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Coupled fixed points in complex partial metric spaces

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Abstract

In this paper, we obtain coupled fixed point theorems in complex partial metric spaces under the different contractive conditions. Examples are provided to support our results.

Keywords: Coupled fixed point, complex valued metric space, complex partial metric space.

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1. Introduction

In 1992, Matthews ([8, 9]) introduced the notion of partial metric space as a part of the study of denotational semantics of data flow programming languages in computer sciences and also extended Banach contraction principle in this space. Subsequently, many authors reported its topological properties and obtained a number of fixed point theorems (for more details and references, we refer to [6]). Azam et al. [3] introduced complex valued metric spaces and established the existence of common fixed point theorems of rational type contractions. Recently, Dhivya and Marudai [5] introduced complex partial metric space and established the existence of common fixed point theorems under the contraction condition of rational expression. Bhaskar and Lakshmikantham [4] introduced the concept of coupled fixed point. Círíc and Lakshmikantham [7] investigated some more coupled fixed point theorems in partially ordered sets. Aydi [2] proved some coupled fixed point results for mapping satisfying different contractive conditions on complete partial metric spaces. Also, Shatanawi et al. [13] extended the results of Aydi [2] in partial metric spaces. For more results obtained in this direction, we refer to ([1, 11–13]).

2. Preliminaries

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\preceq$ on $\mathbb{C}$ as follows: $z_1 \preceq z_2$ if and only if $\text{Re}(z_1) \leqslant \text{Re}(z_2)$, $\text{Im}(z_1) \leqslant \text{Im}(z_2)$. It follows that $z_1 \preceq z_2$ if one of the following conditions is satisfied:

(C1) $\text{Re}(z_1) = \text{Re}(z_2)$, $\text{Im}(z_1) < \text{Im}(z_2)$;
(C2) $\text{Re}(z_1) < \text{Re}(z_2)$, $\text{Im}(z_1) = \text{Im}(z_2)$;
(C3) $\text{Re}(z_1) < \text{Re}(z_2)$, $\text{Im}(z_1) < \text{Im}(z_2)$;
(C4) $\text{Re}(z_1) = \text{Re}(z_2)$, $\text{Im}(z_1) = \text{Im}(z_2)$.

In particular, we write $z_1 \precsim z_2$ if $z_1 \neq z_2$ and one of (C1), (C2), and (C3) is satisfied and also we write $z_1 \prec z_2$ if only (C3) is satisfied. Notice that

(a) $0 \preceq z_1 \precsim z_2$, then $|z_1| < |z_2|$;
(b) $z_1 \preceq z_2$ and $z_2 \prec z_3$, then $z_1 \prec z_3$;
(c) $0 \leqslant a \leqslant b$ and $z_1 \preceq z_2$, then $az_1 \preceq bz_2$.

We collect some definitions with examples.

Definition 2.1 ([3]). Let X be a non-empty set. A mapping $d_c : X \times X \to \mathbb{C}$ is called a complex valued metric on X if it satisfies:

$(d_c 1)$ $0 \preceq d_c(z_1, z_2)$, for all $z_1, z_2 \in X$ and $d_c(z_1, z_2) = 0 \Leftrightarrow z_1 = z_2$;
$(d_c 2)$ $d_c(z_1, z_2) = d_c(z_2, z_1)$, for all $z_1, z_2 \in X$;
$(d_c 3)$ $d_c(z_1, z_2) \preceq d_c(z_1, z_3) + d_c(z_3, z_2)$, for all $z_1, z_2, z_3 \in X$.

Then, (X, d_c) is called a complex valued metric space.

Example 2.2. Let $X = \mathbb{C}$ be a set of complex number. Define $d_c : X \times X \to \mathbb{C}$ by

(i) $d_c(z_1, z_2) = |x_1 - x_2| + i|y_1 - y_2|$;
(ii) $d_c(z_1, z_2) = e^{i\theta}|z_1 - z_2|$, where $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2, 0 \leqslant \theta \leqslant \frac{\pi}{2}$.

Then, (X, d_c) is a complex valued metric space.

Definition 2.3 ([8]). A partial metric on a non-empty set X is a function $\sigma : X \times X \to \mathbb{R}^+$ such that for all $p, q, r \in X$:

(σ_1) $p = q \Longleftrightarrow \sigma(p, p) = \sigma(p, q) = \sigma(p, q)$;
(σ_2) $\sigma(p, p) \leqslant \sigma(p, q)$;
(σ_3) $\sigma(p, q) = \sigma(q, p)$;
(σ_4) $\sigma(p, q) \leqslant \sigma(p, q) \leqslant \sigma(p, r) + \sigma(r, q) - \sigma(r, r)$.

Then, the pair (X, σ) is called a partial metric space. Note that if $\sigma(p, q) = 0$, then from $(\sigma 1)$ and $(\sigma 2)$, $p = q$, however the converse may not be true.

Moreover, if σ is a partial metric on X, then the function $d_\sigma : X \times X \to \mathbb{R}^+$ defined by

$$d_\sigma(p, q) = 2\sigma(p, q) - \sigma(p, p) - \sigma(q, q)$$

is a metric on X.

Example 2.4 ([8, 9]). Let $X = \mathbb{R}^+$ and function $\sigma : X \times X \to \mathbb{R}^+$ be defined by $\sigma(p, q) = \max\{p, q\}$, for all $p, q \in X$. Then, (X, σ) is a partial metric space.

Example 2.5. Let $X = \{0, 1, 2\}$ and function $\sigma : X \times X \to \mathbb{R}^+$ be defined by

$\sigma(0,0) = \sigma(1,1) = \sigma(2,2) = 1$; $\sigma(0,1) = \sigma(1,0) = \sigma(0,2) = \sigma(2,0) = 2$; $\sigma(1,2) = \sigma(2,1) = 3$.

Note that (σ_1), (σ_2), and (σ_3) are easy to verify. For (σ_4), we have the following cases.

Case (i): For $p = q$, $\sigma(p, q) = 1$.

Case (ii): For $p = 0, q = 1$,

$$2 = \sigma(0, 1) \leqslant \sigma(0, 2) + \sigma(2, 1) - \sigma(2, 2) = 4.$$

Case (iii): For $p = 0, q = 2$,

$$2 = \sigma(0, 2) \leqslant \sigma(0, 1) + \sigma(1, 2) - \sigma(1, 1) = 4.$$

Case (iv): For $p = 1, q = 2$,

$$3 = \sigma(1, 2) \leqslant \sigma(1, 0) + \sigma(0, 2) - \sigma(0, 0) = 3.$$

Then, (X, σ) is a partial metric space.

Definition 2.6 ([5]). A complex partial metric on a non-empty set Y is a function $\sigma_c : Y \times Y \to \mathbb{C}^+$ such that for all $p, r, s \in Y$:

$(\sigma_c 1)$ $0 \preceq \sigma_c(p, p) \preceq \sigma_c(p, r)$ (small self-distances);
$(\sigma_c 2)$ $\sigma_c(p, r) = \sigma_c(r, p)$ (symmetry);
$(\sigma_c 3)$ $\sigma_c(p, p) = \sigma_c(p, r) = \sigma_c(r, r) \Leftrightarrow p = r$ (equality);
$(\sigma_c 4)$ $\sigma_c(p, r) \preceq \sigma_c(p, s) + \sigma_c(s, r) - \sigma_c(s, s)$ (triangularity).

Then, (Y, σ_c) is called a complex partial metric space and in short, we write CPMS.

For the complex partial metric σ_c on Y, the function $\sigma_c^t : Y \times Y \to \mathbb{C}^+$ given by $\sigma_c^t(p, r) = 2\sigma_c(p, r) - \sigma_c(p, p) - \sigma_c(r, r)$ is a (usual) metric on Y. Each complex partial metric σ_c on Y generates a topology τ_{σ_c} on Y with the base family of open σ_c-balls $\{B_{\sigma_c}(p, \epsilon) : p \in Y, \epsilon \succ 0\}$, where $B_{\sigma_c}(p, \epsilon) = \{r \in Y : \sigma_c(p, r) \prec \sigma_c(p, p) + \epsilon\}$, for all $p \in Y$ and $0 \prec \epsilon \in \mathbb{C}^+$.

Note that a complex valued metric space is a CPMS. But a CPMS need not be a complex valued metric space. We discuss the following examples for complex partial metric spaces.

Example 2.7 ([5]). Let $Y = [0, \infty)$ endowed with complex partial metric $\sigma_c : Y \times Y \to \mathbb{C}^+$ with $\sigma_c(p, q) = \max\{p, q\}(1 + i)$, for all $p, q \in Y$. Obviously (Y, σ_c) is a complex partial metric space. Note that $\sigma_c(p, p) = p + ip \neq 0$, for all $p \neq 0$. Therefore, (Y, σ_c) is not a complex valued metric space.

Example 2.8. Let $Y = \mathbb{R}$ and function $\sigma_c : Y \times Y \to \mathbb{C}^+$ be defined by $\sigma_c(p, q) = \{|p - q| + \tau\}e^{i\theta}$, for all $p, q \in Y$, where $\tau > 0$, $0 \leqslant \theta \leqslant \frac{\pi}{2}$. It may be noted that $\sigma_c(p, p) = \tau e^{i\theta} \neq 0$, for all $p \in Y$. Obviously, one can verify that σ_c satisfies $(\sigma_c 1)$, $(\sigma_c 2)$, and $(\sigma_c 3)$ hold. For $(\sigma_c 4)$, we obtain

$$
\begin{aligned}
\sigma_c(p, q) &= \{|p - q| + \tau\}e^{i\theta} \\
&= \{|(p - r) + (r - q)| + \tau\}e^{i\theta} \\
&\preceq \{|p - r| + |r - q| + \tau\}e^{i\theta} \\
&= \left(\{|p - r| + \tau\} + \{|r - q| + \tau\} - \tau\right)e^{i\theta} \\
&= \sigma_c(p, r) + \sigma_c(r, q) - \sigma_c(r, r).
\end{aligned}
$$

Therefore, (Y, σ_c) is a complex partial metric space.

Example 2.9. Let $X = \{0, 1, 2, 3\}$ and $\sigma_c : X \times X \to \mathbb{C}^+$ by

$$
\begin{aligned}
\sigma_c(0, 0) &= \sigma_c(1, 1) = \sigma_c(2, 2) = \sigma_c(3, 3) = e^{i\theta}; \\
\sigma_c(0, 1) &= \sigma_c(1, 0) = \sigma_c(0, 2) = \sigma_c(2, 0) = \sigma_c(0, 3) = \sigma_c(3, 0) = 2e^{i\theta}; \\
\sigma_c(1, 2) &= \sigma_c(2, 1) = \sigma_c(1, 3) = \sigma_c(3, 1) = \sigma_c(2, 3) = \sigma_c(3, 2) = 3e^{i\theta},
\end{aligned}
$$

where $0 \leqslant \theta \leqslant \frac{\pi}{2}$. One can verify that $(\sigma_c 1)$, $(\sigma_c 2)$, and $(\sigma_c 3)$. We verify $(\sigma_c 4)$ with the following cases.

Case (i): For $p = q \in X$, $\sigma_c(p, q) = e^{i\theta}$.

Case (ii): For $p = 0, q = 1, r \in \{2, 3\}$, we obtain

$$2e^{i\theta} = \sigma_c(0, 1) \preceq \sigma_c(0, r) + \sigma_c(r, 1) - \sigma_c(r, r) = 4e^{i\theta}.$$

Case (iii): For $p = 0, q = 2, r \in \{1, 3\}$, we obtain

$$2e^{i\theta} = \sigma_c(0, 2) \preceq \sigma_c(0, r) + \sigma_c(r, 2) - \sigma_c(r, r) = 4e^{i\theta}.$$

Case (iv): For $p = 0, q = 3, r \in \{1, 2\}$, we obtain

$$2e^{i\theta} = \sigma_c(0, 3) \preceq \sigma_c(0, r) + \sigma_c(r, 3) - \sigma_c(r, r) = 4e^{i\theta}.$$

Case (v): For $p = 1, q = 2, r \in \{0, 3\}$, we obtain

$$3e^{i\theta} = \sigma_c(1, 2) \preceq \sigma_c(1, r) + \sigma_c(r, 2) - \sigma_c(r, r) = 3e^{i\theta}(or, 5e^{i\theta}).$$

Case (vi): For $p = 1, q = 3, r \in \{0, 2\}$, we obtain

$$3e^{i\theta} = \sigma_c(1, 3) \preceq \sigma_c(1, r) + \sigma_c(r, 3) - \sigma_c(r, r) = 3e^{i\theta}(or, 5e^{i\theta}).$$

Case (vii): For $p = 2, q = 3, r \in \{0, 1\}$, we obtain

$$3e^{i\theta} = \sigma_c(2, 3) \preceq \sigma_c(2, r) + \sigma_c(r, 3) - \sigma_c(r, r) = 3e^{i\theta}(or, 5e^{i\theta}).$$

Therefore, (X, σ_c) is a complex partial metric space.

Definition 2.10 ([5]). Let (Y, σ_c) be a complex partial metric space.

(1) A sequence $\{y_n\}$ in (Y, σ_c) converges to $p \in Y$, if for every $0 \prec \epsilon \in \mathbb{C}^+$ there is $N \in \mathbb{N}$ such that for all $n > N$, we get $y_n \in B_{\sigma_c}(p, \epsilon)$. Then p is said to be a limit of $\{y_n\}$, which is denoted by $\lim_{n \to \infty} y_n = p$ or, $y_n \to p$.
(2) A sequence (y_n) in (Y, σ_c) is called Cauchy if there is $a \in \mathbb{C}^+$ such that for every $\epsilon > 0$ there is $N \in \mathbb{N}$ such that for all $n, m \geqslant N$, $|\sigma_c(y_n, y_m) - a| < \epsilon$.

Definition 2.11 ([5]). Let (Y, σ_c) be a complex partial metric space.

(1) (Y, σ_c) is said to be complete if every Cauchy sequence $\{y_n\}$ in Y converges, with respect to τ_{σ_c}, to a point $p \in Y$ such that $\sigma_c(p, p) = \lim_{n,m \to \infty} \sigma_c(y_n, y_m)$.
(2) A mapping $H : Y \to Y$ is said to be continuous at $p_0 \in Y$ if for every $\epsilon \succ 0$, there exists $\delta \succ 0$ such that $H(B_{\sigma_c}(p_0, \delta)) \subset B_{\sigma_c}(H(p_0), \epsilon)$.

Lemma 2.12 ([5]). *Let (Y, σ_c) be a complex partial metric space. A sequence $\{y_n\}$ in (Y, σ_c) converges to $p \in Y$ if and only if $\sigma_c(p, p) = \lim_{n \to \infty} \sigma_c(p, y_n)$.*

Lemma 2.13 ([5]). *Let (Y, σ_c) be a complex partial metric space. A sequence $\{y_n\}$ is Cauchy sequence in (Y, σ_c), then $\{y_n\}$ is Cauchy in a metric space (Y, σ_c^t).*

Lemma 2.14 ([10]). *Let (Y, σ_c) be a complex partial metric space.*

(1) *A sequence $\{y_n\}$ is Cauchy sequence in (Y, σ_c) if and only if $\{y_n\}$ is Cauchy in a metric space (Y, σ_c^t).*
(2) *(Y, σ_c) is complete if and only if (Y, σ_c^t) is complete. Moreover, $\lim_{n \to \infty} \sigma_c^t(y, y_n) = 0 \Leftrightarrow \sigma_c(y, y) = \lim_{n \to \infty} \sigma_c(y, y_n) = \lim_{n,m \to \infty} \sigma_c(y_n, y_m)$.*

Definition 2.15. Let (Y, σ_c) be a complex partial metric space. Then an element $(p, r) \in Y \times Y$ is said to be a coupled fixed point of the mapping $F : Y \times Y \to Y$ if $F(p, r) = p$ and $F(r, p) = r$.

In this paper, we also extend the coupled fixed point results of Aydi [2] and Shatanawi et al. [13] in complex partial metric spaces.

3. Main results

Theorem 3.1. *Let* (Y, σ_c) *be a complete CPMS. Suppose that the mapping* $\psi : Y \times Y \to Y$ *satisfies*

$$\sigma_c(\psi(p, q), \psi(r, s)) \preceq k\sigma_c(p, r) + l\sigma_c(q, s),$$

for all $p, q, r, s \in Y$, *where* k *and* l *are non-negative constants with* $k + l < 1$. *Then* ψ *has a unique coupled fixed point.*

Proof. Choose $p_0, q_0 \in Y$ and set $p_1 = \psi(p_0, q_0)$ and $q_1 = \psi(q_0, p_0)$. Continuing in this process and setting $p_{n+1} = \psi(p_n, q_n)$ and $q_{n+1} = \psi(q_n, p_n)$, we obtain

$$\sigma_c(p_n, p_{n+1}) = \sigma_c(\psi(p_{n-1}, q_{n-1}), \psi(p_n, q_n)) \preceq k\sigma_c(p_{n-1}, p_n) + l\sigma_c(q_{n-1}, q_n),$$

which implies that

$$|\sigma_c(p_n, p_{n+1})| \leqslant k|\sigma_c(p_{n-1}, p_n)| + l|\sigma_c(q_{n-1}, q_n)|. \tag{3.1}$$

Similarly, one can prove that

$$|\sigma_c(q_n, q_{n+1})| \leqslant k|\sigma_c(q_{n-1}, q_n)| + l|\sigma_c(p_{n-1}, p_n)|. \tag{3.2}$$

From (3.1) and (3.2), we get

$$|\sigma_c(p_n, p_{n+1})| + |\sigma_c(q_n, q_{n+1})| \leqslant (k + l)(|\sigma_c(q_{n-1}, q_n)| + |\sigma_c(p_{n-1}, p_n)|)$$
$$= \alpha(|\sigma_c(q_{n-1}, q_n)| + |\sigma_c(p_{n-1}, p_n)|),$$

where $\alpha = k + l < 1$. Also, we obtain

$$|\sigma_c(p_{n+1}, p_{n+2})| \leqslant k|\sigma_c(p_n, p_{n+1})| + l|\sigma_c(q_n, q_{n+1})|, \tag{3.3}$$
$$|\sigma_c(q_{n+1}, q_{n+2})| \leqslant k|\sigma_c(q_n, q_{n+1})| + l|\sigma_c(p_n, p_{n+1})|. \tag{3.4}$$

From (3.3) and (3.4), we obtain

$$|\sigma_c(p_{n+1}, p_{n+2})| + |\sigma_c(q_{n+1}, q_{n+2})| \leqslant (k + l)(|\sigma_c(q_n, q_{n+1})| + |\sigma_c(p_n, p_{n+1})|)$$
$$= \alpha(|\sigma_c(q_n, q_{n+1})| + |\sigma_c(p_n, p_{n+1})|).$$

Repeating this way, we obtain

$$|\sigma_c(p_n, p_{n+1})| + |\sigma_c(q_n, q_{n+1})| \leqslant \alpha(|\sigma_c(q_{n-1}, q_n)| + |\sigma_c(p_{n-1}, p_n)|)$$
$$\leqslant \alpha^2(|\sigma_c(q_{n-2}, q_{n-1})| + |\sigma_c(p_{n-2}, p_{n-1})|)$$
$$\vdots$$
$$\leqslant \alpha^n(|\sigma_c(q_0, q_1)| + |\sigma_c(p_0, p_1)|).$$

Now, if $|\sigma_c(p_n, p_{n+1})| + |\sigma_c(q_n, q_{n+1})| = s_n$, then

$$s_n \leqslant \alpha s_{n-1} \leqslant \cdots \leqslant \alpha^n s_0.$$

If $s_0 = 0$, then $|\sigma_c(p_0, p_1)| + |\sigma_c(q_0, q_1)| = 0$. Hence $p_0 = p_1 = \psi(p_0, q_0)$ and $q_0 = q_1 = \psi(q_0, p_0)$, which implies that (p_0, q_0) is a coupled fixed point of ψ. Let $s_0 > 0$. For each $n \geqslant m$, we have

$$\sigma_c(p_n, p_m) \preceq \sigma_c(p_n, p_{n-1}) + \sigma_c(p_{n-1}, p_{n-2}) - \sigma_c(p_{n-1}, p_{n-1})$$
$$+ \sigma_c(p_{n-2}, p_{n-3}) + \sigma_c(p_{n-3}, p_{n-4}) - \sigma_c(p_{n-3}, p_{n-3})$$

$$+ \cdots + \sigma_c(p_{m+2}, p_{m+1}) + \sigma_c(p_{m+1}, p_m) - \sigma_c(p_{m+1}, p_{m+1})$$
$$\preceq \sigma_c(p_n, p_{n-1}) + \sigma_c(p_{n-1}, p_{n-2}) + \cdots + \sigma_c(p_{m+1}, p_m),$$

which implies that

$$|\sigma_c(p_n, p_m)| \leqslant |\sigma_c(p_n, p_{n-1})| + |\sigma_c(p_{n-1}, p_{n-2})| + \cdots + |\sigma_c(p_{m+1}, p_m)|.$$

Similarly, one can prove that

$$|\sigma_c(q_n, q_m)| \leqslant |\sigma_c(q_n, q_{n-1})| + |\sigma_c(q_{n-1}, q_{n-2})| + \cdots + |\sigma_c(q_{m+1}, q_m)|.$$

Thus,

$$|\sigma_c(p_n, p_m)| + |\sigma_c(q_n, q_m)| \leqslant s_{n-1} + s_{n-2} + s_{n-3} + \cdots + s_m$$
$$\leqslant (\alpha^{n-1} + \alpha^{n-2} + \cdots + \alpha^m)s_0 \leqslant \frac{\alpha^m}{1-\alpha}s_0 \to 0, \quad n \to \infty,$$

which implies that $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences in (Y, σ_c). Since the complex partial metric space (Y, σ_c) is complete, there exist $p, q \in Y$ such that $\{p_n\} \to p$ and $q_n \to q$ as $n \to \infty$ and $\sigma_c(p, p) = \lim_{n \to \infty} \sigma_c(p, p_n) = \lim_{n,m \to \infty} \sigma_c(p_n, p_m) = 0$, $\sigma_c(q, q) = \lim_{n \to \infty} \sigma_c(q, q_n) = \lim_{n,m \to \infty} \sigma_c(q_n, q_m) = 0$. We now show that $p = \psi(p, q)$. We suppose on the contrary that $p \neq \psi(p, q)$ and $q \neq \psi(q, p)$ so that $0 \prec \sigma_c(p, \psi(p, q)) = l_1$ and $0 \prec \sigma_c(q, \psi(q, p)) = l_2$, then

$$l_1 = \sigma_c(p, \psi(p, q)) \preceq \sigma_c(p, p_{n+1}) + \sigma_c(p_{n+1}, \psi(p, q))$$
$$= \sigma_c(p, p_{n+1}) + \sigma_c(\psi(p_n, q_n), \psi(p, q))$$
$$\preceq \sigma_c(p, p_{n+1}) + k\sigma_c(p_n, p) + l\sigma_c(q_n, q),$$

which implies that

$$|l_1| \leqslant |\sigma_c(p, p_{n+1})| + k|\sigma_c(p_n, p)| + l|\sigma_c(q_n, q)|.$$

As $n \to \infty$, $|l_1| \leqslant 0$ which is a contradiction, therefore $|\sigma_c(p, \psi(p, q))| = 0 \implies p = \psi(p, q)$. Similarly we can prove that $q = \psi(q, p)$. Thus (p, q) is a coupled fixed point of ψ. Now, if (g, h) is another coupled fixed point of ψ, then

$$\sigma_c(p, g) = \sigma_c(\psi(p, q), \psi(g, h)) \preceq k\sigma_c(p, g) + l\sigma(q, h).$$

Thus,

$$\sigma_c(p, g) \preceq \frac{l}{1-k}\sigma_c(q, h),$$

which implies that

$$|\sigma_c(p, g)| \leqslant \frac{l}{1-k}|\sigma_c(q, h)|. \tag{3.5}$$

Similarly,

$$|\sigma_c(q, h)| \leqslant \frac{l}{1-k}|\sigma_c(p, g)|. \tag{3.6}$$

From (3.5) and (3.6), we obtain

$$|\sigma_c(p, g)| + |\sigma_c(q, h)| \leqslant \frac{l}{1-k}[|\sigma_c(p, g)| + |\sigma_c(q, h)|], \qquad \left(1 - \frac{l}{1-k}\right)(|\sigma_c(p, g)| + |\sigma_c(q, h)|) \leqslant 0.$$

Since $k + l < 1$, this implies that $|\sigma_c(p, g)| + |\sigma_c(q, h)| \leqslant 0$. Therefore $p = g$ and $q = h \implies (p, q) = (g, h)$. Thus ψ has a unique coupled fixed point. $\qquad \square$

Corollary 3.2. *Let* (Y, σ_c) *be a complete CPMS. Suppose that the mapping* $\psi : Y \times Y \to Y$ *satisfies*

$$\sigma_c(\psi(p, q), \psi(r, s)) \preceq \frac{k}{2}(\sigma_c(p, r) + \sigma_c(q, s)), \tag{3.7}$$

for all $p, q, r, s \in Y$, *where* k *is a non-negative constant with* $k < 1$. *Then* ψ *has a unique coupled fixed point.*

Example 3.3. Let $Y = [0, \infty)$ endowed with the usual complex partial metric $\sigma_c : Y \times Y \to [0, \infty)$ defined by $\sigma_c(p, q) = \max\{p, q\}(1 + i)$. The complex partial metric space (Y, σ_c) is complete because (Y, σ_c^t) is complete. Indeed, for any $p, q \in Y$,

$$\sigma_c^t = 2\sigma_c(p, r) - \sigma_c(p, p) - \sigma_c(r, r) = 2\max\{p, q\}(1 + i) - (p + ip) - (q + iq) = |p - q| + i|p - q|.$$

Thus, (Y, σ_c) is the Euclidean complex partial metric space which is complete. Consider the mapping $\psi : Y \times Y \to Y$ defined by $\psi(p, q) = \frac{p+q}{12}$. For any $p, q, g, h \in Y$, we have

$$\sigma_c(\psi(p, q), \psi(g, h)) = \frac{1}{12}\max\{p + q, g + h\}(1 + i)$$

$$\preceq \frac{1}{12}[\max\{p, g\} + \max\{q, h\}](1 + i) = \frac{1}{12}[\sigma_c(p, u) + \sigma_c(q, h)],$$

which is the contractive condition (3.7) for $k = \frac{1}{6}$. Therefore, by Corollary 3.2, ψ has a unique coupled fixed point, which is $(0, 0)$. Note that if the mapping $\psi : Y \times Y \to Y$ is given by $\psi(p, q) = \frac{p+q}{2}$, then ψ satisfies the contractive condition (3.7) for $k = 1$, that is,

$$\sigma_c(\psi(p, q), \psi(g, h)) = \frac{1}{2}\max\{p + q, g + h\}(1 + i)$$

$$\leqslant \frac{1}{2}[\max\{p, g\} + \max\{q, h\}](1 + i)$$

$$= \frac{1}{2}[\sigma_c(p, g) + \sigma_c(q, h)].$$

In this case, $(0, 0)$ and $(1, 1)$ are both coupled fixed points of ψ, and, hence, the coupled fixed point of ψ is not unique. This shows that the condition $k < 1$ in Corollary 3.2 and hence $k + l < 1$ in Theorem 3.1 cannot be omitted in the statement of the aforesaid results.

Theorem 3.4. *Let* (Y, σ_c) *be a complete CPMS. Suppose that the mapping* $\psi : Y \times Y \to Y$ *satisfies*

$$\sigma_c(\psi(p, q), \psi(r, s)) \preceq k\sigma_c(\psi(p, q), r) + l\sigma_c(\psi(r, s), p),$$

for all $p, q, r, s \in Y$, *where* k *and* l *are non-negative constants with* $k + 2l < 1$. *Then* ψ *has a unique coupled fixed point.*

Proof. Choose $p_0, q_0 \in Y$ and set $p_1 = \psi(p_0, q_0)$ and $q_1 = \psi(q_0, p_0)$. Continuing in this process and setting $p_{n+1} = \psi(p_n, q_n)$ and $q_{n+1} = \psi(q_n, p_n)$, we obtain

$$\begin{aligned}
\sigma_c(p_n, p_{n+1}) &= \sigma_c(\psi(p_{n-1}, q_{n-1}), \psi(p_n, q_n)) \\
&\preceq k\sigma_c(\psi(p_{n-1}, q_{n-1}), p_n) + l\sigma_c(\psi(p_n, q_n), p_{n-1}) \\
&= k\sigma_c(p_n, p_n) + l\sigma_c(p_{n+1}, p_{n-1}) \\
&\preceq k\sigma_c(p_n, p_{n+1}) + l\sigma_c(p_{n+1}, p_{n-1}) \\
&\preceq k\sigma_c(p_n, p_{n+1}) + l(\sigma_c(p_{n+1}, p_n) + \sigma_c(p_n, p_{n-1}) - \sigma_c(p_n, p_n)) \\
&\preceq k\sigma_c(p_n, p_{n+1}) + l(\sigma_c(p_{n+1}, p_n) + \sigma_c(p_n, p_{n-1})) \\
&\preceq \frac{l}{1 - (k + l)}\sigma_c(p_n, p_{n-1}),
\end{aligned}$$

which implies that

$$|\sigma_c(p_n, p_{n+1})| \leqslant \frac{l}{1-(k+l)}|\sigma_c(p_n, p_{n-1})|. \tag{3.8}$$

Similarly, one can prove that

$$|\sigma_c(q_n, q_{n+1})| \leqslant \frac{l}{1-(k+l)}|\sigma_c(q_n, q_{n-1})|. \tag{3.9}$$

From (3.8) and (3.9), we obtain

$$|\sigma_c(p_n, p_{n+1})| + |\sigma_c(q_n, q_{n+1})| \leqslant \frac{l}{1-(k+l)}(|\sigma_c(p_n, p_{n-1})| + |\sigma_c(q_n, q_{n-1})|)$$
$$= \alpha(|\sigma_c(p_n, p_{n-1})| + |\sigma_c(q_n, q_{n-1})|),$$

where $\alpha = \frac{l}{1-(k+l)} < 1$. Also,

$$|\sigma_c(p_{n+1}, p_{n+2})| \leqslant \frac{l}{1-(k+l)}|\sigma_c(p_n, p_{n-1})|, \tag{3.10}$$

$$|\sigma_c(q_{n+1}, q_{n+2})| \leqslant \frac{l}{1-(k+l)}|\sigma_c(q_n, q_{n-1})|. \tag{3.11}$$

From (3.10) and (3.11), we obtain

$$|\sigma_c(p_{n+1}, p_{n+2})| + |\sigma_c(q_{n+1}, q_{n+2})| \leqslant \frac{l}{1-(k+l)}(|\sigma_c(p_n, p_{n-1})| + |\sigma_c(q_n, q_{n-1})|)$$
$$= \alpha(|\sigma_c(p_n, p_{n-1})| + |\sigma_c(q_n, q_{n-1})|).$$

Repeating this way, we obtain

$$|\sigma_c(p_n, p_{n+1})| + |\sigma_c(q_n, q_{n+1})| \leqslant \alpha(|\sigma_c(p_n, p_{n-1})| + |\sigma_c(q_n, q_{n-1})|)$$
$$\leqslant \alpha^2(|\sigma_c(q_{n-2}, q_{n-1})| + |\sigma_c(p_{n-2}, p_{n-1})|)$$
$$\vdots$$
$$\leqslant \alpha^n(|\sigma_c(q_0, q_1)| + |\sigma_c(p_0, p_1)|).$$

Now, if $|\sigma_c(p_n, p_{n+1})| + |\sigma_c(q_n, q_{n+1})| = s_n$, then

$$s_n \leqslant \alpha s_{n-1} \leqslant \cdots \leqslant \alpha^n s_0.$$

If $s_0 = 0$ then $|\sigma_c(p_0, q_1)| + |\sigma_c(q_0, q_1)| = 0$. Hence $p_0 = p_1 = \psi(p_0, q_0)$ and $q_0 = q_1 = \psi(q_0, p_0)$, which implies that (p_0, q_0) is a coupled fixed point of ψ. Let $s_0 > 0$. For each $n \geqslant m$, we have

$$\sigma_c(p_n, p_m) \preceq \sigma_c(p_n, p_{n-1}) + \sigma_c(p_{n-1}, p_{n-2}) - \sigma_c(p_{n-1}, p_{n-1})$$
$$+ \sigma_c(p_{n-2}, p_{n-3}) + \sigma_c(p_{n-3}, p_{n-4}) - \sigma_c(p_{n-3}, p_{n-3})$$
$$+ \cdots + \sigma_c(p_{m+2}, p_{m+1}) + \sigma_c(p_{m+1}, p_m) - \sigma_c(p_{m+1}, p_{m+1})$$
$$\preceq \sigma_c(p_n, p_{n-1}) + \sigma_c(p_{n-1}, p_{n-2}) + \cdots + \sigma_c(p_{m+1}, p_m),$$

which implies that

$$|\sigma_c(p_n, p_m)| \leqslant |\sigma_c(p_n, p_{n-1})| + |\sigma_c(p_{n-1}, p_{n-2})| + \cdots + |\sigma_c(p_{m+1}, p_m)|.$$

Similarly, one can prove that

$$|\sigma_c(q_n, q_m)| \leqslant |\sigma_c(q_n, q_{n-1})| + |\sigma_c(q_{n-1}, q_{n-2})| + \cdots + |\sigma_c(q_{m+1}, q_m)|.$$

Thus,

$$|\sigma_c(p_n, p_m)| + |\sigma_c(q_n, q_m)| \leqslant s_{n-1} + s_{n-2} + s_{n-3} + \cdots + s_m$$
$$\leqslant (\alpha^{n-1} + \alpha^{n-2} + \cdots + \alpha^m)s_0 \leqslant \frac{\alpha^m}{1-\alpha}s_0 \to 0, \quad n \to \infty,$$

which implies that $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences in (Y, σ_c). Since the complex partial metric space (Y, σ_c) is complete, there exist $p, q \in Y$ such that $\{p_n\} \to p$ and $q_n \to q$ as $n \to \infty$ and $\sigma_c(p, p) = \lim_{n\to\infty} \sigma_c(p, p_n) = \lim_{n,m\to\infty} \sigma_c(p_n, p_m) = 0$, $\sigma_c(q, q) = \lim_{n\to\infty} \sigma_c(q, q_n) = \lim_{n,m\to\infty} \sigma_c(q_n, q_m) = 0$. We now show that $p = \psi(p, q)$. We suppose on the contrary that $p \neq \psi(p, q)$ and $q \neq \psi(q, p)$ so that $0 \prec \sigma_c(p, \psi(p, q)) = l_1$ and $0 \prec \sigma_c(q, \psi(q, p)) = l_2$, then

$$l_1 = \sigma_c(p, \psi(p, q)) \preceq \sigma_c(p, p_{n+1}) + \sigma_c(p_{n+1}, \psi(p, q))$$
$$= \sigma_c(p, p_{n+1}) + \sigma_c(\psi(p_n, q_n), \psi(p, q))$$
$$\preceq \sigma_c(p, p_{n+1}) + k\sigma_c(\psi(p_n, q_n), p) + l\sigma_c(\psi(p, q), p_n)$$
$$= \sigma_c(p, p_{n+1}) + k\sigma_c(p_{n+1}, p) + l\sigma_c(\psi(p, q), p_n),$$

which implies that

$$|l_1| \leqslant |\sigma_c(p, p_{n+1})| + k|\sigma_c(p_n, p)| + l|\sigma_c(\psi(p, q), p_n)|.$$

As $n \to \infty$, $|l_1| \leqslant 0$ which is a contradiction, therefore $|\sigma_c(p, \psi(p, q))| = 0 \implies p = \psi(p, q)$. Similarly, we can prove that $q = \psi(q, p)$. Thus (p, q) is a coupled fixed point of ψ. Now, if (g, h) is another coupled fixed point of ψ, then we obtain

$$\sigma_c(p, g) = \sigma_c(\psi(p, q), \psi(g, h)) \preceq k\sigma_c(\psi(p, q), g) + l\sigma(\psi(g, h), p).$$

Thus,

$$(1 - (k + l))\sigma_c(p, g) \preceq 0,$$

which implies that

$$(1 - (k + l))|\sigma_c(p, g)| \leqslant 0. \tag{3.12}$$

Similarly,

$$(1 - (k + l))|\sigma_c(q, h)| \leqslant 0. \tag{3.13}$$

From (3.12) and (3.13), we obtain

$$(1 - (k + l))\Big(|\sigma_c(p, g)| + |\sigma_c(q, h)|\Big) \leqslant 0.$$

Since $k + l < 1$, therefore $|\sigma_c(p, g)| + |\sigma_c(q, h)| = 0$ and hence, $p = g$ and $q = h \implies (p, q) = (g, h)$. Thus, ψ has a unique coupled fixed point. $\square$

From Theorem 3.4 with $k = l$, we get the following corollary.

Corollary 3.5. *Let (Y, σ_c) be a complete CPMS. Suppose that the mapping $\psi : Y \times Y \to Y$ satisfies*

$$\sigma_c(\psi(p, q), \psi(r, s)) \preceq k(\sigma_c(\psi(p, q), r) + \sigma_c(\psi(r, s), p)),$$

for all $p, q, r, s \in Y$, where k is a non-negative constant with $k < \frac{1}{3}$. Then ψ has a unique coupled fixed point.

Theorem 3.6. *Let* (Y, σ_c) *be a complete CPMS. Suppose that the mapping* $\psi : Y \times Y \to Y$ *satisfies*

$$\sigma_c(\psi(p, q), \psi(r, s)) \preceq r \max\{\sigma_c(p, r), \sigma_c(q, s), \sigma_c(\psi(p, q), p), \sigma_c(\psi(r, s), r)\},$$

for all $p, q, r, s \in Y$, *where* $r \in [0, 1)$. *Then* ψ *has a unique coupled fixed point.*

Proof. Choose $p_0, q_0 \in Y$ and set $p_1 = \psi(p_0, q_0)$ and $q_1 = \psi(q_0, p_0)$. Continuing this process, set $p_{n+1} = \psi(p_n, q_n)$ and $q_{n+1} = \psi(q_n, p_n)$. Then,

$$
\begin{aligned}
\sigma_c(p_{n+1}, p_{n+2}) =& \sigma_c(\psi(p_n, q_n), \psi(p_{n+1}, q_{n+1})) \\
\preceq& r \max(\sigma_c(p_n, p_{n+1}), \sigma_c(q_n, q_{n+1}), \sigma_c(\psi(p_n, q_n), p_n), \sigma_c(\psi(p_{n+1}, q_{n+1}), p_{n+1}) \\
=& r \max(\sigma_c(p_n, p_{n+1}), \sigma_c(q_n, q_{n+1}), \sigma_c(p_{n+1}, p_n), \sigma_c(p_{n+2}, p_{n+1})) \\
\preceq& r \max(\sigma_c(p_n, p_{n+1}), \sigma_c(q_n, q_{n+1})),
\end{aligned}
$$

which implies that

$$|\sigma_c(p_{n+1}, p_{n+2})| \leqslant r \max\{|\sigma_c(p_n, p_{n+1})|, |\sigma_c(q_n, q_{n+1})|\}. \tag{3.14}$$

Similarly, one can prove that

$$|\sigma_c(q_{n+1}, q_{n+2})| \leqslant r \max\{|\sigma_c(q_n, q_{n+1})|, |\sigma_c(p_n, p_{n+1})|\}. \tag{3.15}$$

From (3.14) and (3.15), we get

$$\max\{|\sigma_c(p_{n+1}, p_{n+2})|, |\sigma_c(q_{n+1}, q_{n+2})|\} \leqslant r \max\{|\sigma_c(q_n, q_{n+1})|, |\sigma_c(p_n, p_{n+1})|\}.$$

Continuing this process, we obtain

$$
\begin{aligned}
\max\{|\sigma_c(p_n, p_{n+1})|, |\sigma_c(q_n, q_{n+1})|\} \leqslant& r \max\{|\sigma_c(q_{n-1}, q_n)|, |\sigma_c(p_{n-1}, p_n)|\} \\
\leqslant& r^2 \max\{|\sigma_c(q_{n-2}, q_{n-1})|, |\sigma_c(p_{n-2}, p_{n-1})|\} \\
\leqslant& r^n \max\{|\sigma_c(q_0, q_1)|, |\sigma_c(p_0, p_1)|\}.
\end{aligned}
$$

Therefore

$$\lim_{n \to \infty} \max\{|\sigma_c(p_n, p_{n+1})|, |\sigma_c(q_n, q_{n+1})|\} = 0,$$

which in turn gives

$$\lim_{n \to \infty} |\sigma_c(p_n, p_{n+1})| = 0 \text{ and } \lim_{n \to \infty} |\sigma_c(q_n, q_{n+1})| = 0.$$

For each $n > m$, we have

$$
\begin{aligned}
\sigma_c(p_n, p_m) \preceq& \sigma_c(p_n, p_{n-1}) + \sigma_c(p_{n-1}, p_{n-2}) - \sigma_c(p_{n-1}, p_{n-1}) \\
&+ \sigma_c(p_{n-2}, p_{n-3}) + \sigma_c(p_{n-3}, p_{n-4}) - \sigma_c(p_{n-3}, p_{n-3}) \\
&+ \cdots + \sigma_c(p_{m+2}, p_{m+1}) + \sigma_c(p_{m+1}, p_m) - \sigma_c(p_{m+1}, p_{m+1}) \\
\preceq& \sigma_c(p_n, p_{n-1}) + \sigma_c(p_{n-1}, p_{n-2}) + \cdots + \sigma_c(p_{m+1}, p_m),
\end{aligned}
$$

which implies that

$$|\sigma_c(p_n, p_m)| \leqslant |\sigma_c(p_n, p_{n-1})| + |\sigma_c(p_{n-1}, p_{n-2})| + \cdots + |\sigma_c(p_{m+1}, p_m)|.$$

As $n, m \to \infty$, we obtain

$$\lim_{n,m\to\infty} |\sigma_c(p_n, p_m)| = 0.$$

Similarly, one can prove that

$$\lim_{n\to\infty} |\sigma_c(q_n, q_m)| = 0,$$

which implies that $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences in (Y, σ_c). Since, complex partial metric space (Y, σ_c) is complete, there exist $p, q \in Y$ such that $\{p_n\} \to p$ and $q_n \to q$ as $n \to \infty$ and $\sigma_c(p, p) = \lim_{n\to\infty} \sigma_c(p, p_n) = \lim_{n,m\to\infty} \sigma_c(p_n, p_m) = 0$, $\sigma_c(q, q) = \lim_{n\to\infty} \sigma_c(q, q_n) = \lim_{n,m\to\infty} \sigma_c(q_n, q_m) = 0$. We now show that $p = \psi(p, q)$. Now,

$$\begin{aligned}
\sigma_c(p, \psi(p, q)) &\preceq \sigma_c(p, p_{n+1}) + \sigma_c(p_{n+1}, \psi(p, q)) \\
&= \sigma_c(p, p_{n+1}) + \sigma_c(\psi(p_n, q_n), \psi(p, q)) \\
&\preceq \sigma_c(p, p_{n+1}) + r\max\{\sigma_c(p_n, p), \sigma_c(q_n, q), \sigma_c(\psi(p_n, q_n), p_n), \sigma_c(\psi(p, q), p)\} \\
&= \sigma_c(p, p_{n+1}) + r\max\{\sigma_c(p_n, p), \sigma_c(q_n, q), \sigma_c(p_{n+1}, p_n), \sigma_c(\psi(p, q), p)\},
\end{aligned}$$

which implies that

$$|\sigma_c(p, \psi(p, q))| \leqslant |\sigma_c(p, p_{n+1})| + r\max\{|\sigma_c(p_n, p)|, |\sigma_c(q_n, q)|, |\sigma_c(p_{n+1}, p_n)|, |\sigma_c(\psi(p, q), p)|\}.$$

As $n \to \infty$, $|\sigma_c(p, \psi(p, q))| \leqslant r|\sigma_c(\psi(p, q), p)|$. Since $r \in [0, 1)$, therefore $|\sigma_c(p, \psi(p, q))| = 0 \implies p = \psi(p, q)$. Similarly we can prove that $q = \psi(q, p)$. Thus (p, q) is a coupled fixed point of ψ. Now, if (g, h) is another coupled fixed point of ψ, then

$$\begin{aligned}
\sigma_c(p, g) = \sigma_c(\psi(p, q), \psi(g, h)) &\preceq r\max\{\sigma_c(p, g), \sigma_c(q, h), \sigma_c(\psi(p, q), p), \sigma_c(\psi(g, h), g)\} \\
&\preceq r\max\{\sigma_c(p, g), \sigma_c(q, h), \sigma_c(p, p), \sigma_c(g, g)\}.
\end{aligned}$$

Since $\sigma_c(p, p) \preceq \sigma_c(p, g)$ and $\sigma_c(g, g) \preceq \sigma_c(p, g)$, we have

$$\sigma_c(p, g) \preceq r\max\{\sigma_c(p, g), \sigma_c(q, h)\}, \qquad |\sigma_c(p, g)| \leqslant r\max\{|\sigma_c(p, g)|, |\sigma_c(q, h)|\}. \tag{3.16}$$

Similarly, we can prove

$$|\sigma_c(q, h)| \leqslant r\max\{|\sigma_c(p, g)|, |\sigma_c(q, h)|\}. \tag{3.17}$$

From (3.16) and (3.17), we have

$$\max\{|\sigma_c(p, g)|, |\sigma_c(q, h)|\} \leqslant r\max\{|\sigma_c(p, g)|, |\sigma_c(q, h)|\}.$$

Since $0 \leqslant r < 1$, we have $\max\{|\sigma_c(p, g)|, |\sigma_c(q, h)|\} = 0$ implies that $\sigma_c(p, g) = 0$ and $\sigma_c(q, h) = 0$. Therefore, $p = g$ and $q = h \implies (p, q) = (g, h)$. Thus, ψ has a unique coupled fixed point. $\square$

Corollary 3.7. *Let* (Y, σ_c) *be a complete CPMS. Suppose that the mapping* $\psi : Y \times Y \to Y$ *satisfies*

$$\sigma_c(\psi(p, q), \psi(r, s)) \preceq a\sigma_c(p, r) + b\sigma_c(q, s) + c\sigma_c(\psi(p, q), p) + d\sigma_c(\psi(r, s), r),$$

for all $p, q, r, s \in Y$, *where* a, b, c *and* d *are non-negative constants with* $a + b + c + d < 1$, *then* ψ *has a unique coupled fixed point.*

Proof. The proof follows from Theorem 3.6. Note that

$$\begin{aligned}
a\sigma_c(p, r) &+ b\sigma_c(q, s) + c\sigma_c(\psi(p, q), p) + d\sigma_c(\psi(r, s), r) \\
&\leqslant r\max\{\sigma_c(p, r), \sigma_c(q, s), \sigma_c(\psi(p, q), p), \sigma_c(\psi(r, s), r)\},
\end{aligned}$$

where $r = a + b + c + d < 1$. $\square$

Example 3.8. Let $Y = [0, \infty)$ endowed with the usual complex partial metric $\sigma_c : Y \times Y \to [0, \infty)$ defined by $\sigma_c(p, q) = \max\{p, q\}(1 + i)$. The complex partial metric space (Y, σ_c) is complete because (Y, σ_c^t) is complete. Indeed, for any $p, q \in Y$,

$$\sigma_c^t = 2\sigma_c(p, r) - \sigma_c(p, p) - \sigma_c(r, r) = 2\max\{p, q\}(1 + i) - (p + ip) - (q + iq) = |p - q| + i|p - q|.$$

Thus, (Y, σ_c) is the Euclidean complex metric space which is complete. Consider the mapping $\psi : Y \times Y \to Y$ defined by $\psi(p, q) = \frac{|p - q|}{2}$. For any $p, q, g, h \in Y$, we have

$$
\begin{aligned}
\sigma_c(\psi(p, q), \psi(g, h)) &= \frac{1}{2} \max\{|p - q|, |g - h|\}(1 + i) \\
&= \frac{1}{2} \max\{p - q, q - p, g - h, h - g\}(1 + i) \\
&\preceq \frac{1}{2} \max\{p, q, g, h\}(1 + i) \\
&= \frac{1}{2} \max\{\sigma_c(p, g), \sigma_c(q, h)\} \\
&\preceq \frac{1}{2} \max\{\sigma_c(p, g), \sigma_c(q, h), \sigma_c(\psi(p, q), p), \sigma_c(\psi(g, h), g)\}.
\end{aligned}
$$

Thus, ψ has a unique coupled fixed point. Here, $(0, 0)$ is the unique fixed point of ψ.

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