Expressions and dynamical behavior of solutions of a class of rational difference equations of fifteenth-order

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Abstract

The main goal of this paper, is to obtain the forms of the solutions of the following nonlinear fifteenth-order difference equations

$$x_{n+1} = \frac{x_{n-14}}{\pm 1 \pm x_{n-2}x_{n-5}x_{n-8}x_{n-11}x_{n-14}}, \quad n = 0, 1, 2, \ldots,$$

where the initial conditions $x_{-14}, x_{-13}, \ldots, x_0$ are arbitrary real numbers. Moreover, we investigate stability, boundedness, oscillation and the periodic character of these solutions. Finally, we confirm the results with some numerical examples and graphs by using Matlab program.

Keywords: Recursive sequence, oscillation, semicycles, stability, periodicity, solutions of difference equations.

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1. Introduction

Difference equations appear as natural descriptions of observed evolution phenomena because most measurements of time evolving variables are discrete and as such these equations are in their own right important mathematical models. More importantly, difference equations also appear in the study of discretization methods for differential equations. Several results in the theory of difference equations have been obtained as more or less natural discrete analogues of corresponding results of differential equations.

Recently there has been a lot of interest in studying the global attractivity, boundedness character, periodicity and the solution form of nonlinear difference equations. For some results in this area, for example, Ahmed et al. [5] obtained the expressions of solutions of the class of difference equations

$$x_{n+1} = \frac{x_{n-2k+1}}{\pm 1 \pm \Pi_{i=1}^{k} x_{n-2i+1}}, \quad n = 0, 1, 2, \ldots,$$

with conditions posed on the initial values x_{-j}, $j = 0, 1, 2, \ldots, 2k - 1$, where $k \in \{1, 2, \ldots\}$. Elsayed et al [13] obtained the solutions of the difference equations

$$x_{n+1} = \frac{x_{n-11}}{\pm 1 \pm x_{n-2} x_{n-5} x_{n-8} x_{n-11}}, \quad n = 0, 1, 2, \ldots,$$

where the initial conditions are arbitrary real numbers. Cinar [8–10] obtained the solutions of the difference equations

$$x_{n+1} = \frac{x_{n-1}}{1 + x_n x_{n-1}}, \quad x_{n+1} = \frac{x_{n-1}}{-1 + x_n x_{n-1}}, \quad x_{n+1} = \frac{a x_{n-1}}{1 + b x_n x_{n-1}}.$$

Elsayed [12] studied the difference equation

$$x_{n+1} = \frac{x_{n-5}}{-1 + x_{n-2} x_{n-5}}.$$

For other related papers, see [1–4, 6, 7, 11, 14–17].

In this paper, we obtain the solutions of the following nonlinear difference equations

$$x_{n+1} = \frac{x_{n-14}}{\pm 1 \pm x_{n-2} x_{n-5} x_{n-8} x_{n-11} x_{n-14}}, \quad n = 0, 1, 2, \ldots,$$

with conditions posed on the initial values x_{-j}, $j = 0, 1, 2, \ldots, 14$. Moreover, we investigate stability, boundedness, oscillation and the periodic character of these solutions. Finally, we confirm the results with some numerical examples and graphs by using Matlab program.

Let I be some interval of real numbers and let

$$F : I^{k+1} \to I,$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, x_{-k+1}, \ldots, x_0 \in I$, the difference equation

$$x_{n+1} = F(x_n, x_{n-1}, x_{n-2}, \ldots, x_{n-k}), \quad n = 0, 1, \ldots, \tag{1.1}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

Definition 1.1. A point $\overline{x} \in I$ is called an equilibrium point of Eq. (1.1) if

$$\overline{x} = F(\overline{x}, \overline{x}, \ldots, \overline{x}),$$

that is,

$$x_n = \overline{x} \text{ for all } n \geqslant -k$$

is a solution of Eq. (1.1), or equivalently, $\overline{x}$ is a fixed point of $G(x) = F(x, \ldots, x)$.

Definition 1.2. A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geqslant -k$.

Definition 1.3.

(i) The equilibrium point $\overline{x}$ of Eq. (1.1) is locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + \cdots + |x_0 - \overline{x}| < \delta,$$

we have

$$|x_n - \overline{x}| < \epsilon \quad \text{for all} \quad n \geqslant -k.$$

(ii) The equilibrium point $\bar{x}$ of Eq. (1.1) is locally asymptotically stable if $\bar{x}$ is locally stable of Eq. (1.1) and there exists $\gamma > 0$, such that for all $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0 \in I$ with

$$|x_{-k} - \bar{x}| + |x_{-k+1} - \bar{x}| + \cdots + |x_0 - \bar{x}| < \gamma,$$

we have

$$\lim_{n \to \infty} x_n = \bar{x}.$$

(iii) The equilibrium point $\bar{x}$ of Eq. (1.1) is global attractor if for all $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0 \in I$, we have

$$\lim_{n \to \infty} x_n = \bar{x}.$$

(iv) The equilibrium point $\bar{x}$ of Eq. (1.1) is globally asymptotically stable if $\bar{x}$ is locally stable, and $\bar{x}$ is also a global attractor of Eq. (1.1).

(v) The equilibrium point $\bar{x}$ of Eq. (1.1) is unstable if $\bar{x}$ is not locally stable.

Definition 1.4.

(i) A sequence $\{x_n\}$ is said to oscillate about zero or simply to oscillate if the terms x_n are neither eventually all positive nor eventually all negative. Otherwise the sequence is called nonoscillatory. A sequence $\{x_n\}$ is called strictly oscillatory if for every $n_0 \geqslant 0$, there exist $n_1, n_2 \geqslant n_0$ such that $x_{n_1} x_{n_2} < 0$.

(ii) A sequence $\{x_n\}$ is said to oscillate about $\bar{x}$ if the sequence $\{x_n - \bar{x}\}$ oscillates. The sequence $\{x_n\}$ is called strictly oscillatory about $\bar{x}$ if the sequence $\{x_n - \bar{x}\}$ is strictly oscillatory.

Definition 1.5. A positive semicycle of a solution $\{x_n\}$ of Eq. (1.1) consists of a "string" of terms $\{x_l, x_{l+1}, \ldots, x_m\}$, all greater than or equal to $\bar{x}$, with $l \geqslant -k$ and $m \leqslant \infty$ and such that

$$\text{either } l = -k \text{ or } l > -k \text{ and } x_{l-1} < \bar{x},$$

and

$$\text{either } m = \infty \text{ or } m < \infty \text{ and } x_{m+1} < \bar{x}.$$

A negative semicycle of a solution $\{x_n\}$ of Eq. (1.1) consists of a "string" of terms $\{x_l, x_{l+1}, \ldots, x_m\}$, all less than $\bar{x}$, with $l \geqslant -k$ and $m \leqslant \infty$ and such that

$$\text{either } l = -k \text{ or } l > -k \text{ and } x_{l-1} \geqslant \bar{x},$$

and

$$\text{either } m = \infty \text{ or } m < \infty \text{ and } x_{m+1} \geqslant \bar{x}.$$

Definition 1.6. Suppose that the function F is continuously differentiable in some open neighborhood of an equilibrium point $\bar{x}$. Let

$$p_i = \frac{\partial F}{\partial u_i}(\bar{x}, \bar{x}, \ldots, \bar{x}) \quad \text{for} \quad i = 0, 1, \ldots, k,$$

denote the partial derivatives of $F(u_0, u_1, \ldots, u_k)$ evaluated at the equilibrium $\bar{x}$ of Eq. (1.1). Then the equation

$$y_{n+1} = p_0 y_n + p_1 y_{n-1} + \cdots + p_k y_{n-k}, \quad n = 0, 1, \ldots, \tag{1.2}$$

is called the linearized equation associated of Eq. (1.1) about the equilibrium point $\bar{x}$ and the equation

$$\lambda^{k+1} - p_0 \lambda^k - \cdots - p_{k-1} \lambda - p_k = 0, \tag{1.3}$$

is called the characteristic equation of Eq. (1.2) about $\bar{x}$.

The following result known as the linearized stability theorem is very useful in determining the local stability character of the equilibrium point $\bar{x}$ of Eq. (1.1).

Definition 1.7. Let $\bar{x}$ be an equilibrium of Eq. (1.1).

  (i) $\bar{x}$ is called a hyperbolic equilibrium if Eq. (1.3) has no roots with absolute value equal to one.
 (ii) $\bar{x}$ is called a nonhyperbolic equilibrium if there exists a root of Eq. (1.3) with absolute value equal to one.

Throughout this paper, we define $\mod(k, 3) = k - 3\left[\frac{k}{3}\right]$, where $[x]$ be the greatest integer less than or equal to the real number x.

2. The difference equation $x_{n+1} = \dfrac{x_{n-14}}{1 + x_{n-2}x_{n-5}x_{n-8}x_{n-11}x_{n-14}}$

In this section we give a specific form of the solutions of the first equation in the form

$$x_{n+1} = \frac{x_{n-14}}{1 + x_{n-2}x_{n-5}x_{n-8}x_{n-11}x_{n-14}}, \quad n = 0, 1, 2, \ldots, \tag{2.1}$$

with conditions posed on the initial values x_{-j}, $j = 0, 1, 2, \ldots, 14$. Also, we investigate the stability and boundedness of these solutions.

Theorem 2.1. *Let $\{x_n\}_{n=-14}^{\infty}$ be a solution of the difference Eq. (2.1). Then for $n = 0, 1, 2, \ldots,$*

$$x_{15n-k} = a_k \prod_{i=0}^{n-1} \left(\frac{1 + (5i + M_k - 1)\,P_k}{1 + (5i + M_k)\,P_k} \right), \tag{2.2}$$

where $P_k = \prod_{j=0}^{4} a_{\mod(k,3)+3j}$, $M_k = 5 - \left[\frac{k}{3}\right]$ and $x_{-k} = a_k$, with $rP_k \neq -1$ such that $r \in \{1, 2, 3, \ldots\}$, $k = 0, 1, 2, \ldots, 14$.

Proof. For $n = 0$, the result holds. Now suppose that $n > 0$ and that our assumption holds for $n - 1$. That is

$$x_{15n-15-k} = a_k \prod_{i=0}^{n-2} \left(\frac{1 + (5i + M_k - 1)\,P_k}{1 + (5i + M_k)\,P_k} \right). \tag{2.3}$$

Now, it follows from Eq. (2.1) and using Eq. (2.3) that

$$x_{15n-14} = \frac{x_{15n-29}}{1 + x_{15n-17}x_{15n-20}x_{15n-23}x_{15n-26}x_{15n-29}}$$

$$= \frac{a_{14} \prod_{i=0}^{n-2} \left(\frac{1+(5i+M_{14}-1)P_{14}}{1+(5i+M_{14})P_{14}} \right)}{1 + \prod_{j=0}^{4} \left(a_{3j+2} \prod_{i=0}^{n-2} \left(\frac{1+(5i+M_{3j+2}-1)P_{3j+2}}{1+(5i+M_{3j+2})P_{3j+2}} \right) \right)} = \frac{a_{14} \prod_{i=0}^{n-2} \left(\frac{1+(5i)a_2 a_5 a_8 a_{11} a_{14}}{1+(5i+1)a_2 a_5 a_8 a_{11} a_{14}} \right)}{1 + a_2 a_5 a_8 a_{11} a_{14} \prod_{i=0}^{n-2} \left(\frac{1+(5i)a_2 a_5 a_8 a_{11} a_{14}}{1+(5i+5)a_2 a_5 a_8 a_{11} a_{14}} \right)}.$$

Hence, we have

$$x_{15n-14} = a_{14} \prod_{i=0}^{n-1} \left(\frac{1 + (5i)a_2 a_5 a_8 a_{11} a_{14}}{1 + (5i + 1)\,a_2 a_5 a_8 a_{11} a_{14}} \right).$$

Also, it follows from Eq. (2.1) and using Eq. (2.3) that

$$x_{15n-13} = \frac{x_{15n-28}}{1 + x_{15n-16}x_{15n-19}x_{15n-22}x_{15n-25}x_{15n-28}}$$

$$= \frac{a_{13} \prod_{i=0}^{n-2} \left(\frac{1+(5i+M_{13}-1)P_{13}}{1+(5i+M_{13})P_{13}} \right)}{1 + \prod_{j=0}^{4} \left(a_{3j+1} \prod_{i=0}^{n-2} \left(\frac{1+(5i+M_{3j+1}-1)P_{3j+1}}{1+(5i+M_{3j+1})P_{3j+1}} \right) \right)} = \frac{a_{13} \prod_{i=0}^{n-2} \left(\frac{1+(5i)a_1 a_4 a_7 a_{10} a_{13}}{1+(5i+1)a_1 a_4 a_7 a_{10} a_{13}} \right)}{1 + a_1 a_4 a_7 a_{10} a_{13} \prod_{i=0}^{n-2} \left(\frac{1+(5i)a_1 a_4 a_7 a_{10} a_{13}}{1+(5i+5)a_1 a_4 a_7 a_{10} a_{13}} \right)}.$$

Hence, we have

$$x_{15n-13} = a_{13} \prod_{i=0}^{n-1} \left(\frac{1 + (5i) a_1 a_4 a_7 a_{10} a_{13}}{1 + (5i+1) a_1 a_4 a_7 a_{10} a_{13}} \right).$$

Also, it follows from Eq. (2.1) and using Eq. (2.3) that

$$x_{15n-12} = \frac{x_{15n-27}}{1 + x_{15n-15} x_{15n-18} x_{15n-21} x_{15n-24} x_{15n-27}}$$

$$= \frac{a_{12} \prod_{i=0}^{n-2} \left(\frac{1 + (5i + M_{12} - 1) P_{12}}{1 + (5i + M_{12}) P_{12}} \right)}{1 + \prod_{j=0}^{4} \left(a_{3j} \prod_{i=0}^{n-2} \left(\frac{1 + (5i + M_{3j} - 1) P_{3j}}{1 + (5i + M_{3j}) P_{3j}} \right) \right)} = \frac{a_{12} \prod_{i=0}^{n-2} \left(\frac{1 + (5i) a_0 a_3 a_6 a_9 a_{12}}{1 + (5i+1) a_0 a_3 a_6 a_9 a_{12}} \right)}{1 + a_0 a_3 a_6 a_9 a_{12} \prod_{i=0}^{n-2} \left(\frac{1 + (5i) a_0 a_3 a_6 a_9 a_{12}}{1 + (5i+5) a_0 a_3 a_6 a_9 a_{12}} \right)}.$$

Hence, we have

$$x_{15n-12} = a_{12} \prod_{i=0}^{n-1} \left(\frac{1 + (5i) a_0 a_3 a_6 a_9 a_{12}}{1 + (5i+1) a_0 a_3 a_6 a_9 a_{12}} \right).$$

Similarly, one can easily obtain the other relations for Eq. (2.2). Hence, the proof is completed. □

Theorem 2.2. *Assume that the initial values of the difference Eq.* (2.1) $x_{-14}, x_{-13}, \ldots, x_0 \in [0, \infty)$, *then every solution of Eq.* (2.1) *is bounded.*

Proof. Let $\{x_n\}_{n=-14}^{\infty}$ be a solution of Eq. (2.1). It follows from Eq. (2.1) that

$$0 \leqslant x_{n+1} = \frac{x_{n-14}}{1 + x_{n-2} x_{n-5} x_{n-8} x_{n-11} x_{n-14}} \leqslant x_{n-14} \quad \text{for all} \quad n \geqslant 0.$$

Then the sequence $\{x_{15n-i}\}_{n=0}^{\infty}$, $i = 0, 1, \ldots, 14$ is decreasing and so is bounded from above by $M = \max\{x_{-14}, x_{-13}, \ldots, x_0\}$. □

Theorem 2.3. *The only equilibrium point* $\overline{x}$ *of Eq.* (2.1) *is* $\overline{x} = 0$.

Proof. From Eq. (2.1), we can write $\overline{x} = \frac{\overline{x}}{1 + \overline{x}^5}$. Then we have $\overline{x} + \overline{x}^6 = \overline{x}$, or, $\overline{x}^6 = 0$. Thus, the only equilibrium point of Eq. (2.1) is $\overline{x} = 0$. □

Theorem 2.4. *Assume that the initial values of the difference Eq.* (2.1) $x_{-14}, x_{-13}, \ldots, x_0 \in [0, \infty)$, *then the equilibrium point* $\overline{x} = 0$ *of Eq.* (2.1) *is locally stable.*

Proof. Let $\epsilon > 0$, and let $\{x_n\}_{n=-14}^{\infty}$ be a solution of Eq. (2.1) such that

$$\sum_{j=0}^{14} \left| x_{-j} \right| < \epsilon.$$

It suffices to show that $|x_1| < \epsilon$. Now

$$0 < x_1 = \frac{x_{-14}}{1 + x_{-2} x_{-5} x_{-8} x_{-11} x_{-14}} \leqslant x_{-14} < \epsilon,$$

and so the proof is completed. □

Theorem 2.5. *Assume that the initial values of the difference Eq.* (2.1) $x_{-14}, x_{-13}, \ldots, x_0 \in [0, \infty)$, *then the equilibrium point* $\overline{x} = 0$ *of Eq.* (2.1) *is globally asymptotically stable.*

Proof. We know by Theorem 2.4 that the equilibrium point $\bar{x} = 0$ of Eq. (2.1) is locally stable. So let $\{x_n\}_{n=-14}^{\infty}$ be a positive solution of Eq. (2.1). It suffices to show that $\lim_{n \to \infty} x_n = \bar{x} = 0$. From Theorem 2.2 we have $x_{n+1} < x_{n-14}$ for all $n \geqslant 0$, so the sequences $\{x_{15n-i}\}_{n=0}^{\infty}$, $i = 0, 1, \ldots, 14$ are decreasing and bounded which implies that the sequences $\{x_{15n-i}\}_{n=0}^{\infty}$, $i = 0, 1, \ldots, 14$ converge to limit (say $L_i > 0$). So

$$L_{14} = \frac{L_{14}}{1 + L_2 L_5 L_8 L_{11} L_{14}} = 0, \quad L_{13} = \frac{L_{13}}{1 + L_1 L_4 L_7 L_{10} L_{13}} = 0, \ldots, L_0 = \frac{L_0}{1 + L_0 L_3 L_6 L_9 L_{12}} = 0,$$

which implies that $L_0 = L_1 = \cdots = L_{14} = 0$, from which the result follows. $\qquad \square$

3. The difference equation $x_{n+1} = \dfrac{x_{n-14}}{1 - x_{n-2} x_{n-5} x_{n-8} x_{n-11} x_{n-14}}$

In this section we give a specific form of the solutions of the second equation in the form

$$x_{n+1} = \frac{x_{n-14}}{1 - x_{n-2} x_{n-5} x_{n-8} x_{n-11} x_{n-14}}, \quad n = 0, 1, 2, \ldots, \tag{3.1}$$

with conditions posed on the initial values x_{-j}, $j = 0, 1, 2, \ldots, 14$.

Theorem 3.1. *Let $\{x_n\}_{n=-14}^{\infty}$ be a solution of the difference Eq. (3.1). Then for $n = 0, 1, 2, \ldots,$*

$$x_{15n-k} = a_k \prod_{i=0}^{n-1} \left(\frac{-1 + (5i + M_k - 1) P_k}{-1 + (5i + M_k) P_k} \right), \tag{3.2}$$

where $P_k = \prod_{j=0}^{4} a_{\mathrm{mod}(k,3)+3j}$, $M_k = 5 - \left[\frac{k}{3}\right]$ and $x_{-k} = a_k$, with $r P_k \neq 1$ such that $r \in \{1, 2, 3, \ldots\}$, $k = 0, 1, 2, \ldots, 14$.

Proof. For $n = 0$, the result holds. Now suppose that $n > 0$ and that our assumption holds for $n - 1$. That is

$$x_{15n-15-k} = a_k \prod_{i=0}^{n-2} \left(\frac{-1 + (5i + M_k - 1) P_k}{-1 + (5i + M_k) P_k} \right). \tag{3.3}$$

Now, it follows from Eq. (3.1) and using Eq. (3.3) that

$$x_{15n-14} = \frac{x_{15n-29}}{1 - x_{15n-17} x_{15n-20} x_{15n-23} x_{15n-26} x_{15n-29}}$$

$$= \frac{a_{14} \prod_{i=0}^{n-2} \left(\frac{-1 + (5i + M_{14} - 1) P_{14}}{-1 + (5i + M_{14}) P_{14}} \right)}{1 - \prod_{j=0}^{4} \left(a_{3j+2} \prod_{i=0}^{n-2} \left(\frac{-1 + (5i + M_{3j+2} - 1) P_{3j+2}}{-1 + (5i + M_{3j+2}) P_{3j+2}} \right) \right)} = \frac{a_{14} \prod_{i=0}^{n-2} \left(\frac{-1 + (5i) a_2 a_5 a_8 a_{11} a_{14}}{-1 + (5i+1) a_2 a_5 a_8 a_{11} a_{14}} \right)}{1 - a_2 a_5 a_8 a_{11} a_{14} \prod_{i=0}^{n-2} \left(\frac{-1 + (5i) a_2 a_5 a_8 a_{11} a_{14}}{-1 + (5i+5) a_2 a_5 a_8 a_{11} a_{14}} \right)}.$$

Hence, we have

$$x_{15n-14} = a_{14} \prod_{i=0}^{n-1} \left(\frac{-1 + (5i) a_2 a_5 a_8 a_{11} a_{14}}{-1 + (5i + 1) a_2 a_5 a_8 a_{11} a_{14}} \right).$$

Also, it follows from Eq. (3.1) and using Eq. (3.3) that

$$x_{15n-13} = \frac{x_{15n-28}}{1 - x_{15n-16} x_{15n-19} x_{15n-22} x_{15n-25} x_{15n-28}}$$

$$= \frac{a_{13} \prod_{i=0}^{n-2} \left(\frac{-1 + (5i + M_{13} - 1) P_{13}}{-1 + (5i + M_{13}) P_{13}} \right)}{1 - \prod_{j=0}^{4} \left(a_{3j+1} \prod_{i=0}^{n-2} \left(\frac{-1 + (5i + M_{3j+1} - 1) P_{3j+1}}{-1 + (5i + M_{3j+1}) P_{3j+1}} \right) \right)} = \frac{a_{13} \prod_{i=0}^{n-2} \left(\frac{-1 + (5i) a_1 a_4 a_7 a_{10} a_{13}}{-1 + (5i+1) a_1 a_4 a_7 a_{10} a_{13}} \right)}{1 - a_1 a_4 a_7 a_{10} a_{13} \prod_{i=0}^{n-2} \left(\frac{-1 + (5i) a_1 a_4 a_7 a_{10} a_{13}}{-1 + (5i+5) a_1 a_4 a_7 a_{10} a_{13}} \right)}.$$

Hence, we have

$$x_{15n-13} = a_{13} \prod_{i=0}^{n-1} \left(\frac{-1 + (5i)\, a_1 a_4 a_7 a_{10} a_{13}}{-1 + (5i+1)\, a_1 a_4 a_7 a_{10} a_{13}} \right).$$

Also, it follows from Eq. (3.1) and using Eq. (3.3) that

$$x_{15n-12} = \frac{x_{15n-27}}{1 - x_{15n-15} x_{15n-18} x_{15n-21} x_{15n-24} x_{15n-27}}$$

$$= \frac{a_{12} \prod_{i=0}^{n-2} \left(\frac{-1 + (5i + M_{12} - 1) P_{12}}{-1 + (5i + M_{12}) P_{12}} \right)}{1 - \prod_{j=0}^{4} \left(a_{3j} \prod_{i=0}^{n-2} \left(\frac{-1 + (5i + M_{3j} - 1) P_{3j}}{-1 + (5i + M_{3j}) P_{3j}} \right) \right)} = \frac{a_{12} \prod_{i=0}^{n-2} \left(\frac{-1 + (5i)\, a_0 a_3 a_6 a_9 a_{12}}{-1 + (5i+1)\, a_0 a_3 a_6 a_9 a_{12}} \right)}{1 - a_0 a_3 a_6 a_9 a_{12} \prod_{i=0}^{n-2} \left(\frac{-1 + (5i)\, a_0 a_3 a_6 a_9 a_{12}}{-1 + (5i+5)\, a_0 a_3 a_6 a_9 a_{12}} \right)}.$$

Hence, we have

$$x_{15n-12} = a_{12} \prod_{i=0}^{n-1} \left(\frac{-1 + (5i)\, a_0 a_3 a_6 a_9 a_{12}}{-1 + (5i+1)\, a_0 a_3 a_6 a_9 a_{12}} \right).$$

Similarly, one can easily obtain the other relations for Eq. (3.2). Hence, the proof is completed. □

Theorem 3.2. *Eq.* (3.1) *has a unique equilibrium point* $\overline{x} = 0$, *which is a non hyperbolic equilibrium point.*

Proof. From Eq. (3.1), we can write $\overline{x} = \frac{\overline{x}}{1 - \overline{x}^5}$. Then we have $\overline{x} - \overline{x}^6 = \overline{x}$, or, $\overline{x}^6 = 0$. Thus the only equilibrium point of Eq. (3.1) is $\overline{x} = 0$. Let $f : (0, \infty)^5 \to (0, \infty)$ be a function defined by

$$f(x, y, z, u, v) = \frac{x}{1 - xyzuv}.$$

Then we have

$$f_x(x, y, z, u, v) = \frac{1}{(1 - xyzuv)^2}, \qquad\qquad f_y(x, y, z, u, v) = \frac{x^2 zuv}{(1 - xyzuv)^2},$$

$$f_z(x, y, z, u, v) = \frac{x^2 yuv}{(1 - xyzuv)^2}, \qquad\qquad f_u(x, y, z, u, v) = \frac{x^2 yzv}{(1 - xyzuv)^2},$$

$$f_v(x, y, z, u, v) = \frac{x^2 yzu}{(1 - xyzuv)^2}.$$

which implies that

$$f_x(\overline{x}, \overline{x}, \overline{x}, \overline{x}, \overline{x}) = 1, \ f_y(\overline{x}, \overline{x}, \overline{x}, \overline{x}, \overline{x}) = f_z(\overline{x}, \overline{x}, \overline{x}, \overline{x}, \overline{x}) = f_u(\overline{x}, \overline{x}, \overline{x}, \overline{x}, \overline{x}) = f_v(\overline{x}, \overline{x}, \overline{x}, \overline{x}, \overline{x}) = 0.$$

So, the linearized equation of Eq. (3.1) about the equilibrium point $\overline{x} = 0$ is

$$z_{n+1} = z_{n-14}, \tag{3.4}$$

and the characteristic equation of Eq. (3.4) about the equilibrium point $\overline{x} = 0$ is

$$\lambda^{15} - 1 = 0,$$

which implies that

$$|\lambda_i| = 1 \ ; i = 1, 2, \dots, 15,$$

so, $\overline{x}$ is a non hyperbolic equilibrium point. □

Open Problem: Investigate the global behavior of solutions of Eq. (3.1) about the equilibrium point $\overline{x} = 0$.

4. The difference equation $x_{n+1} = \dfrac{x_{n-14}}{-1 + x_{n-2}x_{n-5}x_{n-8}x_{n-11}x_{n-14}}$

In this section we give a specific form of the solutions of the third equation in the form

$$x_{n+1} = \frac{x_{n-14}}{-1 + x_{n-2}x_{n-5}x_{n-8}x_{n-11}x_{n-14}}, \quad n = 0, 1, 2, \ldots, \tag{4.1}$$

with conditions posed on the initial values x_{-j}, $j = 0, 1, 2, \ldots, 14$. Also, we investigate the oscillation and periodicity of these solutions.

Theorem 4.1. *Every solution of Eq.* (4.1) *is periodic with period* 30.

Proof. From Eq. (4.1), we have

$$x_{n+30} = \frac{x_{n+15}}{-1 + \prod_{i=0}^{4} x_{n+15+3i}}. \tag{4.2}$$

Since

$$-1 + \prod_{i=0}^{4} x_{n+15+3i} = -1 + \prod_{i=0}^{3} x_{n+15+3i} \, \frac{x_{n+12}}{-1 + \prod_{i=0}^{4} x_{n+12+3i}}$$

$$= \frac{1}{-1 + \prod_{i=0}^{4} x_{n+12+3i}}$$

$$\Rightarrow -1 + \prod_{i=0}^{4} x_{n+15+3i} = \frac{1}{-1 + \prod_{i=0}^{4} x_{n+12+3i}}.$$

Similarly

$$-1 + \prod_{i=0}^{4} x_{n+12+3i} = \frac{1}{-1 + \prod_{i=0}^{4} x_{n+9+3i}},$$

so

$$-1 + \prod_{i=0}^{4} x_{n+15+3i} = -1 + \prod_{i=0}^{4} x_{n+9+3i}.$$

Similarly

$$-1 + \prod_{i=0}^{4} x_{n+9+3i} = -1 + \prod_{i=0}^{4} x_{n+3+3i} \text{ and } -1 + \prod_{i=0}^{4} x_{n+3+3i} = \frac{1}{-1 + \prod_{i=0}^{4} x_{n+3i}},$$

which imply from Eq. (4.1) and Eq. (4.2) that

$$x_{n+30} = \frac{\left(\dfrac{x_n}{-1 + \prod_{i=0}^{4} x_{n+3i}} \right)}{\left(\dfrac{1}{-1 + \prod_{i=0}^{4} x_{n+3i}} \right)} = x_n, \, n = 0, 1, 2, \ldots \,.$$

$\square$

Theorem 4.2. *The periodic* 30 *solution of Eq.* (4.1) *has the form*

$$x_{30n-k} = \frac{a_{k-(q(k))(15)^{q(k)}}}{(-1+p_{k-15})^{\alpha(k)q(k)}}; \quad k = 0, 1, \ldots, 29 \text{ and } n = 1, 2, \ldots,$$

where $x_{-j} = a_j$, $p_j = \prod\limits_{i=0}^{4} a_{\mathrm{mod}(j,3)+3i}$, *with* $p_j \neq 1$, $j = 0, 1, 2, \ldots, 14$, $p_{-l} = 0, l = 1, 2, \ldots, 15, q(k) = \frac{1}{2}((-1)^{[\frac{k}{15}]+1} + 1)$ *and* $\alpha(k) = (-1)^{[\frac{k}{3}]+1}$.

Proof. From the definition of $q(k)$, we can see that

$$q(0) = q(1) = \cdots = q(14) = 0 \text{ and } q(15) = q(16) = \cdots = q(29) = 1.$$

Also,

$$\alpha(i+6r) = -1, \; \alpha(i+3+6r) = 1; \quad i = 0, 1, 2 \text{ and } r = 0, 1, 2, 3, 4.$$

So,

$$x_1 = \frac{x_{-14}}{-1+x_{-2}x_{-5}x_{-8}x_{-11}x_{-14}} = \frac{a_{14}}{-1+a_2a_5a_8a_{11}a_{14}},$$

$$x_2 = \frac{x_{-13}}{-1+x_{-1}x_{-4}x_{-7}x_{-10}x_{-13}} = \frac{a_{13}}{-1+a_1a_4a_7a_{10}a_{13}},$$

$$x_3 = \frac{x_{-12}}{-1+x_0x_{-3}x_{-6}x_{-9}x_{-12}} = \frac{a_{12}}{-1+a_0a_3a_6a_9a_{12}},$$

$$x_4 = \frac{x_{-11}}{-1+x_1x_{-2}x_{-5}x_{-8}x_{-11}} = a_{11}(-1+a_2a_5a_8a_{11}a_{14}),$$

$$x_5 = \frac{x_{-10}}{-1+x_2x_{-1}x_{-4}x_{-7}x_{-10}} = a_{10}(-1+a_1a_4a_7a_{10}a_{13}),$$

$$x_6 = \frac{x_{-9}}{-1+x_3x_0x_{-3}x_{-6}x_{-9}} = a_9(-1+a_0a_3a_6a_9a_{12}),$$

$$x_7 = \frac{x_{-8}}{-1+x_4x_1x_{-2}x_{-5}x_{-8}} = \frac{a_8}{-1+a_2a_5a_8a_{11}a_{14}},$$

$$x_8 = \frac{x_{-7}}{-1+x_5x_2x_{-1}x_{-4}x_{-7}} = \frac{a_7}{-1+a_1a_4a_7a_{10}a_{13}},$$

$$x_9 = \frac{x_{-6}}{-1+x_6x_3x_0x_{-3}x_{-6}} = \frac{a_6}{-1+a_0a_3a_6a_9a_{12}},$$

$$x_{10} = \frac{x_{-5}}{-1+x_7x_4x_1x_{-2}x_{-5}} = a_5(-1+a_2a_5a_8a_{11}a_{14}),$$

$$x_{11} = \frac{x_{-4}}{-1+x_8x_5x_2x_{-1}x_{-4}} = a_4(-1+a_1a_4a_7a_{10}a_{13}),$$

$$x_{12} = \frac{x_{-3}}{-1+x_9x_6x_3x_0x_{-3}} = a_3(-1+a_0a_3a_6a_9a_{12}),$$

$$x_{13} = \frac{x_{-2}}{-1+x_{10}x_7x_4x_1x_{-2}} = \frac{a_2}{-1+a_2a_5a_8a_{11}a_{14}},$$

$$x_{14} = \frac{x_{-1}}{-1+x_{11}x_8x_5x_2x_{-1}} = \frac{a_1}{-1+a_1a_4a_7a_{10}a_{13}},$$

$$x_{15} = \frac{x_0}{-1+x_{12}x_9x_6x_3x_0} = \frac{a_0}{-1+a_0a_3a_6a_9a_{12}},$$

$$x_{16} = \frac{x_1}{-1+x_{13}x_{10}x_7x_4x_1} = a_{14},$$

$$x_{17} = \frac{x_2}{-1+x_{14}x_{11}x_8x_5x_2} = a_{13},$$

$$x_{18} = \frac{x_3}{-1+x_{15}x_{12}x_9x_6x_3} = a_{12},$$

$$x_{19} = \frac{x_4}{-1+x_{16}x_{13}x_{10}x_7x_4} = a_{11},$$

$$x_{20} = \frac{x_5}{-1+x_{17}x_{14}x_{11}x_8x_5} = a_{10},$$

$$x_{21} = \frac{x_6}{-1+x_{18}x_{15}x_{12}x_9x_6} = a_9,$$

$$x_{22} = \frac{x_7}{-1+x_{19}x_{16}x_{13}x_{10}x_7} = a_8,$$

$$x_{23} = \frac{x_8}{-1+x_{20}x_{17}x_{14}x_{11}x_8} = a_7,$$

$$x_{24} = \frac{x_9}{-1+x_{21}x_{18}x_{15}x_{12}x_9} = a_6,$$

$$x_{25} = \frac{x_{10}}{-1+x_{22}x_{19}x_{16}x_{13}x_{10}} = a_5,$$

$$x_{26} = \frac{x_{11}}{-1+x_{23}x_{20}x_{17}x_{14}x_{11}} = a_4,$$

$$x_{27} = \frac{x_{12}}{-1+x_{24}x_{21}x_{18}x_{15}x_{12}} = a_3,$$

$$x_{28} = \frac{x_{13}}{-1+x_{25}x_{22}x_{19}x_{16}x_{13}} = a_2,$$

$$x_{29} = \frac{x_{14}}{-1+x_{26}x_{23}x_{20}x_{17}x_{14}} = a_1,$$

$$x_{30} = \frac{x_{15}}{-1+x_{27}x_{24}x_{21}x_{18}x_{15}} = a_0,$$

and the result follows by induction. □

Theorem 4.3. *Eq.* (4.1) *has two equilibrium points* 0 *and* $\sqrt[5]{2}$, *which are non hyperbolic equilibrium points.*

Proof. The proof is similar to the proof of Theorem 3.2, and will be omitted. □

Theorem 4.4. *Eq.* (4.1) *is periodic of period* 15 *iff* $p_k = 2; k = 0, 1, 2$ *and will take the form*

$$x_{15n-k} = a_k; \quad k = 0, 1, \ldots, 14 \text{ and } n = 0, 1, 2, \ldots.$$

Proof. The proof follows immediately from Theorem 4.2. □

Theorem 4.5. *Assume that* $a_0, a_1, \ldots, a_{14} \in (0,1)$. *Then the solution* $\{x_n\}_{n=-14}^{\infty}$ *of the difference Eq.* (4.1) *oscillates about the equilibrium point* $\overline{x} = 0$, *with positive semicycles of length* 15, *and negative semicycles of length* 15.

Proof. From Theorem 4.2, we have $x_1, x_2, \ldots, x_{15} < 0$ and $x_{16}, x_{17}, \ldots, x_{30} > 0$, and the result follows by induction. □

5. The difference equation $x_{n+1} = \dfrac{x_{n-14}}{-1 - x_{n-2}x_{n-5}x_{n-8}x_{n-11}x_{n-14}}$

In this section we give a specific form of the solutions of the fourth equation in the form

$$x_{n+1} = \frac{x_{n-14}}{-1 - x_{n-2}x_{n-5}x_{n-8}x_{n-11}x_{n-14}}, \quad n = 0, 1, 2, \ldots, \tag{5.1}$$

with conditions posed on the initial values x_{-j}, $j = 0, 1, 2, \ldots, 14$. Also, we investigate the oscillation and periodicity of these solutions.

Theorem 5.1. *Every solution of Eq.* (5.1) *is periodic with period* 30.

Proof. The proof is similar to the proof of Theorem 4.1, and will be omitted. □

Theorem 5.2. *The periodic* 30 *solution of Eq.* (5.1) *has the form*

$$x_{30n-k} = \frac{a_{k-(q(k))(15)^{q(k)}}}{(-1 - p_{k-15})^{\alpha(k)q(k)}} \quad ; k = 0, 1, \ldots, 29 \text{ and } n = 1, 2, \ldots,$$

where $x_{-j} = a_j$, $p_j = \prod\limits_{i=0}^{4} a_{\mathrm{mod}(j,3)+3i}$, *with* $p_j \neq -1$, $j = 0, 1, 2, \ldots, 14$, $p_{-l} = 0, l = 1, 2, \ldots, 15, q(k) = \frac{1}{2}((-1)^{[\frac{k}{15}]+1} + 1)$ *and* $\alpha(k) = (-1)^{[\frac{k}{3}]+1}$.

Proof. From the definition of $q(k)$, we can see that

$$q(0) = q(1) = \cdots = q(14) = 0 \text{ and } q(15) = q(16) = \cdots = q(29) = 1.$$

Also,

$$\alpha(i+6r) = -1, \ \alpha(i+3+6r) = 1 \ ; i = 0, 1, 2 \text{ and } r = 0, 1, 2, 3, 4.$$

So,

$$x_1 = \frac{x_{-14}}{-1 - x_{-2}x_{-5}x_{-8}x_{-11}x_{-14}} = -\frac{a_{14}}{1 + a_2 a_5 a_8 a_{11} a_{14}},$$

$$x_2 = \frac{x_{-13}}{-1 - x_{-1}x_{-4}x_{-7}x_{-10}x_{-13}} = -\frac{a_{13}}{1 + a_1 a_4 a_7 a_{10} a_{13}},$$

$$x_3 = \frac{x_{-12}}{-1 - x_0 x_{-3} x_{-6} x_{-9} x_{-12}} = -\frac{a_{12}}{1 + a_0 a_3 a_6 a_9 a_{12}},$$

$$x_4 = \frac{x_{-11}}{-1 - x_1 x_{-2} x_{-5} x_{-8} x_{-11}} = -a_{11}(1 + a_2 a_5 a_8 a_{11} a_{14}),$$

$$x_5 = \frac{x_{-10}}{-1 - x_2 x_{-1} x_{-4} x_{-7} x_{-10}} = -a_{10}(1 + a_1 a_4 a_7 a_{10} a_{13}),$$

$$x_6 = \frac{x_{-9}}{-1 - x_3 x_0 x_{-3} x_{-6} x_{-9}} = -a_9(1 + a_0 a_3 a_6 a_9 a_{12}),$$

$$x_7 = \frac{x_{-8}}{-1 - x_4 x_1 x_{-2} x_{-5} x_{-8}} = -\frac{a_8}{1 + a_2 a_5 a_8 a_{11} a_{14}},$$

$$x_8 = \frac{x_{-7}}{-1 - x_5 x_2 x_{-1} x_{-4} x_{-7}} = -\frac{a_7}{1 + a_1 a_4 a_7 a_{10} a_{13}},$$

$$x_9 = \frac{x_{-6}}{-1 - x_6 x_3 x_0 x_{-3} x_{-6}} = \frac{a_6}{-1 + a_0 a_3 a_6 a_9 a_{12}},$$

$$x_{10} = \frac{x_{-5}}{-1 - x_7 x_4 x_1 x_{-2} x_{-5}} = -a_5(1 + a_2 a_5 a_8 a_{11} a_{14}),$$

$$x_{11} = \frac{x_{-4}}{-1 - x_8 x_5 x_2 x_{-1} x_{-4}} = -a_4(1 + a_1 a_4 a_7 a_{10} a_{13}),$$

$$x_{12} = \frac{x_{-3}}{-1 - x_9 x_6 x_3 x_0 x_{-3}} = -a_3(1 + a_0 a_3 a_6 a_9 a_{12}),$$

$$x_{13} = \frac{x_{-2}}{-1 - x_{10} x_7 x_4 x_1 x_{-2}} = -\frac{a_2}{1 + a_2 a_5 a_8 a_{11} a_{14}},$$

$$x_{14} = \frac{x_{-1}}{-1 - x_{11} x_8 x_5 x_2 x_{-1}} = -\frac{a_1}{1 + a_1 a_4 a_7 a_{10} a_{13}},$$

$$x_{15} = \frac{x_0}{-1 - x_{12} x_9 x_6 x_3 x_0} = -\frac{a_0}{1 + a_0 a_3 a_6 a_9 a_{12}},$$

$$x_{16} = \frac{x_1}{-1 - x_{13} x_{10} x_7 x_4 x_1} = a_{14},$$

$$x_{17} = \frac{x_2}{-1 - x_{14}x_{11}x_8x_5x_2} = a_{13},$$

$$x_{18} = \frac{x_3}{-1 - x_{15}x_{12}x_9x_6x_3} = a_{12},$$

$$x_{19} = \frac{x_4}{-1 - x_{16}x_{13}x_{10}x_7x_4} = a_{11},$$

$$x_{20} = \frac{x_5}{-1 - x_{17}x_{14}x_{11}x_8x_5} = a_{10},$$

$$x_{21} = \frac{x_6}{-1 - x_{18}x_{15}x_{12}x_9x_6} = a_9,$$

$$x_{22} = \frac{x_7}{-1 - x_{19}x_{16}x_{13}x_{10}x_7} = a_8,$$

$$x_{23} = \frac{x_8}{-1 - x_{20}x_{17}x_{14}x_{11}x_8} = a_7,$$

$$x_{24} = \frac{x_9}{-1 - x_{21}x_{18}x_{15}x_{12}x_9} = a_6,$$

$$x_{25} = \frac{x_{10}}{-1 - x_{22}x_{19}x_{16}x_{13}x_{10}} = a_5,$$

$$x_{26} = \frac{x_{11}}{-1 - x_{23}x_{20}x_{17}x_{14}x_{11}} = a_4,$$

$$x_{27} = \frac{x_{12}}{-1 - x_{24}x_{21}x_{18}x_{15}x_{12}} = a_3,$$

$$x_{28} = \frac{x_{13}}{-1 - x_{25}x_{22}x_{19}x_{16}x_{13}} = a_2,$$

$$x_{29} = \frac{x_{14}}{-1 - x_{26}x_{23}x_{20}x_{17}x_{14}} = a_1,$$

$$x_{30} = \frac{x_{15}}{-1 - x_{27}x_{24}x_{21}x_{18}x_{15}} = a_0,$$

and the result follows by induction. □

Theorem 5.3. *Eq. (5.1) has two equilibrium points 0 and $-\sqrt[5]{2}$, which are non hyperbolic equilibrium points.*

Proof. The proof is similar to the proof of Theorem 3.2, and will be omitted. □

Theorem 5.4. *Eq. (5.1) is periodic of period 15 iff $p_k = -2$; $k = 0, 1, 2$ and will be take the form*

$$x_{15n-k} = a_k \quad ; k = 0, 1, \dots, 14 \text{ and } n = 0, 1, 2, \dots .$$

Proof. The proof follows immediately from Theorem 5.2. □

Theorem 5.5. *Assume that $a_0, a_1, \dots, a_{14} \in [0, \infty)$. Then the solution $\{x_n\}_{n=-14}^{\infty}$ of the difference Eq. (5.1) oscillates about the equilibrium point $\overline{x} = 0$, with positive semicycles of length 15, and negative semicycles of length 15.*

Proof. From Theorem 5.2, we have $x_1, x_2, \dots, x_{15} < 0$ and $x_{16}, x_{17}, \dots, x_{30} > 0$, and the result follows by induction. □

6. Numerical examples

To verify the results of this paper, we consider some numerical examples as follows.

Example 6.1. The graph of the difference Eq. (2.1) and the case when $x_{-14} = 7.8$, $x_{-13} = 5.5$, $x_{-12} = 7$, $x_{-11} = 5$, $x_{-10} = 9.8$, $x_{-9} = 8.2$, $x_{-8} = 7.9$, $x_{-7} = 6.6$, $x_{-6} = 1$, $x_{-5} = 6$, $x_{-4} = 3$, $x_{-3} = 8.5$, $x_{-2} = 2.9$, $x_{-1} = 1.3$ and $x_0 = 6.9$ is shown in Figure 1.
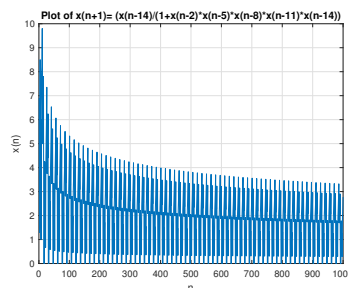

Figure 1: Graph of the difference Eq. (2.1).

Example 6.2. The graph of the difference Eq. (3.1) and the case when $x_{-14} = 25$, $x_{-13} = 16$, $x_{-12} = 4$, $x_{-11} = 9.5$, $x_{-10} = 30$, $x_{-9} = 10$, $x_{-8} = 7.9$, $x_{-7} = 6.6$, $x_{-6} = 1.5$, $x_{-5} = 6$, $x_{-4} = 3$, $x_{-3} = 5.5$, $x_{-2} = 2$, $x_{-1} = 1$ and $x_0 = 0.4$ is shown in Figure 2.
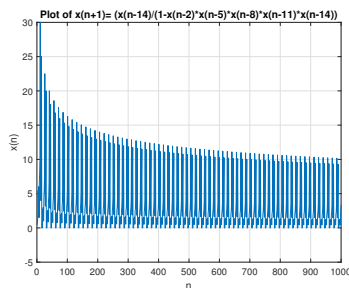
Figure 2: Graph of the difference Eq. (3.1).

Example 6.3. Figure 3 shows the period 30 solution and the oscillation of the solution of the difference Eq. (4.1) where $x_{-14} = 0.15$, $x_{-13} = 0.49$, $x_{-12} = 0.85$, $x_{-11} = 0.59$, $x_{-10} = 0.25$, $x_{-9} = 0.75$, $x_{-8} = 0.8$, $x_{-7} = 0.6$, $x_{-6} = 0.5$, $x_{-5} = 0.9$, $x_{-4} = 0.4$, $x_{-3} = 0.3$, $x_{-2} = 0.7$, $x_{-1} = 0.2$ and $x_0 = 0.1$.
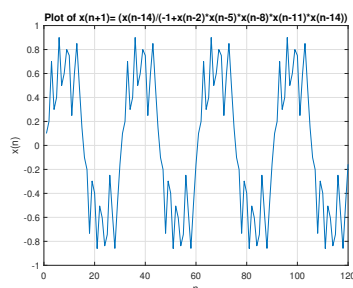

Figure 3: Graph of the difference Eq. (4.1).

Example 6.4. Figure 4 shows the period 30 solution and the oscillation of the solution of the difference Eq. (5.1) where $x_{-14} = 0.8$, $x_{-13} = 0.5$, $x_{-12} = 2.5$, $x_{-11} = 0.9$, $x_{-10} = 0.5$, $x_{-9} = 1$, $x_{-8} = 1.2$, $x_{-7} = 0.3$, $x_{-6} = 0.22$, $x_{-5} = 1.8$, $x_{-4} = 0.85$, $x_{-3} = 1.5$, $x_{-2} = 2$, $x_{-1} = 0.1$ and $x_0 = 0.3$.
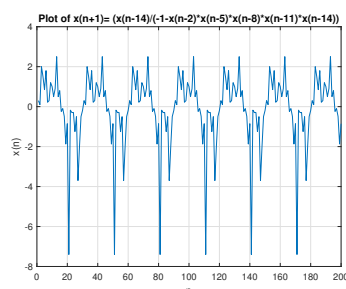

Figure 4: Graph of the difference Eq. (5.1).

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