



Asymptotic behavior of traveling waves for non-quasi-monotone system with delay



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Abstract

This paper is concerned with a population dynamic model with delay. In this work, by rewriting the equation and using the Ikehara's theorem, we show the exact asymptotic behavior of the profile as $\xi \rightarrow -\infty$ for critical speed.

Keywords: Traveling waves, Ikehara's theorem, asymptotic behavior.

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1. Introduction

In this work, we are concerned with the asymptotic behavior of non-monotone traveling waves of the delayed reaction diffusion equation without quasi-monotonicity

$$\begin{cases} u_{1t}(t, x) = du_{1xx}(t, x) - a_{11}u_1(t, x) + a_{12}u_2(t, x), \\ u_{2t}(t, x) = -a_{22}u_2(t, x) + \tilde{g}(u_1(t - \tau, x)), \end{cases} \quad (1.1)$$

where d , a_{11} , a_{12} , and a_{22} are positive constants, $u_1(t, x)$ and $u_2(t, x)$ denote the spatial densities of infectious agents and the infective human population at time $t \geq 0$, respectively. $1/a_{11}$ is the mean lifetime of the agents in the environment, $1/a_{22}$ is the mean infectious period of the infective human, a_{12} is the multiplicative factor of the infectious agents due to the human population, $\tilde{g}(u_1)$ is the force of infection on human population due to a concentration u_1 of the infectious agents and τ is the disease latent period.

Over the years, there have been many important monotone results [9, 11, 15, 16] on traveling wave solutions for reaction-diffusion scalar equations and systems.

However, on the traveling wave solutions of reaction-diffusion equations and systems without quasi-monotonicity, it seems that few results [2, 3, 6, 8]. Wu and Li [10, 12] established the existence of traveling wave solutions for some complicated non-local reaction-diffusion equations with delay and non-local diffusion term equations with delay the idea of auxiliary equations and Schauders fixed point theorem

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[7, 8], respectively. In particular, Wu [12] applied the method to solve the existence of traveling wave solutions of a class of non-monotone integral equations. As an application of this result, the existence of traveling wave solutions of the following epidemic system with distributed delay

$$\begin{cases} u_{1t}(t, x) = u_{1xx}(t, x) - u_1(t, x) + u_2(t, x), \\ u_{2t}(t, x) = -\beta u_2(t, x) + \int_0^\infty g(u_1(t-s, x))P(ds), \end{cases} \tag{1.2}$$

has been obtained. It is easy to see that if $P(\cdot)$ is Dirac function $\delta(\cdot)$ and (1.2) is the system (1.1). To the best of our knowledge, the asymptotic behavior of non-monotone traveling waves of such equations remain open and seem to be very interesting and challenging problems. The purpose of this work is to establish the asymptotic behavior of the non-monotone traveling waves of (1.1) in the non-monotone monostable case.

2. Preliminaries and main results

Notations. Throughout this paper, $C > 0$ denotes a generic constant, $C_i > 0 (i = 1, 2, \dots)$ represents a specific constant. Let I be an interval. $L^2(I)$ is the space of the square integrable functions defined on I , and $H^k(I) (k \geq 0)$ is the Sobolev space of the L^2 -function $h(x)$ defined on the interval I whose derivatives $\frac{d^i}{dx^i} h (i = 1, 2, \dots, k)$ also belong to $L^2(I)$. $L^2_w(I)$ denotes the weighted L^2 -space with a weight function $w(x) > 0$ and its norm is defined by $\|h\|_{L^2_w} = \left(\int_I w(x) |h(x)|^2 dx \right)^{\frac{1}{2}}$, and $H^k_w(I)$ is the weighted Sobolev space with the norm given by

$$\|h\|_{H^k_w} = \left(\sum_{i=0}^k \int_I w(x) \left| \frac{d^i}{dx^i} h(x) \right|^2 dx \right)^{\frac{1}{2}}.$$

Let $T > 0$ be a number and \mathcal{B} be a Banach space. $C([0, T]; \mathcal{B})$ is the space of \mathcal{B} -valued continuous functions on $[0, T]$. $L^2([0, T]; \mathcal{B})$ is the space of \mathcal{B} -valued L^2 -functions on $[0, T]$. The corresponding spaces of \mathcal{B} -valued functions on $[0, \infty)$ are defined similarly.

For convenience, we rewrite (1.1) to the following system

$$\begin{cases} u_{1t}(t, x) = u_{1xx}(t, x) - u_1(t, x) + u_2(t, x), \\ u_{2t}(t, x) = -\beta u_2(t, x) + g(u_1(t - \tau, x)), \end{cases} \tag{2.1}$$

where $\beta = \frac{a_{11}}{a_{22}}$, $g = \frac{a_{12}}{a_{22}} \tilde{g}$.

A traveling wave of Eq. (2.1) connecting u_- and u_+ is a solution $u(t, x) = \phi(x + ct) = \phi(\xi)$, $\xi = x + ct$, satisfying the following ordinary differential equation

$$\begin{cases} c\phi'_1(\xi) = \phi''_1(\xi) - \phi_1(\xi) + \phi_2(\xi), \\ c\phi'_2(\xi) = -\beta\phi_2(\xi) + g(\phi_1(\xi - c\tau)), \\ \phi_1(\pm\infty) = u_{1\pm}, \quad \phi_2(\pm\infty) = u_{2\pm}. \end{cases} \tag{2.2}$$

The characteristic equation associated with the linearized equation of (1.2) at $(0, 0)$ is

$$P(\lambda) = (\lambda^2 - c\lambda - 1)(-c\lambda - \beta) - g'(0).$$

In addition, throughout this paper, we assume that $\beta > 1$ and (1.1) satisfies the initial conditions

$$\begin{cases} u_1(s, x) = u_{10}(s, x), & (s, x) \in [-\tau, \mathbb{R}], \\ u_2(0, x) = u_{20}(x), & x \in \mathbb{R}. \end{cases}$$

We also need the following assumptions for the sake of proving the existence of traveling wave solutions (see [13]):

- (A₁) $g(0) = \beta K - g(K) = 0$ for some $K > 0$, $g'(0) > \beta$, and there exists a $\nu \in (0, 1]$ such that $\limsup_{u \rightarrow 0^+} [g'(0) - \frac{g(u)}{u}]u^{-\nu} < +\infty$;
- (A₂) $\min\{k^*u, K^+\} \geq \frac{1}{\beta}g(u) > 0$ for all $u \in (0, K^+]$ and g is Lipschitz continuous on $[0, K^+]$ for some $K^+ \geq K$, where $k^* = \frac{g'(0)}{\beta}$;
- (A₃) one of the following assumptions holds:
 - (i) $K \geq \frac{1}{\beta}g(u) > u$ for all $u \in (0, K)$;
 - (ii) $u < \frac{1}{\beta}g(u) < 2K - u$ for $u \in [K^-, K)$ and $u > \frac{1}{\beta}g(u) > 2K - u$ for $u \in (K, K^+]$, where $K^- = k^* \inf_{\eta \in (0, K^+]} \{g(\eta) : \frac{1}{\beta}g(\eta) \leq \eta\}$;
 - (iii) $u < \frac{1}{\beta}g(u)$ for $u \in [K^-, K)$, $u > \frac{1}{\beta}g(u)$ for $u \in (K, K^+]$ and there is no pair $0 < \gamma_1 < K < \gamma_2 \leq K^+$ such that $\gamma_1 = \frac{1}{\beta}g(\gamma_2)$ and $\gamma_2 = \frac{1}{\beta}g(\gamma_1)$;
- (A₄) $g''(u) < 0$ for all $u \in (0, \infty)$.

Proposition 2.1 (Existence of traveling waves). *Assume that (A₁)-(A₂) holds. Then, there exists $c_* > 0$ such that*

- (i) *for any $c > c_*$, (1.2) admits a traveling wave solution $\Phi(\xi) := (\phi_1(\xi), \phi_2(\xi))$ satisfying $\phi_i(\xi) = O(e^{\Lambda_1(c)\xi})$ as $\xi \rightarrow -\infty$, $i = 1, 2$, $\phi_1(\xi) \in C(\mathbb{R}, [0, K^+])$ and*

$$0 < K^- \leq \liminf_{\xi \rightarrow +\infty} \phi_1(\xi) \leq \limsup_{\xi \rightarrow +\infty} \phi_1(\xi) \leq K^+,$$

$$0 < \liminf_{\xi \rightarrow +\infty} \phi_2(\xi) \leq \limsup_{\xi \rightarrow +\infty} \phi_2(\xi) \leq K^+,$$

moreover, if (A₃) holds, then $\Phi(+\infty) = (K, K_0)$, where $K_0 = g(K)/\beta$ and $\Lambda_1(c)$ is the smallest solution such that the linearized characteristic equation at $(0, 0)$ of (1.2) has solutions;

- (ii) *for $c = c_*$, (1.2) admits a traveling wave solution with the wave speed c_* ;*
- (iii) *for all $c \in (0, c_*)$, (1.2) admits no such wave solution with the wave speed c .*

Lemma 2.2 ([4]). *Let $F(\lambda) := \int_{-\infty}^0 \phi(\xi)e^{-\lambda\xi}d\xi$, where $\phi(\xi)$ is a positive increasing function for $\xi \in \mathbb{R}$. Suppose that $F(\lambda)$ can be written in the form*

$$F(\lambda) = \frac{H(\lambda)}{(\alpha - \lambda)^{k+1}},$$

where $k > -1$ and $H(\lambda)$ is analytic in the strip $0 < \text{Re}\lambda \leq \alpha$ for some $\alpha > 0$, then

$$\lim_{\xi \rightarrow -\infty} \frac{\phi(\xi)}{|\xi|^k e^{\lambda\xi}} = \frac{H(\lambda)}{\Gamma(\lambda + 1)}.$$

Next, we state our main result about the asymptotic behavior of traveling wave solutions of (1.1).

Theorem 2.3. *Assume that (A₁)-(A₄) hold. Let (ϕ_1, ϕ_2) be the solution of (2.2), then there exist constants $\theta_i = \theta_i(\phi_1, \phi_2)$, $i = 1, 2$, such that*

$$\lim_{\xi \rightarrow -\infty} \frac{\phi_i(\xi + \theta_i)}{|\xi|^k e^{\lambda_1\xi}} = 1,$$

where $k = 0$ for $c > c^$ and $k = 1$ for $c = c^*$.*

Proof. By Proposition 2.1, it is easy to obtain that, for any $0 < k < \lambda_1$, $\phi_i(\xi) = O(e^{k\xi})$ as $\xi \rightarrow -\infty$, $i = 1, 2$, where λ_1 is the smallest positive real root of $P(\lambda) = 0$ for (2.1). Let denote's

$$\Phi_i(\lambda) := \int_{\mathbb{R}} \phi_i(\xi)e^{-\lambda\xi}d\xi, \quad i = 1, 2.$$

Then we get that $\Phi_i(\lambda) < \infty$ for all complex λ with $\operatorname{Re}\lambda \in (0, \lambda_1)$. By integration by parts, we have

$$\int_{\mathbb{R}} \phi_i'(\xi)e^{-\lambda\xi}d\xi = \lambda\Phi_i(\lambda), \quad i = 1, 2, \quad \operatorname{Re}\lambda \in (0, \lambda_1),$$

and

$$\int_{\mathbb{R}} \phi_i''(\xi)e^{-\lambda\xi}d\xi = \lambda^2\Phi_i(\lambda), \quad i = 1, 2, \quad \operatorname{Re}\lambda \in (0, \lambda_1).$$

Equations (2.2) can be rewritten in the following integral form

$$\phi_1(\xi) = \frac{1}{\lambda_1^+ - \lambda_1^-} \left[\int_{-\infty}^{\xi} e^{\lambda_1^-(\xi-s)} \phi_2(s) ds + \int_{\xi}^{+\infty} e^{\lambda_1^+(\xi-s)} \phi_2(s) ds \right], \quad (2.3)$$

$$\phi_2(\xi) = \frac{1}{c} \int_{-\infty}^{\xi} e^{-\frac{\beta}{c}(\xi-s)} g(\phi_1(s - c\tau)) ds, \quad (2.4)$$

where $\lambda_1^- = \frac{c-\sqrt{c^2+4}}{2} < 0$ and $\lambda_1^+ = \frac{c+\sqrt{c^2+4}}{2} > 0$ are the solutions of $\lambda^2 - c\lambda - 1 = 0$. Moreover, the integral form (2.3) and (2.4) imply that

$$\begin{aligned} \phi_1'(\xi) &= \frac{1}{\lambda_1^+ - \lambda_1^-} \left[\lambda_1^- \int_{-\infty}^{\xi} e^{\lambda_1^-(\xi-s)} \phi_2(s) ds + \lambda_1^+ \int_{\xi}^{+\infty} e^{\lambda_1^+(\xi-s)} \phi_2(s) ds \right], \\ \phi_2'(\xi) &= -\frac{\beta}{c^2} \int_{-\infty}^{\xi} e^{-\frac{\beta}{c}(\xi-s)} g(\phi_1(s - c\tau)) ds. \end{aligned}$$

Let $0 < k < \lambda_1$, since $\phi_2(\xi) = O(e^{k\xi})$ as $\xi \rightarrow -\infty$, there exists a $\xi_1 < 0$ such that $|\phi_2(\xi)| \leq Me^{k\xi}$ for $\xi \leq \xi_1$, we obtain

$$\int_{-\infty}^{\xi} |e^{\lambda_1^-(\xi-s)} \phi_2(s)| ds \leq M \int_{-\infty}^{\xi} e^{\lambda_1^-(\xi-s)} e^{ks} ds = \frac{Me^{k\xi}}{k - \lambda_1^-}$$

and

$$\begin{aligned} \int_{\xi}^{+\infty} |e^{\lambda_1^+(\xi-s)} \phi_2(s)| ds &= \int_{\xi}^{\xi_1} |e^{\lambda_1^+(\xi-s)} \phi_2(s)| ds + \int_{\xi_1}^{+\infty} |e^{\lambda_1^+(\xi-s)} \phi_2(s)| ds \\ &\leq M \int_{\xi}^{\xi_1} e^{\lambda_1^+(\xi-s)} e^{ks} ds + e^{\lambda_1^+\xi} \int_{\xi_1}^{+\infty} e^{-\lambda_1^+s} |\phi_2(s)| ds \\ &\leq \frac{Me^{k\xi}}{\lambda_1^+ - k} + M_1 e^{\lambda_1^+\xi}, \end{aligned}$$

where $M_1 = \int_{\xi_1}^{+\infty} e^{-\lambda_1^+s} |\phi_2(s)| ds < \infty$. Thus, it follows that $\phi_1'(\xi) = O(e^{k\xi})$, and similarly $\phi_2'(\xi) = O(e^{k\xi})$ as $\xi \rightarrow -\infty$. Since $g \in C^2$, we get

$$g(\phi_1(\xi - c\tau)) - g'(0)\phi_1(\xi - c\tau) = O(|\phi_1(\xi - c\tau)|^2) = O(|\phi_1(\xi)|^2) = O(e^{2k\xi})$$

as $\xi \rightarrow -\infty$ for any $0 < k < \lambda$. Now we define the functions $Q_i(\lambda)$, $i = 1, 2$,

$$Q_1(\lambda) := \int_{\mathbb{R}} \phi_2(\xi)e^{-\lambda\xi}d\xi, \quad Q_2(\lambda) := \int_{\mathbb{R}} \left[g'(0)\phi_1(\xi - c\tau) - g(\phi_1(\xi - c\tau)) \right] e^{-\lambda\xi}d\xi,$$

are analytic in the strip $\operatorname{Re}\lambda \in (0, 2\lambda_1)$. At the same time, it is easy to see that $Q_i(\lambda) > 0$, $i = 1, 2$, $\operatorname{Re}\lambda \in (0, 2\lambda_1)$. Therefore, (2.2) can be rewritten in the following form

$$\phi_1''(\xi) - c\phi_1'(\xi) - \phi_1(\xi) = -\phi_2(\xi), \quad (2.5)$$

$$-c\phi_2'(\xi) - \beta\phi_2(\xi) + g'(0)\phi_1(\xi - c\tau) = g'(0)\phi_1(\xi - c\tau) - g(\phi_1(\xi - c\tau)). \quad (2.6)$$

Multiplying (2.5) and (2.6) by $e^{-\lambda\xi}$ and integrating both sides of the equations from $-\infty$ to $+\infty$, we get

$$\begin{aligned} (\lambda^2 - c\lambda - 1)\Phi_1(\lambda) &= -Q_1(\lambda), \\ (-c\lambda - \beta)\Phi_2(\lambda) + g'(0)e^{-\lambda c\tau}\Phi_1(\lambda) &= Q_2(\lambda). \end{aligned}$$

Thus,

$$\begin{aligned} \Phi_1(\lambda) &= -\frac{Q_1(\lambda)}{\lambda^2 - c\lambda - 1}, \quad \operatorname{Re}\lambda \in (0, \lambda_1), \\ P(\lambda)\Phi_2(\lambda) &= (\lambda^2 - c\lambda - 1)Q_2(\lambda) + g'(0)[e^{-\lambda c\tau} - 1]Q_1(\lambda), \quad \operatorname{Re}\lambda \in (0, \lambda_1). \end{aligned} \tag{2.7}$$

Denoting $F(\lambda) := \int_{-\infty}^0 \phi_2(\xi)e^{-\lambda\xi}d\xi$, by (2.7), $F(\lambda)$ can be rewritten as

$$F(\lambda) = \frac{(\lambda^2 - c\lambda - 1)Q_2(\lambda) + g'(0)[e^{-\lambda c\tau} - 1]Q_1(\lambda)}{P(\lambda)} - \int_0^{+\infty} \phi_2(\xi)e^{-\lambda\xi}d\xi := \frac{H(\lambda)}{(\lambda_1 - \lambda)^{k+1}},$$

where $k = 0$ for $c > c_*$ and $k = 1$ for $c = c_*$. Let $H(\lambda) := F(\lambda)(\lambda_1 - \lambda)^{k+1}$, then we have

$$H(\lambda) = \frac{(\lambda^2 - c_*\lambda - 1)Q_2(\lambda) + g'(0)[e^{-\lambda c\tau} - 1]Q_1(\lambda)}{P(\lambda)/(\lambda_1 - \lambda)^{k+1}} - (\lambda_1 - \lambda)^{k+1} \int_0^{+\infty} \phi_2(\xi)e^{-\lambda\xi}d\xi,$$

since $\phi_2(\xi) < \infty$ and $(\lambda_1 - \lambda)^{k+1} \int_0^{+\infty} \phi_2(\xi)e^{-\lambda\xi}d\xi$ is analytic for $\operatorname{Re}\lambda > 0$. Note that λ_1 is the smallest positive simple solution of $P(\lambda) = 0$ for $c > c_*$. Therefore, $(\lambda_1 - \lambda)^{k+1}/P(\lambda)$ is analytic for $0 < \operatorname{Re}\lambda \leq \lambda_1$. In addition, we know $(\lambda_1^2 - c\lambda_1 - 1) < 0$, $Q_i(\lambda_1) > 0$ and $P(\lambda)/(\lambda_1 - \lambda)^{k+1} < 0$ at near $\lambda = \lambda_1$ for $i = 1, 2$, it follows that $H(\lambda_1) > 0$.

Notice that $\phi_2(\xi)$ may not be monotone increasing on $(-\infty, 0)$, there exists a constant $m > 0$ such that $\bar{\phi}_2(\xi) := \phi_2(\xi)e^{m\xi}$ is monotone increasing on $(-\infty, 0)$. By (2.4), we have

$$c\bar{\phi}_2'(\xi) = \int_{-\infty}^{\xi} \left(m - \frac{\beta}{c}\right)e^{m\xi - \frac{\beta}{c}(\xi-s)}g(\phi(s - c\tau))ds.$$

Thus $\bar{\phi}_2'(\xi) > 0$ provided $m > \frac{\beta}{c_*}$. This implies that $\bar{\phi}_2(\xi)$ is nondecreasing. It is easy to see that

$$\bar{F}(\lambda) := \int_{-\infty}^0 \bar{\phi}_2(\xi)e^{-\lambda\xi}d\xi = F(\lambda - m).$$

Using the modified Ikehara's theorem in [1] again, we obtain the existence of $\lim_{\xi \rightarrow -\infty} \frac{\bar{\phi}_2(\xi)}{|\xi|^k e^{(m+\lambda_1)\xi}}$, where $k = 0$ as $c > c_*$ and $k = 1$ as $c = c_*$. Denote

$$G(\lambda_1) := \lim_{\xi \rightarrow -\infty} \frac{\bar{\phi}_2(\xi)}{|\xi|^k e^{(m+\lambda_1)\xi}} = \lim_{\xi \rightarrow -\infty} \frac{\phi_2(\xi)}{|\xi|^k e^{\lambda_1\xi}},$$

and for any $\theta_2 \in \mathbb{R}$, we obtain

$$G(\lambda_1) = \lim_{\xi \rightarrow -\infty} \frac{\phi_2(\xi + \theta_2)}{|\xi + \theta_2|^k e^{\lambda_1(\xi + \theta_2)}} = e^{-\lambda_1\theta_2} \lim_{\xi \rightarrow -\infty} \frac{\phi_2(\xi + \theta_2)}{|\xi|^k e^{\lambda_1\xi}},$$

let θ_2 is the constant satisfying $e^{\lambda_1\theta_2}G(\lambda_1) = 1$, then

$$\lim_{\xi \rightarrow -\infty} \frac{\phi_2(\xi + \theta_2)}{|\xi|^k e^{\lambda_1\xi}} = 1,$$

where $k = 0$ as $c > c_*$ and $k = 1$ as $c = c_*$. Similarly, we can obtain that there exists the constant θ_1 such that

$$\lim_{\xi \rightarrow -\infty} \frac{\phi_2(\xi + \theta_2)}{|\xi|^k e^{\lambda_1\xi}} = 1,$$

where $k = 0$ as $c > c_*$ and $k = 1$ as $c = c_*$. This completes the proof. □

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