fuzzy function such that $D_1 y$or$D_1 y$ exists. If y and $D_1 y$ satisfy problem (1), we say y is a (1)-solution of problem (1). Similarly, if y and $D_2 y$ satisfy problem (1), we say y is a (2)-solution of problem (1).

By using theorem 2.4 we can state useful approach for solving FIVP:
Let us suppose α-cut of functions $x(t), x_0, f(t,x)$ are the following form:

$$[x(t)]^{\propto} = [\underline{x}_{\propto}(t), \overline{x}_{\propto}(t)],$$
$$[x_0]^{\propto} = [\underline{x_0}, \overline{x_0}],$$
$$[f(t,x(t))]^{\propto} = [\underline{f}_{\propto}\big(t, \underline{x}_{\propto}, \overline{x}_{\propto}\big), \overline{f}_{\propto}\big(t, \underline{x}_{\propto}, \overline{x}_{\propto}\big)],$$

Then we have two following cases:

Case (I): if $x(t)$ is (1)-differentiable then solvingFIVP (1) translates into the following algorithm:

step (i) solving the following system of ODEs:

$$\begin{cases} \underline{x}'_{\propto}(t) = \underline{f}_{\propto}\big(t, \underline{x}_{\propto}, \overline{x}_{\propto}\big) = F\big(t, \underline{x}, \overline{x}\big), & \underline{x}(t_0) = \underline{x_0} \\ \overline{x}^{'}_{\propto}(t) = \overline{f}_{\propto}\big(t, \underline{x}_{\propto}, \overline{x}_{\propto}\big) = G\big(t, \underline{x}, \overline{x}\big), & \overline{x}(t_0) = \overline{x_0} \end{cases} \tag{2}$$

step (ii) ensure that the solution $[\underline{x}_{\propto}(t), \overline{x}_{\propto}(t)]$ and $[\underline{x}'_{\propto}(t), \overline{x}'_{\propto}(t)]$ are valid level sets.

step (iii) by using the representation theorem again, we construct a (1)-solution $x(t)$ such that $[x(t)]^{\propto} = [\underline{x}_{\propto}(t), \overline{x}_{\propto}(t)]$, for all $\propto \in [0,1]$.

Case (II): if $x(t)$ is (2)-differentiable then solvingFIVP (1) translates into the following algorithm:

step (i) solving the following system of ODEs:

$$\begin{cases} \underline{x}^{'}_{\propto}(t) = \overline{f}_{\propto}\big(t, \underline{x}_{\propto}, \overline{x}_{\propto}\big) = G\big(t, \underline{x}, \overline{x}\big), & \underline{x}(t_0) = \underline{x_0} \\ \overline{x}^{'}_{\propto}(t) = \underline{f}_{\propto}\big(t, \underline{x}_{\propto}, \overline{x}_{\propto}\big) = F\big(t, \underline{x}, \overline{x}\big), & \overline{x}(t_0) = \overline{x_0} \end{cases} \tag{3}$$

step (ii) ensure that the solution $[\underline{x}_{\propto}(t), \overline{x}_{\propto}(t)]$ and $[\underline{x}'_{\propto}(t), \overline{x}'_{\propto}(t)]$ are valid level sets.

step (iii) by using the representation theorem again, we construct a (2)-solution $x(t)$ such that $[x(t)]^{\propto} = [\underline{x}_{\propto}(t), \overline{x}_{\propto}(t)]$, for all $\propto \in [0,1]$.

Now we extend Bede's characterization theorem [4] to fuzzy differential equation under generalized differentiability:

Theorem 3.3. Let us consider the FIVP (1) where $f: I \times \mathbb{R}_F \to \mathbb{R}_F$ is such that

(i) $[f(t,x)]^{\propto} = [\underline{f}_{\propto}(t, \underline{x}_{\propto}, \overline{x}_{\propto}), \overline{f}_{\propto}(t, \underline{x}_{\propto}, \overline{x}_{\propto})]$

(ii) $\underline{f}_{\propto}$, $\overline{f}_{\propto}$ are equicontinuous.

(iii) there exists $L > 0$ such that:

$$\left|\underline{f}_{\propto}(t, x_1, y_1) - \underline{f}_{\propto}(t, x_2, y_2)\right| \leq L max\{|x_1 - x_2|, |y_1 - y_2|\}, \ \forall \propto \in [0,1];$$
$$\left|\overline{f}_{\propto}(t, x_1, y_1) - \overline{f}_{\propto}(t, x_2, y_2)\right| \leq L max\{|x_1 - x_2|, |y_1 - y_2|\}, \quad \forall \propto \in [0,1];$$

Then for(1)-differentiability, the FIVP (1) and the system of ODEs(2) are equivalent and in (2)-differentiability, the FIVP (1) and the system of ODEs(3) are equivalent.

Proof. In the paper [4], the authors proved for (1)-differentiability. The result for (2)-differentiability is obtained analogously by using theorem 2.4.

4. Runge-Kutta method for FDE

In this section we present Runge-Kutta method for solving (1) by the generalized characterization theorem. Here we state the existence theorem for FDE:

Theorem 4.1. Under appropriate conditions, the FIVP (1) considered under generalized differentiability has locally two solutions, and the successive iterations

$$x(0) = x_0, \qquad x_{n+1}(t) = x_0 + \int_{t_0}^{T} f(s, x_n(s))ds$$

and

$$x(0) = x_0, \qquad x_{n+1}(t) = x_0 \ominus (-1)\int_{t_0}^{T} f(s, x_n(s))ds$$

converge to the (1)-solution and the (2)-solution, respectively.

Proof. The authors of [21] proved for (1)-differentiability. The result for (2)-differentiability is obtained in [6].

Based on the generalized characterization theorem, we replace the fuzzy differential equation with its equivalent system and then, for approximating the two fuzzy solutions, we solve numerically two ODE systems which consist of four classic ordinary differential equations with initial conditions.

Now we extend Runge-Kutta method in [7] for finding two fuzzy solutions of FDEs under generalized differentiability.We consider the partition P for interval $[t_0, T]$

$$P: \ t_0 = a_0 < a_1 < \cdots < a_N = T,$$
$$a_i = a_0 + ih, \ \ h = \frac{T - t_0}{N}.$$

Suppose two exact solutions $[Y_1(t)]^{\propto} = \left[\underline{Y_1}(t,\propto), \overline{Y_1}(t,\propto)\right]$ and $[Y_2(t)]^{\propto} = [\underline{Y_2}(t,\propto), \overline{Y_2}(t,\propto)]$are approximated by some$[y_1(t)]^{\propto} = \left[\underline{y_1}(t,\propto), \overline{y_1}(t,\propto)\right]$, $[y_2(t)]^{\propto} = [\underline{y_2}(t,\propto), \overline{y_2}(t,\propto)]$, respectively.

The exact and approximate solution at grid point $a_i, 0 \leq i \leq N$are denoted by $Y_{1_n}(\propto), Y_{2_n}(\propto), y_{1_n}(\propto)$ and $y_{2_n}(\propto)$, respectively .

The generalized Runge-Kutta method based on the second order approximation of $\underline{Y_1}'(t,\propto), \overline{Y_1}'(t,\propto), \underline{Y_2}'(t,\propto), \overline{Y_2}'(t,\propto)$ and equations (2) and (3) is obtained as follows:

$$\begin{cases} \underline{y_1}_{n+1}(\propto) = \underline{y_1}_n(\propto) + \left(1-\frac{1}{2\theta}\right)hF\left(t_n, \underline{y_1}_n(\alpha), \overline{y_1}_n(\alpha)\right) + \left(\frac{1}{2\theta}\right)hF\left(t_n+\theta h, \underline{z_1}^{\alpha}_{n+1}, \overline{z_1}^{\alpha}_{n+1}\right) \\ \overline{y_1}_{n+1}(\propto) = \overline{y_1}_n(\propto) + \left(1-\frac{1}{2\theta}\right)hG\left(t_n, \underline{y_1}_n(\propto), \overline{y_1}_n(\propto)\right) + \left(\frac{1}{2\theta}\right)hG\left(t_n+\theta h, \underline{z_1}^{\alpha}_{n+1}, \overline{z_1}^{\alpha}_{n+1}\right) \\ \underline{y_1}_0(\propto) = \underline{y_0}(\propto) \\ \overline{y_1}_0(\propto) = \overline{y_0}(\propto) \end{cases} \quad (4)$$

$$\begin{cases} \underline{z_1}^{\propto}_{n+1} = \underline{y_1}_n(\propto) + \theta hF\left(t_n, \underline{y_1}_n(\alpha), \overline{y_1}_n(\alpha)\right) \\ \overline{z_1}^{\alpha}_{n+1} = \overline{y_1}_n(\alpha) + \theta hG\left(t_n, \underline{y_1}_n(\alpha), \overline{y_1}_n(\alpha)\right) \end{cases} \quad (5)$$

$$\begin{cases} \underline{y_2}_{n+1}(\propto) = \underline{y_2}_n(\propto) + \left(1-\frac{1}{2\theta}\right)hG\left(t_n, \underline{y_2}_n(\alpha), \overline{y_2}_n(\alpha)\right) + \left(\frac{1}{2\theta}\right)hG\left(t_n+\theta h, \underline{z_2}^{\alpha}_{n+1}, \overline{z_2}^{\alpha}_{n+1}\right) \\ \overline{y_2}_{n+1}(\propto) = \overline{y_2}_n(\propto) + \left(1-\frac{1}{2\theta}\right)hF\left(t_n, \underline{y_2}_n(\propto), \overline{y_2}_n(\propto)\right) + \left(\frac{1}{2\theta}\right)hF\left(t_n+\theta h, \underline{z_2}^{\alpha}_{n+1}, \overline{z_2}^{\alpha}_{n+1}\right) \\ \underline{y_2}_0(\propto) = \underline{y_0}(\propto) \\ \overline{y_2}_0(\propto) = \overline{y_0}(\propto) \end{cases} \quad (6)$$

$$\begin{cases} \underline{z_2}^{\propto}_{n+1} = \underline{y_2}_n(\propto) + \theta hG\left(t_n, \underline{y_2}_n(\alpha), \overline{y_2}_n(\alpha)\right) \\ \overline{z_2}^{\alpha}_{n+1} = \overline{y_2}_n(\alpha) + \theta hF\left(t_n, \underline{y_2}_n(\alpha), \overline{y_2}_n(\alpha)\right) \end{cases} \quad (7)$$

Lemma 4.2. [17] let the sequences of numbers $\{W_n\}_{n=0}^{\infty}$, $\{V_n\}_{n=0}^{\infty}$ satisfy

$$|W_{n+1}| \leq |W_n| + A\max\{|W_n|, |V_n|\} + B,$$
$$|V_{n+1}| \leq |V_n| + A\max\{|W_n|, |V_n|\} + B.$$

for some given positive constants A and B , and denote $U_n = |W_n| + |V_n|, 0 \leq n \leq N$, then

$$U_n \leq (1+2A)^n U_0 + 2B\frac{(1+2A)^n - 1}{(1+2A)-1}, 1 \leq n \leq N.$$

The following theorem shows that the generalized Runge-Kutta approximation pointwise converge to the exact solutions. Let $F(t,u,v)$ and $G(t,u,v)$ be the functions F and G of equations (2) and (3), where u and v are constants and $u \leq v$.The domain where F and G are defined is therefore:

$K = \{(t,u,v)|0 \le t \le A, -\infty < v < \infty, -\infty < u \le v\}$.

Theorem 4.3. Let $F(t,u,v)$ and $G(t,u,v)$ belong to $C^1(K)$ and let the partial derivatives of F and G be bounded over K. Then for arbitrary fixed $\propto: \propto \in [0,1]$ the generalized Runge-Kutta approximation of Eqs. (4) and (6) converge to the exact solution $Y_1(t,\propto)$, $Y_2(t,\propto)$ uniformly in it.

Proof. If we consider (1)-differentiability, then for convergence of Eq. (4) similar to [17] is sufficient to show:

$$\lim_{h\to 0} \underline{y_1}_N(\propto) = \underline{Y_1}(t,\propto), \quad \lim_{h\to 0} \overline{y_1}_N(\propto) = \overline{Y_1}(t,\propto).$$

by using the Taylor theorem, we have:

$$\underline{Y_1}_{n+1}(\alpha) = \underline{Y_1}_n(\alpha) + \left(1-\frac{1}{2\theta}\right) hF\left[t_n, \underline{Y_1}_n(\alpha), \overline{Y}_{1n}(\alpha)\right] + \left(\frac{1}{2\theta}\right) hF\left[t_n+\theta h, \underline{Z_1}_{n+1}{}^{\alpha}, \overline{Z}_{1n+1}{}^{\alpha}\right] + \frac{h^3}{6}\underline{Y_1}'''\left(\underline{\xi_1}_n\right)$$

and

$$\overline{Y}_{1n+1}(\alpha) = \overline{Y}_{1n}(\alpha) + \left(1-\frac{1}{2\theta}\right) hG\left[t_n, \underline{Y_1}_n(\alpha), \overline{Y}_{1n}(\alpha)\right] + \left(\frac{1}{2\theta}\right) hG\left[t_n+\theta h, \underline{Z_1}_{n+1}{}^{\alpha}, \overline{Z}_{1n+1}{}^{\alpha}\right] + \frac{h^3}{6}\overline{Y_1}'''\left(\overline{\xi}_{1n}\right).$$

where $t_n \le \underline{\xi_1}_n, \overline{\xi}_{1n} \le t_{n+1}$. Then we have:

$$\underline{Y_1}_{n+1}(\alpha) - \underline{y_1}_{n+1}(\alpha) = \underline{Y_1}_n(\alpha) - \underline{y_1}_n(\alpha) + \left(1-\frac{1}{2\theta}\right) h\left\{F\left[t_n, \underline{Y_1}_n(\alpha), \overline{Y}_{1n}(\alpha)\right] - F\left[t_n, \underline{y_1}_n(\alpha), \overline{y}_{1n}(\alpha)\right]\right\} + \left(\frac{1}{2\theta}\right) h\left\{F\left[t_n+\theta h, \underline{Z_1}_{n+1}{}^{\alpha}, \overline{Z}_{1n+1}{}^{\alpha}\right] - F\left[t_n+\theta h, \underline{z_1}_{n+1}{}^{\alpha}, \overline{z}_{1n+1}{}^{\alpha}\right]\right\} + \frac{h^3}{6}\underline{Y_1}'''\left(\underline{\xi_1}_n\right)$$

and

$$\begin{aligned}\overline{Y}_{1\,n+1}(\alpha)-\overline{y}_{1\,n+1}(\alpha)&=\overline{Y}_{1\,n}(\alpha)-\overline{y}_{1\,n}(\alpha)\\&+\left(1-\frac{1}{2\theta}\right)h\left\{G\left[t_n,\underline{Y_1}_n(\alpha),\overline{Y}_{1\,n}(\alpha)\right]-G\left[t_n,\underline{y_1}_n(\alpha),\overline{y}_{1\,n}(\alpha)\right]\right\}\\&+\left(\frac{1}{2\theta}\right)h\left\{G\left[t_n+\theta h,\underline{Z_1}_{n+1}{}^{\alpha},\overline{Z}_{1\,n+1}{}^{\alpha}\right]-G\left[t_n+\theta h,\underline{z_1}_{n+1}{}^{\alpha},\overline{z}_{1\,n+1}{}^{\alpha}\right]\right\}\\&+\frac{h^3}{6}\overline{Y}_1'''\left(\overline{\xi}_{1\,n}\right)\end{aligned}$$

Similarly we have:

$$\begin{aligned}\underline{Z_1}_{n+1}(\alpha)-\underline{z_1}_{n+1}(\alpha)&=\underline{Y_1}_n(\alpha)-\underline{y_1}_n(\alpha)\\&+\theta h\left\{F\left[t_n,\underline{Y_1}_n(\alpha),\overline{Y}_{1\,n}(\alpha)\right]-F\left[t_n,\underline{y_1}_n(\alpha),\overline{y}_{1\,n}(\alpha)\right]\right\}\\&+\frac{h^2}{2}\underline{Y_1}''\left(\underline{\eta_1}_n\right).\end{aligned}$$

and

$$\begin{aligned}\overline{Z}_{1\,n+1}(\alpha)-\overline{z}_{1\,n+1}(\alpha)&=\overline{Y}_{1\,n}(\alpha)-\overline{y}_{1\,n}(\alpha)\\&+\theta h\left\{G\left[t_n,\underline{Y_1}_n(\alpha),\overline{Y}_{1\,n}(\alpha)\right]-G\left[t_n,\underline{y_1}_n(\alpha),\overline{y}_{1\,n}(\alpha)\right]\right\}\\&+\frac{h^2}{2}\overline{Y}_1''\left(\overline{\eta}_{1\,n}\right).\end{aligned}$$

where $t_n\le\underline{\eta_1}_n,\overline{\eta}_{1\,n}\le t_{n+1}$.

Now, we define $W_{1_n},V_{1_n},P_{1_n},T_{1_n}$ by the following terms:

$$\begin{aligned}&W_{1_n}=\underline{Y_1}_{n+1}(\alpha)-\underline{y_1}_{n+1}(\alpha),\quad V_{1_n}=\overline{Y}_{1\,n+1}(\alpha)-\overline{y}_{1\,n+1}(\alpha),\\&P_{1_n}=\underline{Z_1}_{n+1}(\alpha)-\underline{z_1}_{n+1}(\alpha),\quad T_{1_n}=\overline{Z}_{1\,n+1}(\alpha)-\overline{z}_{1\,n+1}(\alpha).\end{aligned}$$

Then we have:

$$\begin{aligned}&\left|W_{1_{n+1}}\right|\le\left|W_{1_n}\right|+\left(1-\frac{1}{2\theta}\right)2Lh\max\left\{\left|W_{1_n}\right|,\left|V_{1_n}\right|\right\}+\left(\frac{1}{2\theta}\right)2Lh\max\left\{\left|P_{1_{n+1}}\right|,\left|T_{1_{n+1}}\right|\right\}+\frac{h^3}{6}\underline{N_1}.\\&\left|V_{1_{n+1}}\right|\le\left|V_{1_n}\right|+\left(1-\frac{1}{2\theta}\right)2Lh\max\left\{\left|W_{1_n}\right|,\left|V_{1_n}\right|\right\}+\left(\frac{1}{2\theta}\right)2Lh\max\left\{\left|P_{1_{n+1}}\right|,\left|T_{1_{n+1}}\right|\right\}+\frac{h^3}{6}\overline{N_1}.\\&\left|P_{1_{n+1}}\right|\le\left|W_{1_n}\right|+2Lh\max\left\{\left|W_{1_n}\right|,\left|V_{1_n}\right|\right\}+\frac{h^2}{2}\underline{M_1}.\\&\left|T_{1_{n+1}}\right|\le\left|V_{1_n}\right|+2Lh\max\left\{\left|W_{1_n}\right|,\left|V_{1_n}\right|\right\}+\frac{h^2}{2}\overline{M_1}.\end{aligned}$$

where $\underline{N_1}=\sup\underline{Y_1}'''(t,\alpha)$, $\overline{N_1}=\sup\overline{Y_1}'''(t,\alpha)$, $\underline{M_1}=\sup\underline{Y_1}''(t,\alpha)$, $\overline{M_1}=\sup\overline{Y_1}''(t,\alpha)$ and $L>0$ is a bound for the partial derivatives of F,G.

By substitute $\left|P_{1_{n+1}}\right|, \left|T_{1_{n+1}}\right|$ in $\left|W_{1_{n+1}}\right|, \left|V_{1_{n+1}}\right|$, we have:

$$\left|W_{1_{n+1}}\right| \le \left|W_{1_n}\right| + \left(1-\frac{1}{2\theta}\right)2Lh\max\left\{\left|W_{1_n}\right|,\left|V_{1_n}\right|\right\}$$
$$+\left(\frac{1}{2\theta}\right)2Lh\max\left\{\max\left\{\left|W_{1_n}\right|,\left|V_{1_n}\right|\right\}+2Lh\theta\left(\max\left\{\left|W_{1_n}\right|,\left|V_{1_n}\right|\right\}\right)+\frac{h^2}{2}K_1\right\}$$
$$+\frac{h^3}{6}\underline{N_1}.$$

and

$$\left|V_{1_{n+1}}\right| \le \left|V_{1_n}\right| + \left(1-\frac{1}{2\theta}\right)2Lh\max\left\{\left|W_{1_n}\right|,\left|V_{1_n}\right|\right\}$$
$$+\left(\frac{1}{2\theta}\right)2Lh\max\left\{\max\left\{\left|W_{1_n}\right|,\left|V_{1_n}\right|\right\}+2Lh\theta\left(\max\left\{\left|W_{1_n}\right|,\left|V_{1_n}\right|\right\}\right)+\frac{h^2}{2}K_1\right\}$$
$$+\frac{h^3}{6}\overline{N_1}.$$

where $K_1 = \max\left\{\underline{M_1},\overline{M_1}\right\}$. Now the above term can abbreviate to the following:

$$\left|W_{1_{n+1}}\right| \le \left|W_{1_n}\right| + \frac{h^3}{6}\left(\underline{N_1}+\frac{3L}{\theta}K_1\right)$$
$$+\left(\max\left\{\left|W_{1_n}\right|,\left|V_{1_n}\right|\right\}\right)\left\{\left(1-\frac{1}{2\theta}\right)2Lh+\left(\frac{1}{2\theta}\right)2Lh(1+2Lh)\right\},$$

and

$$\left|V_{1_{n+1}}\right| \le \left|V_{1_n}\right| + \frac{h^3}{6}\left(\overline{N_1}+\frac{3L}{\theta}K_1\right)\left(1-\frac{1}{2\theta}\right)2Lh\max\left\{\left|W_{1_n}\right|,\left|V_{1_n}\right|\right\}$$
$$+\left(\max\left\{\left|W_{1_n}\right|,\left|V_{1_n}\right|\right\}\right)\left\{\left(1-\frac{1}{2\theta}\right)2Lh+\left(\frac{1}{2\theta}\right)2Lh(1+2Lh)\right\}.$$

Then by Lemma 4.2 we have:

$$\left|W_{1_n}\right| \le \left(1+4Lh(1+Lh)\right)^n\left|U_0\right| + \frac{h^3}{3}\left(\underline{N_1}+\frac{3L}{\theta}K_1\right)\frac{\left(1+4Lh(1+Lh)\right)^n-1}{4Lh(1+Lh)},$$
$$\left|V_{1_n}\right| \le \left(1+4Lh(1+Lh)\right)^n\left|U_0\right| + \frac{h^3}{3}\left(\overline{N_1}+\frac{3L}{\theta}K_1\right)\frac{\left(1+4Lh(1+Lh)\right)^n-1}{4Lh(1+Lh)}.$$

where $\left|U_0\right| = \left|W_{1_0}\right| + \left|V_{1_0}\right|$. In particular

$$\left|W_{1_N}\right| \le \left(1+4Lh(1+Lh)\right)^N \left|U_0\right| + \frac{h^3}{3}\left(\underline{N_1} + \frac{3L}{\theta}K_1\right)\frac{\left(1+4Lh(1+Lh)\right)^{(T-t_0)/h}-1}{4Lh(1+Lh)},$$

$$\left|V_{1_N}\right| \le \left(1+4Lh(1+Lh)\right)^N \left|U_0\right| + \frac{h^3}{3}\left(\overline{N_1} + \frac{3L}{\theta}K_1\right)\frac{\left(1+4Lh(1+Lh)\right)^{(T-t_0)/h}-1}{4Lh(1+Lh)}.$$

Since $W_{1_0} = V_{1_0} = 0$ and know for $\delta > -1$, relationship $e^{k\delta} > (1+\delta)^k$ satisfy, then by assumption $k = \frac{T-t_0}{h}$,
$\delta = 4Lh(1+Lh)$ we have:

$$\left|W_{1_N}\right| \le \frac{h^2}{3}\left(\underline{N_1} + \frac{3L}{\theta}K_1\right)\frac{e^{4L(1+Lh)(T-t_0)}-1}{4Lh(1+Lh)},$$

$$\left|V_{1_N}\right| \le \frac{h^2}{3}\left(\overline{N_1} + \frac{3L}{\theta}K_1\right)\frac{e^{4L(1+Lh)(T-t_0)}-1}{4Lh(1+Lh)}.$$

and if $h \to 0$ we get $W_N \to 0, V_N \to 0$ which concludes the proof. □

Now we will present an example to show that our method works.

Example 2.2. Let us consider the nuclear decay equation.

$$x'(t) = -\lambda \odot x(t), \quad x(t_0) = x_0,$$

where $x(t)$ is the number of radionuclides present in a given radioactive material, λ is the decay constant and x_0 is the initial number of radionuclides. In the model, uncertainty is introduced if we have uncertain information on the initial value x_0 of radionuclides present in the material. Note that the phenomenon of nuclear disintegration is considered a stochastic process, uncertainty being introducedby the lack of information on the radioactive material under study. In order to take into account the uncertainty we consider x_0 to be a fuzzy number.

Let $\lambda = 1$, $I = [0, 0.1]$ and $x_0 = [\alpha - 1, 1-\alpha]$ then we have:

$$F\left(t_n, \underline{y}(\propto), \overline{y}(\propto)\right) = -\overline{y}(\propto), \qquad G\left(t_n, \underline{y}(\propto), \overline{y}(\propto)\right) = -\underline{y}(\propto)$$

By using the formulation (2) we get exact solution

$$Y_1(t,\alpha) = [(\alpha-1)e^t, (1-\alpha)e^t]$$

That is a (1)-differentiable solution of the problem (1).
Using the formulation (3),

$$Y_2(t,\alpha) = [(\alpha-1)e^{-t}, (1-\alpha)e^{-t}],$$

is a (2)-differentiable solution of the problem (1). To get the generalized Runge-Kutta approximation we devide I into $N = 10$ equally spaced subintervals and calculate

$$\begin{cases} \underline{y_1}_{n+1}(\propto) = \underline{y_1}_n(\propto) - \left(1 - \frac{1}{2\theta}\right) h\overline{y_1}_n(\alpha) - \left(\frac{1}{2\theta}\right) h\overline{z_1}^{\alpha}_{n+1} \\ \overline{y_1}_{n+1}(\propto) = \overline{y_1}_n(\propto) - \left(1 - \frac{1}{2\theta}\right) h\underline{y_1}_n(\propto) - \left(\frac{1}{2\theta}\right) h\underline{z_1}^{\alpha}_{n+1} \\ \quad \underline{y_1}_0(\propto) = \underline{y_0}(\propto) \\ \quad \overline{y_1}_0(\propto) = \overline{y_0}(\propto) \end{cases}$$

$$\begin{cases} \underline{z_1}^{\propto}_{n+1} = \underline{y_1}_n(\propto) - \theta h\overline{y_1}_n(\alpha) \\ \overline{z_1}^{\alpha}_{n+1} = \overline{y_1}_n(\alpha) - \theta h\underline{y_1}_n(\alpha) \end{cases}$$

for finding the (1)-solution and compute

$$\begin{cases} \underline{y_2}_{n+1}(\propto) = \underline{y_2}_n(\propto) - \left(1 - \frac{1}{2\theta}\right) h\underline{y_2}_n(\alpha) - \left(\frac{1}{2\theta}\right) h\underline{z_2}^{\alpha}_{n+1} \\ \overline{y_2}_{n+1}(\propto) = \overline{y_2}_n(\propto) - \left(1 - \frac{1}{2\theta}\right) h\overline{y_2}_n(\propto) - \left(\frac{1}{2\theta}\right) h\overline{z_2}^{\alpha}_{n+1} \\ \underline{y_2}_0(\propto) = \underline{y_0}(\propto) \\ \overline{y_2}_0(\propto) = \overline{y_0}(\propto) \end{cases}$$

$$\begin{cases} \underline{z_2}^{\propto}_{n+1} = \underline{y_2}_n(\propto) - \theta h\underline{y_2}_n(\alpha) = (1 - \theta h)\underline{y_2}_n(\alpha) \\ \overline{z_2}^{\alpha}_{n+1} = \overline{y_2}_n(\alpha) - \theta h\overline{y_2}_n(\alpha) = (1 - \theta h)\overline{y_2}_n(\alpha) \end{cases}$$

for finding (2)-solution.

By substituting $\underline{z_1}^{\propto}_{n+1}$, $\overline{z_1}^{\alpha}_{n+1}$, $\underline{z_2}^{\propto}_{n+1}$ and $\overline{z_2}^{\alpha}_{n+1}$ in $\underline{y_1}_{n+1}(\propto)$, $\overline{y_1}_{n+1}(\propto)$, $\underline{y_2}_{n+1}(\propto)$ and $\overline{y_2}_{n+1}(\propto)$, we have:

$$\begin{cases} \underline{y_1}_{n+1}(\propto) = \left(1 + \frac{h^2}{2}\right)\underline{y_1}_n(\propto) - h\overline{y_1}_n(\propto), \\ \overline{y_1}_{n+1}(\propto) = -h\underline{y_1}_n(\propto) + \left(1 + \frac{h^2}{2}\right)\overline{y_1}_n(\propto). \end{cases}$$

$$\begin{cases} \underline{y_2}_{n+1}(\propto) = \left(1 - h + \frac{h^2}{2}\right)\underline{y_2}_n(\propto), \\ \overline{y_2}_{n+1}(\propto) = \left(1 - h + \frac{h^2}{2}\right)\overline{y_2}_n(\propto). \end{cases}$$

A comparison between the exact and the approximate solutions at $t = 0.1$and the error of generalized Runge-Kutta and Euler method is shown in the following tables and figures 1 and 2.

Table 1.

∝	$\underline{y}_1$	$\underline{Y}_1$	Runge-Kutta Error	Euler Error	$\overline{y}_1$	$\overline{Y}_1$	Runge-Kutta Error	Euler Error
0	-1.10516909	-1.10517092	-1.828191e-6	-5.487927e-4	1.10516909	1.10517092	1.828191e-6	5.487927e-4
0.1	-0.99465218	-0.99465383	-1.645372e-6	-4.939134e-4	0.99465218	0.99465383	1.645372e-6	4.939134e-4
0.2	-0.88423527	-0.88413673	-1.462553e-6	-4.390341e-4	0.88423527	0.88413673	1.462553e-6	4.390341e-4
0.3	-0.77361836	-0.77361964	-1.279733e-6	-3.841549e-4	0.77361836	0.77361964	1.279733e-6	3.841549e-4
0.4	-0.66310145	-0.66310255	-1.096914e-6	-3.292756e-4	0.66310145	0.66310255	1.096914e-6	3.292756e-4
0.5	-0.55258454	-0.55258546	-9.140953e-7	-2.743963e-4	0.55258454	0.55258546	9.140953e-7	2.743963e-4
0.6	-0.44206764	-0.44206837	-7.312763e-7	-2.195171e-4	0.44206764	0.44206837	7.312763e-7	2.195171e-4
0.7	-0.33155073	-0.33155128	-5.484572e-7	-1.646378e-4	0.33155073	0.33155128	5.484572e-7	1.646378e-4
0.8	-0.22103382	-0.22103418	-3.656381e-7	-1.097585e-4	0.22103382	0.22103418	3.656381e-7	1.097585e-4
0.9	-0.11051691	-0.11051709	-1.828191e-7	-5.487927e-5	0.11051691	0.11051709	1.828191e-7	5.487927e-5
1	0	0	0	0	0	0	0	0

Table 2.

∝	$\underline{y}_2$	$\underline{Y}_2$	Runge-Kutta Error	Euler Error	$\overline{y}_2$	$\overline{Y}_2$	Runge-Kutta Error	Euler Error
0	-0.90483894	-0.90483742	-1.828191e-6	-4.553430e-4	0.90483894	0.90483742	1.828191e-6	4.553430e-4
0.1	-0.81435504	-0.81435368	-1.645372e-6	-4.098087e-4	0.81435504	0.81435368	1.645372e-6	4.098087e-4
0.2	-0.72387115	-0.72386993	-1.462553e-6	-3.642744e-4	0.72387115	0.72386993	1.462553e-6	3.642744e-4
0.3	-0.63338726	-0.63338619	-1.279733e-6	-3.187401e-4	0.63338726	0.63338619	1.279733e-6	3.187401e-4
0.4	-0.54290336	-0.54290245	-1.096914e-6	-2.732058e-4	0.54290336	0.54290245	1.096914e-6	2.732058e-4
0.5	-0.45241947	-0.45241871	-9.140953e-7	-2.276715e-4	0.45241947	0.45241871	9.140953e-7	2.276715e-4
0.6	-0.36193557	-0.36193497	-7.312763e-7	-1.821372e-4	0.36193557	0.36193497	7.312763e-7	1.821372e-4
0.7	-0.27145268	-0.27145123	-5.484572e-7	-1.366029e-4	0.27145268	0.27145123	5.484572e-7	1.366029e-4
0.8	-0.18096779	-0.18096748	-3.656381e-7	-9.106861e-5	0.18096779	0.18096748	3.656381e-7	9.106861e-5
0.9	-0.09048389	-0.09048374	-1.828191e-7	-4.553430e-5	0.09048389	0.09048374	1.828191e-7	4.553430e-5
1	0	0	0	0	0	0	0	0

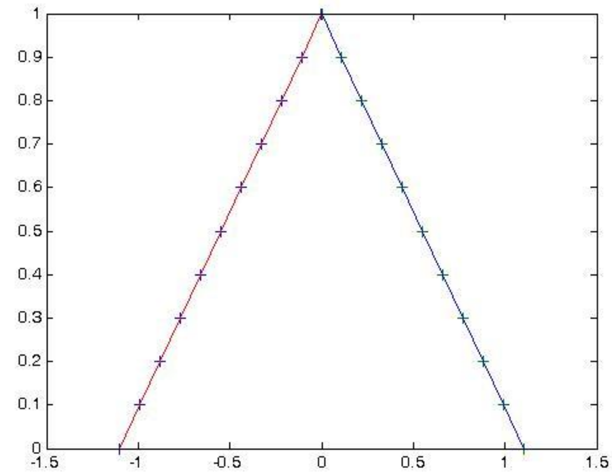

Figure 1. (-) exact (1)-solution, (+) approximated points using Hukuhara differentiability

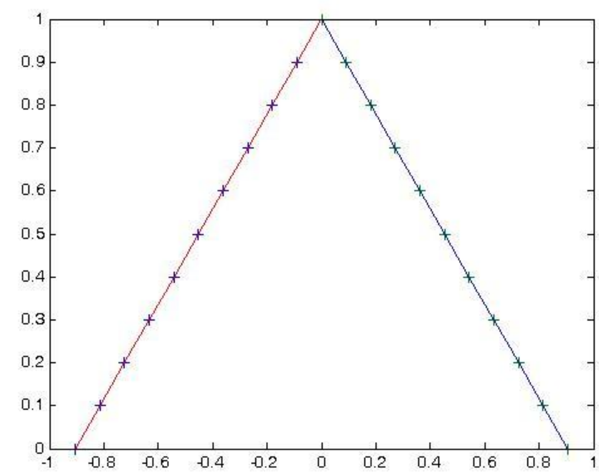

Figure 2. (-) exact (1)-solution, (+) approximated points using (2)-differentiability

Now, if consider the same differential equation under Hukuhara differentiability, then the (1)-solution(it exists and unique by theorems in [21]) has an increasing length of its support, which leads us to the conclusion that there is a possibility that the radioactivity of the system increases as time goes on and even a non-zero possibility that it is negative! Fortunately, the real situation is different, and the radioactivity of a material always decreases with time and it cannot be negative. Then we conclude that the second solution is more efficient than the first one and (2)-solution models the radioactive decay better. This is an advantage of the generalized differentiability that allows us to select better solution.

Also,by comparison the errors of generalized Runge-Kutta and Euler methods in tables 1 and 2 we observe that the error of generalized Runge-Kutta method less than the generalized Euler method. That is the generalized Runge-Kutta method is better than generalized Euler method.

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