J. Math. Computer Sci., 25 (2022), 166–175

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

On the q-Sumudu transform with two variables and some properties

Artan F. Alidema[a], Shkumbin V. Makolli[b,*]

[a]Department of Mathematics, Faculty of Mathematical and Natural Science, University of Prishtina, Mother Theresa p.n, 10000 Prishtina, Kosovo.

[b]Department of Mathematics, Faculty of Mechanical Engeenering, University of Prishtina, Agim Ramadani p.n, 1000 Pristina, Kosovo.

Abstract

In this paper we present some properties of double q-Sumudu transform in q-calculus by using the functions of two variables. Furthermore results on convergence, absolute convergence and convolution are discussed. At the end some examples are given to illustrate use of double q-Sumudu transform.

Keywords: Double q-Sumudu transform, convergence, convolution.

2020 MSC: 33D05, 33D60, 35A22, 42A85.

1. Introduction

The Sumudu transform was introduced by Watugala [24] and he applied it to the solution of ordinary differential equations. Asiru [7] and Belgacem [9] give the general and fundamental properties of the Sumudu transform. Watugala [25] has extended Sumudu transform to functions of two variables, and applied it to solving partial differential equations.

Tchuenche [22] applied the double Sumudu transform to an evolution equation of population dynamics. Kilicman and Gadain [19] gave the relations between double Laplace transform and double Sumudu transform and applied it to the solutions of non-homogenous wave equations. Debnath [11] presented general properties of the double Laplace transform, convolution and its properties. Convergence of double Sumudu transform was proved by Zulfiqar et al. [3].

In 1910, Jackson [17] presented a precise definition of so-called q-Jackson integral and developed q-calculus in a systematic way. It is well known that there are two types of q-Laplace transforms and they have been studied in detail by many authors ([1, 16, 20]).

The theory of q-analysis has been applied in many areas of mathematics, engineering , and physics, like in ordinary fractional calculus, optimal control problems, q-transform analysis and also in finding solutions of the q-difference and q-integral equations (see [2, 8, 18]).

*Corresponding author

Email addresses: artan.alidema@uni-pr.edu (Artan F. Alidema), shkumbin.makolli@uni-pr.edu (Shkumbin V. Makolli)

doi: 10.22436/jmcs.025.02.05

Received: 2021-03-19 *Revised:* 2021-04-12 *Accepted:* 2021-04-24

Purohit and Kalla [20] evaluated the q-Laplace transforms of the q-Bessel functions, and they gave several useful cases of its application. Albayrak et al. [4, 5], have introduced q-analogue of Sumudu transform and investigated the fundamental properties of the q-Sumudu transform of certain q-polynomials. Brahim and Riah [10] introduced the q-analogue of the two dimensional Mellin transform, gave some properties and also proved the inversion formula of the q-two dimensional Mellin transform. Double q-Laplace transform was introduced by Sadjang [21] as well as by Ganie [14]. The latter gave certain results on convergence, absolute convergence, convolution and its properties.

In this paper, we examine some properties of q-double Sumudu transform in q-calculus by using functions of two variables. We have shown some results regarding the convergence, absolute convergence and introduced convolution for q-double Sumudu transform.

The Sumudu transform of function $f(x)$ is defined by Watugala in [24] as:

$$S[f(x); s] = \frac{1}{s}\int_0^\infty e^{-\frac{x}{s}} f(x)dx, s \in (-\tau_1, \tau_2),$$

while $f(x)$ is a function from the set of functions

$$A = [f(x)|\exists M, \tau_1, \tau_2 > 0, |f(x)| < Me^{\frac{x}{\tau_j}}, \text{ if } \quad x \in (-1)^j \times [0, \infty)].$$

The q-analoque of Sumudu transform by the q-Jackson integral is given by Albayrak et al. [6] as follows:

$$F(s) = S_q[f(x); s] = \frac{1}{(1-q)s}\int_0^\infty e_q^{-\frac{x}{s}} f(x)d_qx, s \in (-\tau_1, \tau_2).$$

Over the set of functions

$$B = [f(x)|\exists M, \tau_1, \tau_2 > 0, \ |f(x)| < Me_q^{\frac{|x|}{\tau_j}}, \text{ if } \quad x \in (-1)^j \times [0, \infty)].$$

Let $f(x, y)$ be the function of two variables in the positive quadrant of Oxy plane then its double Sumudu transform is given by [22] as:

$$S[f(x, t); (p, s)] = \frac{1}{ps}\int_0^\infty \int_0^\infty e^{(-\frac{x}{p}-\frac{t}{s})} f(x, t)dxdt.$$

Double inverse Sumudu transform can be written as:

$$S^{-1}(p.s) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\frac{x}{p}} dp \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} e^{\frac{t}{s}} f(p, s)ds,$$

where $f(p, s)$ is analytic functions for all p and s in the region defined by the inequalities Re $(p) \geqslant \gamma$ and Re $(s) \geqslant \delta$, while γ and δ are real constants.

2. Auxiliary results

In this section we summarize the basic definitions and mathematical notations.

The q-factorials for $q \in (0, 1)$ and $a \in C$ are defined as

$$(a; q)_0 = 1, \quad (a; q)_n = \prod_{k=0}^{n-1}(1 - aq)^k, \ n = 1, 2, \ldots, \quad (a; q)_\infty = \lim_{n\to\infty}(a; q)_n = \prod_{k=0}^{n-1}(1 - aq)^k.$$

Also we write $[a]_q = \frac{1-q^a}{1-q}$, $[a]_q! = \frac{(q;q)_n}{(1-q)^n}$, $n \in N$.

The q-derivatives of a function are given by [18]

$$(D_qf)(x) = \frac{(f(x) - f(qx))}{(1 - q)x}, \quad \text{if } \ x \neq 0, \qquad (D_qf)(0) = f'(0)$$

provided that $f'(0)$ exists. If f is differentiable, then $(D_q f)(x)$ tends to $f'(x)$ as q tends to 1. For $n \in N$ we have $D_q^1 = D_q$, $(D_q^+)^1 = D_q^+$. The q-derivative of the product of two functions is defined as

$$D_q(f \cdot g)(x) = g(x)D_q f(x) + f(qx)D_q g(x).$$

The q-integrals from 0 to a and from 0 to ∞ known as q-Jackson integrals are defined in [17]

$$\int_0^a f(x)d_q(x) = (1-q)a \sum_{n=-\infty}^{\infty} f(aq^n)q^n, \quad \int_0^\infty f(x)d_q(x) = (1-q) \sum_{n=-\infty}^{\infty} f(q^n)q^n,$$

provided these sums converge absolutely. The integration by parts in terms of q-calculus is given by:

$$\int_a^b g(x)D_q f(x)d_q x = f(b)g(b) - f(a)g(a) - \int_a^b f(qx)D_q g(x)d_q x.$$

The q-analogues of the exponential functions are defined in [15, 18],

$$E_q^z = \sum_{n=0}^{\infty} q^{n(n-1)/2}\frac{z^n}{[n]_q!} = (-(1-q)z;q)_\infty, \quad e_q^z = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} = \frac{1}{((1-q)z;q)_\infty}, |z| < \frac{1}{1-q}.$$

The q-exponentials are analogues of classical exponential functions and satisfy the relations $D_q e_q^z = e_q^z$, $D_q E_q^z = E_q^{qz}$, and $e_q^z E_q^{-z} = E_q^{-z} e_q^z = 1$. Jacskson also defined q-analogue of the gamma function $\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}dx$, while many properties are found in [23, 26–28],

$$\Gamma_q(t) = \frac{(q;q)_\infty}{(q^t;q)_\infty}(1-q)^{1-t}, \quad t \neq 0, -1, -2, \ldots,$$

that satisfies

$$\Gamma_q(t+1) = [t]_q \Gamma(t), \quad \Gamma_q(1) = 1,$$

and $\lim_{q\to 1^-} \Gamma_q(t) = \Gamma(t)$, $\quad \mathbb{R}(t) > 0$. The Γ_q function has the q-integral representation as

$$\Gamma_q(s) = \int_0^{1/(1-q)} t^{s-1}E_q^{-qt}d_q t = \int_0^{\infty/(1-q)} t^{s-1}E_q^{-qt}d_q t.$$

The q-integral representation of Γ_q based on q-exponential function e_q^x and q-integral representation of q-beta function are defined in [12] as: for all $s, t > 0$, we have

$$\Gamma_q(s) = K_q(s)\int_0^{\infty/(1-q)} t^{s-1}e_q^{-t}d_q t$$

and $B_q(t,s) = K_q(t)\int_0^\infty x^{t-1}\frac{(-xq^{s+t};q)_\infty}{(-x;q)_\infty}d_q x$, where in [10], $K_q(t) = \frac{(-q,-1;q)_\infty}{(-q^t,-q^{1-t};q)_\infty}$. If $\frac{\log(1-q)}{\log(q)} \in Z$, we obtain

$$\Gamma_q(s) = K_q(s)\int_0^\infty t^{s-1}e_q^{-t}d_q t = \int_0^\infty t^{s-1}E_q^{-qt}d_q t.$$

3. Two dimensional q-Sumudu transform

Definition 3.1. Let $q = (q_1, q_2) \in (0,1)^2, (s,t) \in \mathbb{C}^2$ and let f be a function of two variables x and y defined on $\mathbb{R}_{q_1,+} \times .\mathbb{R}_{q_2,+}$. Then the q-two dimensional Sumudu transform of f is defined by the double integral is given as: ([13])

$$S_q(f)(s,t) = S_q[f(x,y)](s,t) = \frac{1}{(1-q)^2}\frac{1}{ps}\int_0^\infty \int_0^\infty e_q^{(-\frac{x}{p}-\frac{t}{s})}f(x,t)d_q x d_q t$$

(provided that this integral exists), where $R_{q,+} = \left\{q^n, n \in \mathbb{Z}\right\}$ and

$$\begin{cases} e_q^{-x} = [1-(1-q)x]^{1/1-q}, & \text{for } 0 < x < \frac{1}{1-q},\ q < 1, \\ e_q^{-x} = [1-(1-q)x]^{-1/1-q}, & \text{for } x \geqslant 0,\ q > 1. \end{cases}$$

4. Convergence of q-double Sumudu transform

Theorem 4.1. *Let $f(x,y)$ be a function of two variables continuous in $\mathbb{R}_{q_1,+}\times\mathbb{R}_{q_2,+}$ or continuous in the positive quadrant of xy plane. If the integral*

$$\frac{1}{(1-q)^2}\frac{1}{ps}\int_0^\infty\int_0^\infty e_q^{(-\frac{x}{p}-\frac{t}{s})}f(x,t)d_qxd_qt \tag{4.1}$$

converges at $s=s_0$ and $t=t_0$, then the integral in (4.1) converges for $s<s_0, p<p_0$.

We prove this Theorem by using following Lemma.

Lemma 4.2. *If the integral*

$$\frac{1}{(1-q)}\frac{1}{s}\int_0^\infty e_q^{-\frac{t}{s}}f(x,t)d_qt$$

converges for $s=s_0$, then the integral converge for $s<s_0$.

Proof. Let us assume that, $\alpha(x,t)=\frac{1}{(1-q)}\frac{1}{s_0}\int_0^t e_q^{(-\frac{u}{s_0})}f(x,u)d_qu$. It is obvious that $\alpha(x,0)=0$ and $\lim_{t\to\infty}\alpha(x,t)$ exists, because integral $\frac{1}{(1-q)}\frac{1}{s}\int_0^\infty e_q^{(-\frac{t}{s})}f(x,t)d_qt$ converges at $s=s_0$.

Now let us denote

$$\alpha_t(x,t)=\frac{1}{(1-q)}\frac{1}{s_0}e_q^{(-\frac{t}{s_0})}f(x,t).$$

For ζ and R such that $0<\zeta<R$ we will consider

$$\begin{aligned}\frac{1}{(1-q)}\frac{1}{s}\int_\zeta^R e_q^{-\frac{t}{s}}f(x,t)d_qt&=\frac{1}{(1-q)}\frac{1}{s}\int_\zeta^R e_q^{-\frac{t}{s}}(1-q)s_0e_q^{\frac{t}{s_0}}\alpha_t(x,t)d_qt\\&=\frac{s_0}{s}\int_\zeta^R e_q^{-\frac{t}{s}}e_q^{\frac{t}{s_0}}\alpha_t(x,t)d_qt=\frac{s_0}{s}\int_\zeta^R e_q^{-\frac{t(s_0-s)}{s_0s}}\alpha_t(x,t)d_qt.\end{aligned}$$

If we use partial integration we obtain

$$\begin{aligned}&=\frac{s_0}{s}\left[e_q^{\frac{-t(s_0-s)}{ss_0}}\alpha(x,t)|_\zeta^R+\frac{s_0-s}{ss_0}\int_\zeta^R e_q^{\frac{-t(s_0-s)}{ss_0}}\alpha(x,qt)d_qt\right]\\&=\frac{s_0}{s}\left[e_q^{\frac{-R(s_0-s)}{ss_0}}\alpha(x,R)-e_q^{\frac{-\zeta(s_0s)}{ss_0}}\alpha(x,\zeta)+\frac{s_0-s}{ss_0}\int_\zeta^R e_q^{\frac{-t(s_0-s)}{ss_0}}\alpha(x,qt)d_qt\right].\end{aligned}$$

If in the last expression we put $\zeta\to0$, then due to $\lim_{\zeta\to0}\alpha(x,\zeta)=\alpha(x,0)=0$, we get

$$\frac{1}{(1-q)s}\int_0^R e_q^{\frac{-t}{s}}f(x,t)d_qt=\frac{s_0}{s}\left[e_q^{\frac{-R(s_0-s)}{ss_0}}\alpha(x,R)+\frac{s_0-s}{ss_0}\int_0^R e_q^{\frac{-t(s_0-s)}{ss_0}}\alpha(x,qt)d_qt\right].$$

Now if we let $R\to\infty$, then the first term on the right tends to zero when $s<s_0$, subsequently we get

$$e_q^{\frac{-R(s_0-s)}{ss_0}}\alpha(x,R)\to0$$

and the left expression is equal to

$$\frac{1}{(1-q)s}\int_0^\infty e_q^{\frac{-t}{s}}f(x,t)d_qt=\frac{s_0-s}{s^2}\int_0^\infty e_q^{\frac{-t(s_0-s)}{ss_0}}\alpha(x,qt)d_qt.$$

According to limit test for convergence of improper integrals we can prove that

$$\frac{(s_0-s)}{s^2}\int_0^\infty e_q^{\frac{-t(s_0-s)}{ss_0}}\alpha(x,qt)d_qt$$

converges if the following limit converges

$$\lim_{t\to\infty} t^p e_q^{\frac{-t(s_0-s)}{ss_0}} \alpha(x,qt), \quad \text{for, } p>1.$$

It is obvious that for $p=2$ and $s<s_0$ the above limit converges, considering the fact that $\lim_{t\to\infty}\alpha(x,qt)$ converges. Therefore the following integral

$$\frac{1}{(1-q)}\frac{1}{s}\int_0^\infty e_q^{-\frac{t}{s}} f(x,t)d_qt$$

converges for $s<s_0$. □

Lemma 4.3. *If the integral*

$$\frac{1}{(1-q)}\frac{1}{s}\int_0^\infty e_q^{-\frac{t}{s}} f(x,t)d_qt$$

converges for $s\leqslant s_0$ *and the integral*

$$\frac{1}{(1-q)}\frac{1}{p}\int_0^\infty e_q^{-\frac{x}{p}} f(x,s)d_qx \tag{4.2}$$

converges for $p=p_0$*, then integral* (4.2) *converges for* $p<p_0$.

Proof. Same as above Lemma. □

Proof of the Theorem 4.1.

$$\begin{aligned}\frac{1}{(1-q)^2}\frac{1}{ps}\int_0^\infty\int_0^\infty e_q^{-\frac{x}{p}-\frac{t}{s}} f(x,t)d_qxd_qt &= \frac{1}{(1-q)}\frac{1}{p}\int_0^\infty e_q^{-\frac{x}{p}}\left\{\frac{1}{(1-q)}\frac{1}{s}\int_0^\infty e_q^{-\frac{t}{s}} f(x,t)d_qt\right\}d_qx\\ &= \frac{1}{(1-q)}\frac{1}{p}\int_0^\infty e_q^{-\frac{x}{p}} h(x,t)d_qx,\end{aligned}$$

where $h(x,t)=\frac{1}{(1-q)}\frac{1}{s}\int_0^\infty e_q^{-\frac{t}{s}} f(x,t)d_qt$ and according to Lemma 4.3 it converges for $s<s_0$, while in accordance with Lemma 4.2 the following $\frac{1}{(1-q)}\frac{1}{p}\int_0^\infty e_q^{-\frac{x}{p}} h(x,t)d_qx$ converges for $p<p_0$. Therefore, the initial integral, $\frac{1}{(1-q)^2}\frac{1}{ps}\int_0^\infty\int_0^\infty e_q^{-\frac{x}{p}-\frac{t}{s}} f(x,t)d_qxd_qt$ converges for $s<s_0,\ p<p_0$. □

Theorem 4.4 (Absolute convergence)**.** *If integral*

$$\frac{1}{(1-q)^2}\frac{1}{ps}\int_0^\infty\int_0^\infty e_q^{(-\frac{x}{p}\frac{-t}{s})} f(x,t)d_qxd_qt$$

converges absolute for $s=s_0$ *and* $p=p_0$*, then it converges for* $s\leqslant s_0$ *and* $p\leqslant p_0$.

Proof.

$$e_q^{(-\frac{x}{p}-\frac{t}{s})}|f(x,t)|\leqslant e_q^{(-\frac{x}{p_0}-\frac{t}{s_0})}|f(x,t)| \text{ for } (p\leqslant p_0<+\infty, s\leqslant s_0<+\infty),$$

therefore

$$\begin{aligned}&\leqslant \frac{1}{(1-q)^2}\frac{1}{ps}\int_0^\infty\int_0^\infty e_q^{(-\frac{x}{p}-\frac{t}{s})}|f(x,t)|d_qtd_qx\\ &\leqslant \frac{1}{(1-q)^2}\frac{1}{ps}\int_0^\infty\int_0^\infty e_q^{(-\frac{x}{p_0}-\frac{t}{s_0})}|f(x,t)|d_qtd_qx\\ &\leqslant \frac{1}{(1-q)^2}\frac{p_0s_0}{ps}\frac{1}{p_0s_0}\int_0^\infty\int_0^\infty e_q^{(-\frac{x}{p_0}-\frac{t}{s_0})}|f(x,t)|d_qtd_qx.\end{aligned}$$

The last integral converges based on the hypothesis, which means that the initial integral converges for $p\leqslant p_0$ and $s\leqslant s_0$. □

Theorem 4.5. *If $f(x,y)$ is a periodic function of periods a and b, $f(x+a,y+b) = f(x,y)$, and if $S_q[f(x,y)]$ exists, then*

$$S_q[f(x,y)](p,s) = \frac{[1-e^{-\frac{a}{p}-\frac{b}{s}}]^{-1}}{(1-q)^2ps}\int_0^a\int_0^b e_q^{(-\frac{x}{p}-\frac{t}{s})}f(x,t)d_qxd_qt.$$

Proof.

$$\begin{aligned}
S_q(f(x,y)) &= \frac{1}{(1-q)^2ps}\int_0^\infty\int_0^\infty e_q^{(-\frac{x}{p}-\frac{t}{s})}f(x,t)d_qxd_qt\\
&= \frac{1}{(1-q)^2ps}\int_0^a\int_0^b e_q^{(-\frac{x}{p}-\frac{t}{s})}f(x,t)d_qxd_qt + \frac{1}{(1-q)^2ps}\int_a^\infty\int_b^\infty e_q^{(-\frac{x}{p}-\frac{t}{s})}f(x,t)d_qxd_qt
\end{aligned}$$

if we put $x = u + a$ and $t = v + b$, we get

$$= \frac{1}{(1-q)^2ps}\int_0^a\int_0^b e_q^{(-\frac{x}{p}-\frac{t}{s})}f(x,t)d_qxd_qt + \frac{1}{(1-q)^2ps}\cdot e_q^{\frac{-a}{p}-\frac{b}{s}}\int_0^\infty\int_0^\infty e_q^{(-\frac{u}{p}-\frac{v}{s})}f(u,v)d_qud_qv.$$

Therefore

$$\begin{aligned}
S_qf(x,t) &= \frac{1}{(1-q)^2ps}\int_0^a\int_0^b e_q^{(-\frac{x}{p}-\frac{t}{s})}f(x,t)d_qxd_qt + e_q^{(-\frac{a}{p}-\frac{b}{s})}S_qf(x,t)\\
&= [1-e_q^{(-\frac{a}{p}-\frac{b}{s})}]S_qf(x,t) = \frac{1}{(1-q)^2ps}\int_0^a\int_0^b e_q^{(-\frac{x}{p}-\frac{t}{s})}f(x,t)d_qxd_qt,\\
S_qf(x,t) &= \frac{[1-e_q^{-\frac{a}{p}-\frac{b}{s}}]^{-1}}{(1-q)^2ps}\int_0^a\int_0^b e_q^{(-\frac{x}{p}-\frac{t}{s})}f(x,t)d_qxd_qt,
\end{aligned}$$

S_q is a double Sumudu transform of a periodic function. □

5. Double q-Sumudu convolution product

Definition 5.1. The convolution of $f(x,y)$ and $g(x,y)$ is defined as

$$(f**g)(x,y) = \frac{1}{(1-q)^2ps}\int_0^x\int_0^y f(\zeta,\mu)g(x-\zeta,y-\mu)d_q\zeta d_q\mu.$$

Theorem 5.2 (Convolution Theorem). *Let $f_1(x,t)$ and $f_2(x,t)$ be two functions having double q-double Sumudu transform. Then q-double Sumudu transform of the double convolution is given by:*

$$S_q\{f_1(x,t)**f_2(x,t)\}(p,s) = S_q[f_1(x,t)]S_q[f_2(x,t)].$$

Proof.

$$\begin{aligned}
&S_q\{f_1(x,t)**f_2(x,t)\}(p,s)\\
&\quad= \frac{1}{(1-q)^4ps}\int_0^\infty\int_0^\infty e_q^{(-\frac{x}{p}-\frac{t}{s})}\left\{\int_0^x\int_0^t f_1(\zeta,\mu)f_2(x-\zeta,t-\mu)d_q\zeta d_q\mu\right\}d_qxd_qt\\
&\quad= \frac{1}{(1-q)^4p^2s^2}\int_0^\infty\int_0^\infty \left\{\int_0^x\int_0^t \left[1+(q-1)\left(\frac{x}{p}+\frac{t}{s}\right)\right]^{-\frac{1}{q-1}} f_1(\zeta,\mu)f_2(x-\zeta,t-\mu)d_q\zeta d_q\mu\right\}d_qxd_qt\\
&\quad= \frac{1}{(1-q)^4p^2s^2}\int_0^\infty\int_0^\infty f_1(\zeta,\mu)\left\{\int_0^x\int_0^t \left[1+(q-1)\left(\frac{x}{p}+\frac{t}{s}\right)\right]^{-\frac{1}{q-1}} f_2(x-\zeta,t-\mu)d_q\zeta d_q\mu\right\}d_qxd_qt.
\end{aligned}$$

Let us replace $x-\zeta = u$ and $t-\mu = v$, and if we denote

$$I = \int_0^x\int_0^t \left[1+(q-1)\left(\frac{x}{p}+\frac{t}{s}\right)\right]^{-\frac{1}{q-1}} f_2(x-\zeta,y-\mu)d_q\zeta d_q\mu$$

in light of new variables, and if we consider upper bounds for x and t, the integral I can also be written as following

$$\begin{aligned} I = &\left[1+(q-1)\left(\frac{\zeta}{p}+\frac{\mu}{s}\right)\right]^{-\frac{1}{q-1}} \int_{-\zeta}^{\infty}\int_{-\mu}^{\infty}\left[1+(q-1)\left(\frac{u}{p}+\frac{v}{s}\right)\right]^{-\frac{1}{q-1}} \\ &\times \left[1+(q-1)\left(\frac{\zeta}{p}+\frac{\mu}{s}\right)\right]^{\frac{1}{q-1}}\left[1+(q-1)\left(\frac{u}{p}+\frac{v}{s}\right)\right]^{\frac{1}{q-1}}\left[1+(q-1)\left(\frac{u+\zeta}{p}+\frac{v+\mu}{s}\right)\right]^{-\frac{1}{q-1}} f_2(u,v)d_qu d_qv. \end{aligned}$$

Since both functions $f_1(x,t)$ and $f_2(x,t)$ are defined in the positive quadrant of the Oxy plane, it is obvious that

$$\begin{aligned} I = &\left[1+(q-1)\left(\frac{\zeta}{p}+\frac{\mu}{s}\right)\right]^{-\frac{1}{q-1}} \int_{0}^{\infty}\int_{0}^{\infty}\left[1+(q-1)\left(\frac{u}{p}+\frac{v}{s}\right)\right]^{-\frac{1}{q-1}} \\ &\times \left[1+(q-1)\left(\frac{\zeta}{p}+\frac{\mu}{s}\right)\right]^{\frac{1}{q-1}}\left[1+(q-1)\left(\frac{u}{p}+\frac{v}{s}\right)\right]^{\frac{1}{q-1}}\left[1+(q-1)\left(\frac{u+\zeta}{p}+\frac{v+\mu}{s}\right)\right]^{-\frac{1}{q-1}} f_2(u,v)d_qu d_qv. \end{aligned}$$

If we now write

$$\left[1+(q-1)\left(\frac{\zeta}{p}+\frac{\mu}{s}\right)\right]^{\frac{1}{q-1}}\left[1+(q-1)\left(\frac{u}{p}+\frac{v}{s}\right)\right]^{\frac{1}{q-1}}\left[1+(q-1)\left(\frac{u+\zeta}{p}+\frac{v+\mu}{s}\right)\right]^{-\frac{1}{q-1}} f_2(u,v) = f_2^*(u,v),$$

from the last relation we can then express $f_2(u,v)$ as

$$f_2(u,v) = \left[1+(q-1)\left(\frac{\zeta}{p}+\frac{\mu}{s}\right)\right]^{-\frac{1}{q-1}}\left[1+(q-1)\left(\frac{u}{p}+\frac{v}{s}\right)\right]^{-\frac{1}{q-1}}\left[1+(q-1)\left(\frac{u+\zeta}{p}+\frac{v+\mu}{s}\right)\right]^{\frac{1}{q-1}} f_2^*(u,v),$$

and after replacing it in the initial integral we will get:

$$\begin{aligned} S_q\left\{f_1(x,t)**f_2(x,t)\right\} &= \frac{1}{(1-q)^4p^2s^2}\int_0^{\infty}\int_0^{\infty} f_1(\zeta,\mu)\left[1+(q-1)\left(\frac{\zeta}{p}+\frac{\mu}{s}\right)\right]^{-\frac{1}{q-1}} \\ &\quad\times\left\{\int_0^{\infty}\int_0^{\infty}\left[1+(q-1)\left(\frac{u}{p}+\frac{v}{s}\right)\right]^{-\frac{1}{q-1}} f_2^*(u,v)d_qud_qv\right\}d_q\zeta d_q\mu \\ &= S_q\{f_1\{(\zeta,\mu)\}\left\{\frac{1}{(1-q)^2ps}\int_0^{\infty}\int_0^{\infty}\left[1+(q-1)\left(\frac{u}{p}+\frac{v}{s}\right)\right]^{-\frac{1}{q-1}} f_2^*(u,v)d_qud_qv\right\} \\ &= S_q\{f_1(\zeta,\mu)\}\cdot S_q\{f_2^*(u,v)\}, \\ S_q\{f_1(x,t)**f_2(x,t)\}(p,s) &= S_q[f_1(x,t)]S_q[f_2(x,t)]. \end{aligned}$$

□

5.1. *Properties of double q-Sumudu transform method*

Some properties of q-double Sumudu transform are given as following.

a) Scaling: For a real number k,

$$\begin{aligned} S_q[kf(x,y)](p,s) &= \frac{1}{(1-q)^2}\frac{1}{ps}\int_0^{\infty}\int_0^{\infty} k e_q^{(-\frac{x}{p}-\frac{y}{s})} f(x,y)d_qxd_qy \\ &= \frac{1}{(1-q)^2}\frac{k}{ps}\int_0^{\infty}\int_0^{\infty} e_q^{(-\frac{x}{p}-\frac{y}{s})} f(x,y)d_qxd_qy kS_q[f(x,y)](p,s). \end{aligned}$$

b) Linearity:

$$S_q[mf(x,y)+S_q mf(x,y)](p,s) = mS_q[f(x,y)] + nS_q[f(x,y)](p,s),$$

$$\begin{aligned}
S_q[mf(x,y)+nf(x,y)](p,s) &= \frac{1}{(1-q)^2}\frac{1}{ps}\int_0^\infty\int_0^\infty [mf(x,y)+nf(x,y)]e_q^{(-\frac{x}{p}-\frac{y}{s})}d_qxd_qy\\
&\quad\times\frac{1}{(1-q)^2}\frac{1}{ps}\left\{\int_0^\infty\int_0^\infty mf(x,y)e_q^{(-\frac{x}{p}-\frac{y}{s})}d_qxd_qy\right.\\
&\quad\left.+\int_0^\infty\int_0^\infty nf(x,y)e_q^{(-\frac{x}{p}-\frac{y}{s})}d_qxd_qy\right\}\\
&= mS_q[f(x,y)](p,s)+nS_q[f(x,y)](p,s)].
\end{aligned}$$

c): For $a>0$, $b>0$, and if we denote $\overline{\overline{G_q}}(p,s)=S_q[f(x,y)(p,s)]$, we have:

$$\begin{aligned}
S_q[e_q^{-\frac{x}{a}-\frac{y}{b}}f(x,y)] &= \frac{ab}{(a+p)(b+s)}\overline{\overline{G_q}}\left(\frac{ap}{a+p},\frac{bs}{b+s}\right),\\
S_q[e_q^{-\frac{x}{a}-\frac{y}{b}}f(x,y)] &= \frac{1}{(1-q)^2}\frac{1}{ps}\int_0^\infty\int_0^\infty e_q^{(-\frac{x}{p}-\frac{y}{s})}e_q^{(-\frac{x}{a}-\frac{y}{b})}f(x,y)d_qxd_qy\\
&= \frac{1}{(1-q)^2}\frac{1}{ps}\int_0^\infty\int_0^\infty e_q^{-\frac{(a+p)x}{ap}}e_q^{-\frac{(b+s)y}{bs}}f(x,y)d_qxd_qy\\
&= \frac{1}{(1-q)^2}\frac{1}{ps}\int_0^\infty\int_0^\infty e_q^{-\frac{(a+p)x}{ap}-\frac{(b+s)y}{bs}}f(x,y)d_qxd_qy = \frac{ab}{(a+p)(b+s)}\overline{\overline{G_q}}\left(\frac{ap}{a+p},\frac{bs}{b+s}\right).
\end{aligned}$$

d): $S_q[f(x)]=\frac{1}{(1-q)}\overline{G_q}(p)$, where $\overline{G_q}(p)=\frac{1}{(1-q)}\frac{1}{p}\int_0^\infty e_q^{-\frac{x}{p}}f(x)d_qx$,

$$\begin{aligned}
S_q[f(x)] &= \frac{1}{(1-q)^2}\frac{1}{ps}\int_0^\infty\int_0^\infty e_q^{-\frac{x}{p}-\frac{y}{s}}f(x)d_qxd_qy\\
&= \frac{1}{(1-q)^2}\frac{1}{ps}\int_0^\infty e_q^{-\frac{y}{s}}d_qy\int_0^\infty e_q^{-\frac{x}{p}}f(x)d_qx\\
&= \frac{1}{(1-q)^2}\frac{1}{ps}(-s)\left[e_q^{-\frac{y}{s}}\right]\Big|_0^\infty\int_0^\infty e_q^{-\frac{x}{p}}f(x)d_qx\\
&= \frac{1}{(1-q)}\left[\frac{1}{(1-q)}\frac{1}{p}\int_0^\infty e_q^{-\frac{x}{p}}f(x)d_qx\right]=\frac{1}{(1-q)}\overline{G_q}(p).
\end{aligned}$$

5.2. Examples

1. If $f(x,y)=1$ for $x>0, y>0$, then for $1<q<2$

$$\begin{aligned}
S_q[1] &= \frac{1}{(1-q)^2}\frac{1}{ps}\int_0^\infty\int_0^\infty e_q^{(-\frac{x}{p}-\frac{t}{s})}d_qxd_qt\\
&= \frac{1}{(1-q)^2}\frac{1}{ps}\int_0^\infty\int_0^\infty e_q^{-\frac{x}{p}}e_q^{-\frac{t}{s}}d_qxd_qt\\
&= \frac{1}{(1-q)^2}\frac{1}{ps}\int_0^\infty e_q^{-\frac{x}{p}}\left\{\int_0^\infty\left[1-(1-q)\frac{t}{s}\right]^{\frac{1}{1-q}}d_qt\right\}d_qx\\
&= \frac{1}{(1-q)^2}\frac{1}{ps}\int_0^\infty e_q^{-\frac{x}{p}}\left\{\frac{\left[1-(1-q)\frac{t}{s}\right]^{\frac{2-q}{1-q}}}{\frac{2-q}{1-q}(1-q)}s\Big|_0^\infty\right\}d_qx\\
&= \frac{1}{(1-q)^2}\frac{1}{p}\int_0^\infty e_q^{-\frac{x}{p}}\frac{1}{(q-2)}d_qx
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{(1-q)^2}\frac{1}{p}\frac{1}{(q-2)}\int_0^\infty e_q^{-\frac{x}{p}} d_q x \\
&= \frac{1}{(1-q)^2}\frac{1}{p}\frac{1}{(q-2)}\int_0^\infty \left[1-(1-q)\frac{x}{p}\right]^{\frac{1}{1-q}} d_q x \\
&= \frac{1}{(1-q)^2}\frac{1}{p}\frac{1}{(q-2)}\left\{\frac{\left[1-(1-q)\frac{x}{p}\right]^{\frac{2-q}{1-q}}}{\frac{2-q}{1-q}(1-q)} p\Big|_0^\infty\right\} = \frac{1}{(1-q)^2}\frac{1}{(q-2)^2}.
\end{aligned}
$$

2. If $f(x,t) = \cos_q(\frac{x}{a}+\frac{t}{b})$,

$$
S_q\left[\cos_q\left(\frac{x}{a}+\frac{t}{b}\right)\right] = \frac{1}{(1-q)^2}\frac{1}{ps}\int_0^\infty\int_0^\infty e_q^{(-\frac{x}{p}-\frac{t}{s})}\left[\frac{e_q^{i(\frac{x}{a}+\frac{t}{b})}+e_q^{-i(\frac{x}{a}+\frac{t}{b})}}{2}\right] d_q x d_q t.
$$

The last integral can be divided in two parts

$$
= \frac{1}{(1-q)^2}\frac{1}{ps}\int_0^\infty\int_0^\infty \frac{e_q^{(-\frac{x}{p}-\frac{t}{s})}e_q^{i(\frac{x}{a}+\frac{t}{b})}}{2} d_q x d_q t + \frac{1}{(1-q)^2}\frac{1}{ps}\int_0^\infty\int_0^\infty \frac{e_q^{(-\frac{x}{p}-\frac{t}{s})}e_q^{-i(\frac{x}{a}+\frac{t}{b})}}{2} d_q x d_q t.
$$

Similarly as above we get

$$
= \frac{1}{2}\left[\frac{ba}{(1-q)^2(b-si)(a-pi)} + \frac{ba}{(1-q)^2(b+si)(a+pi)}\right] = \frac{ba(ba-sp)}{(1-q)^2(b^2+s^2)(a^2+p^2)}.
$$

3. If $f(x,t) = (xt)^n$,

$$
S_q[(xt)^n)] = \frac{1}{(1-q)^2}\frac{1}{ps}\int_0^\infty\int_0^\infty e_q^{(-\frac{x}{p}-\frac{t}{s})}(xt)^n d_q x d_q t = \frac{1}{(1-q)^2}\frac{1}{ps}\int_0^\infty e_q^{(-\frac{x}{p})}x^n d_q x\int_0^\infty e_q^{(-\frac{t}{s})}t^n d_q t.
$$

If we substitute $\frac{x}{p} = u$, we get

$$
\begin{aligned}
&= \frac{1}{(1-q)^2}\frac{1}{ps}\int_0^\infty e_q^{-u}(pu)^n p d_q u\int_0^\infty e_q^{(-\frac{t}{s})}t^n d_q t \\
&= \frac{1}{(1-q)^2}\frac{1}{s}p^n\int_0^\infty e_q^{-u}u^n d_q u\int_0^\infty e_q^{(-\frac{t}{s})}t^n d_q t = \frac{1}{(1-q)^2}\frac{p^n}{s}\Gamma_q(n+1)\int_0^\infty e_q^{(-\frac{t}{s})}t^n d_q t.
\end{aligned}
$$

In a similar manner if we substitute $\frac{t}{s} = v$, we will get

$$
= \frac{p^n s^n}{(1-q)^2}\Gamma_q(n+1)\Gamma_q(n+1).
$$

6. Conclusion

In this paper we have introduced some properties of double q-Sumudu transform and its convolution. Convergence and absolute convergence were also discussed, as well as the q-Sumudu transform of periodic functions. The results proved in this paper appear to be new and with certain applications to solving q-difference and q-integral equations.

References

[1] W. H. Abdi *On q-Laplace Transforms*, Proc. Nat. Acad. Sci. India Sect. A, **29** (1960), 389–408. 1

[2] M. H. Abu Risha, M. H. Annaby, H. E. H. Ismail, Z. S. Mansour, *Linear q-difference equations*, Z. Anal. Anwend., **26** (2007), 481–494. 1

[3] Z. Ahmed, M. I. Idrees, F. B. M. Belgacem, Z. Perveen, *On the convergence of double Sumudu transform*, J. Nonlinear Sci. Appl., **13** 2019, 154–162. 1

[4] D. Albayrak, S. D .Purohit, F. Ucar, *On* q*-Sumudu transforms of certain* q*-polynomials*, Filomat, **27** (2013), 411–427. 1
[5] D. Albayrak, S. D. Purohit, F. Ucar, *On* q*-analogues of Sumudu transforms*, An. Stiint. Univ. "Ovidius" Constanţa Ser. Mat., **21** (2013), 239–259. 1
[6] D. Albayrak, S. D. Purohit, F. Ucar, *Certain Inversion and Representation formulas for* q*-Sumudu Transforms*, Hacet. J. Math. Stat., **43** (2014), 699–713. 1
[7] M. A. Asiru, *Further Properties of the Sumudu Transform and its Applications*, Internat. J. Math. Ed. Sci. Tech., **33** (2002), 441–449. 1
[8] G. Bangerezako, *Variational calculus on* q*-nonuniform lattices*, J. Math. Anal. Appl., **306** (2005), 161–179. 1
[9] F. B. M. Belgacem, A. A. Karaballi, S. L. Kalla, *Analytical investigations of the Sumudu transform and applications to integral production equations*, Math. Probl. Eng., **2003** (2003), 103–118. 1
[10] K. Brahim, L. Riahi, *Two dimensional Mellin transform in quantum Calculus*, Acta Math. Sci. Ser. B (Engl. Ed.), **38** (2018), 546–560. 1, 2
[11] L. Debnath, *The Double Laplace Transforms and Their Properties with Applications to Functional, Integral and Partial Differential Equations*, Int. J. Appl. Comput. Math., **2** (2016), 223–241. 1
[12] A. De Sole, V. G. Kac, *On integral representation of* q*-gamma and* q*-beta functions*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl., **16** (2005), 11–29. 2
[13] J. A. Ganie, A. Ahmad, R. Jain *Basic Analogue of Double Sumudu Transform and its Applicability in Population Dynamics*, Asian J. Math. Stat., **11** (2018), 12–17. 3.1
[14] J. V. Ganie, R. Jain, *On a system of* q*-Laplace transform of two variables with applications*, J. Comput. Appl. Math., **366** (2020), 12 pages. 1
[15] G. Gasper, M. Rahmen, *Basic Hypergeometric Series*, Second ed., Cambridge University Press, Cambridge, (2004). 2
[16] W. Hahn, *Beitrage Zur Theorie der Heineschen Reihen, die 24 Inte-grale der hypergeometrischen* q*-Diferenzengleichung, das* q*-Analog on der Laplace Transformation*, Math. Nachr., **2** (1949), 340–379. 1
[17] D. O. Jackson, T. Fukuda, O. Dunn, E. Majors, *On* q*-definite integrals*, Quart. J. Pure Appl. Math., **41** (1910), 193–203. 1, 2
[18] V. Kac, P. Cheung, *Quantum Calculus*, Springer-Verlag, New York, (2002). 1, 2
[19] A. Kilicman, H. E. Gadian, *On the application of Laplace and Sumudo transforms*, J. Franklin Inst., **347** (2010), 848–862. 1
[20] S. D. Purohit, S. L. Kalla ,*On* q*-Laplace transforms of the* q*-Bessel functions*, Fract. Calc. Appl. Anal., **10** (2007), 189–196. 1
[21] P. N. Sadjang, *On double* q*-Laplace transform and application*, arXiv, **2019** (2019), 27 pages. 1
[22] J. M. Tchuenche, N. S. Mbare, *An Application of the double Sumudu Transform*, J. Math. Anal. Appl. (Ruse), **1** (2007), 31–39. 1
[23] M. K. Wang, Y. M. Chu, *Refinements of transformation inequalities for zero-balanced hypergeometric functions*, Acta Math. Sci. Ser. B (Engl. Ed.), **37** (2017), 607–622. 2
[24] G. K. Watugala, *Sumudu transform: a new integral transform to solve differential equations and control engineering problems*, Internat. J. Math. Ed. Sci. Tech., **24** (1993), 35–43. 1
[25] G. K. Watugula, *The Sumudu transform for functions of two variables*, Math. Engrg. Indust., **8** (2002), 293–302. 1
[26] Z.-H. Yang, Y.-M. Chu, *Asymptotic formulas for gamma function with applications*, Appl. Math. Comput., **270** (2015), 665–680. 2
[27] Z.-H. Yang, W.-M. Qian, Y.-M. Chu, W. Zhang, *On rational bounds for the gamma function*, J. Inequal. Appl., **2017** (2017), 17 pages.
[28] Z.-H. Yang, W. Zhang, Y.-M. Chu, *Sharp Gautschi inequality for parameter* $0 < p < 1$ *with applications*, Math. Inequal. Appl., **20** (2017), 1107–1120. 2
[29] T.-H. Zhao, Y.-M. Chu, H. Wang, *Logarithmically complete monotonicity properties relating to the gamma function*, Abstr. Appl. Anal., **2011** (2011), 13 pages.