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θ_s-open sets and θ_s-continuity of maps in the product space

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Abstract

In this study, the concept of θ_s-open set is introduced. The topology formed by θ_s-open sets is strictly finer than the topology formed by θ-open sets but is not comparable with the topology formed by ω_θ-open sets. Related concepts such as θ_s-open and θ_s-closed functions, θ_s-continuous function, θ_s-connected space, and some versions of separation axioms are defined and characterized. Finally, the concept of θ_s-continuous function from an arbitrary topological space into the product space is investigated further.

Keywords: θ_s-open, θ_s-closed, θ_s-connected, θ_s-continuous, θ_s-Hausdorff, θ_s-regular, θ_s-normal.

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1. Introduction and Preliminaries

Introducing new versions of open sets in a topological space which may acquire either weaker or stronger properties is often studied. The first attempt was done by Levine [21], where he introduced the concepts of semi-open set, semi-closed set and semi-continuity of a function.

A subset O of a space X is semi-open if $O \subseteq Cl(Int(O))$. Equivalently, O is semi-open if there exists an open set G in X such that $G \subseteq O \subseteq Cl(G)$. A subset F of X is semi-closed if its complement $X \setminus F$ is semi-open in X. Let A be a subset of a space X. A point $p \in X$ is a semi-closure point of A if for every semi-open set G in X containing x, $G \cap A \neq \varnothing$. We denote by $sCl(A)$ the set of all semi-closure points of A.

In 1968, Velicko [24] introduced the concept of θ-continuity between topological spaces and defined the concepts of θ-closure and θ-interior of a set. The concepts of θ-open sets and its related topological concepts had been also studied by numerous authors, see [1, 5, 6, 13, 14, 17–19, 22, 23].

Let $(X, \mathfrak{T})$ be a topological space and $A \subseteq X$. The θ-*closure* and θ-*interior* of A are, respectively, defined by $Cl_\theta(A) = \{x \in X : Cl(U) \cap A \neq \varnothing$ for every open set U containing $x\}$ and $Int_\theta(A) = \{x \in X : Cl(U) \subseteq A$ for some open set U containing $x\}$, where $Cl(U)$ is the closure of U in X. A is θ-*closed* if $Cl_\theta(A) = A$

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and θ-*open* if $\mathrm{Int}_\theta(A) = A$. Equivalently, A is θ-open if and only if $X \setminus A$ is θ-closed. It is known that the collection $\mathcal{T}_\theta$ of all θ-open sets forms a topology on X, which is strictly coarser that $\mathcal{T}$.

In 1982, Hdeib [16] introduced the concepts of ω-open, ω-closed sets and ω-closed mappings on a topological space. He showed that ω-closed mappings are strictly weaker than closed mappings. The collection $\mathcal{T}_\omega$ of all ω-open sets forms a topology on X, which is strictly finer than $\mathcal{T}$. Several mathematicians studied the concepts related to ω-open sets and its corresponding topological concepts, see [2–4, 7–12, 20].

In 2010, Ekici et al. [15] introduced the concepts of ω_θ-open and ω_θ-closed sets on a topological space. Then the concepts of ω_θ-interior, ω_θ-closure, ω_θ-continuity and ω_θ-connectedness were subsequently defined.

A point x of a topological space X is called a condensation point of $A \subseteq X$ if for each open set G containing x, $G \cap A$ is uncountable. A subset B of X is ω-closed if it contains all of its condensation points. The complement of B is ω-open. Equivalently, a subset U of X is ω-open (resp., ω_θ-open) if and only if for each $x \in U$, there exists an open set O containing x such that $O \setminus U$ (resp., $O \setminus \mathrm{Int}_\theta(U)$) is countable. A subset B of X is ω_θ-closed if its complement $X \setminus B$ is ω_θ-open. It is worth noting that the collection $\mathcal{T}_{\omega_\theta}$ of ω_θ-open sets forms a topology on X that is strictly finer than $\mathcal{T}_\theta$ but is not comparable with $\mathcal{T}$.

Let $\mathcal{A}$ be an indexing set and $\{Y_\alpha : \alpha \in \mathcal{A}\}$ be a family of topological spaces. For each $\alpha \in \mathcal{A}$, let $\mathcal{T}_\alpha$ be the topology on Y_α. The Tychonoff topology on $\Pi\{Y_\alpha : \alpha \in \mathcal{A}\}$ is the topology generated by a subbase consisting of all sets $p_\alpha^{-1}(U_\alpha)$, where the projection map $p_\alpha : \Pi\{Y_\alpha : \alpha \in \mathcal{A}\} \to Y_\alpha$ is defined by $p_\alpha(\langle y_\beta \rangle) = y_\alpha$, U_α ranges over all members of $\mathcal{T}_\alpha$, and α ranges over all elements of $\mathcal{A}$. Corresponding to $U_\alpha \subseteq Y_\alpha$, denote $p_\alpha^{-1}(U_\alpha)$ by $\langle U_\alpha \rangle$. Similarly, for finitely many indices $\alpha_1, \alpha_2, \dots, \alpha_n$, and sets $U_{\alpha_1} \subseteq Y_{\alpha_1}$, $U_{\alpha_2} \subseteq Y_{\alpha_2}, \dots, U_{\alpha_n} \subseteq Y_{\alpha_n}$, the subset $\langle U_{\alpha_1} \rangle \cap \langle U_{\alpha_2} \rangle \cap \cdots \cap \langle U_{\alpha_n} \rangle = p_{\alpha_1}^{-1}(U_{\alpha_1}) \cap p_{\alpha_2}^{-1}(U_{\alpha_2}) \cap \cdots \cap p_{\alpha_n}^{-1}(U_{\alpha_n})$ is denoted by $\langle U_{\alpha_1}, U_{\alpha_2}, \dots, U_{\alpha_n} \rangle$. We note that for each open set U_α subset of Y_α, $\langle U_\alpha \rangle = p_\alpha^{-1}(U_\alpha) = U_\alpha \times \Pi_{\beta \neq \alpha} Y_\beta$. Hence, a basis for the Tychonoff topology consists of sets of the form $\langle B_{\alpha_1}, B_{\alpha_2}, \dots, B_{\alpha_k} \rangle$, where B_{α_i} is open in Y_{α_i} for every $i \in K = \{1, 2, \dots, k\}$.

Now, the projection map $p_\alpha : \Pi\{Y_\alpha : \alpha \in \mathcal{A}\} \to Y_\alpha$ is defined by $p_\alpha(\langle y_\beta \rangle) = y_\alpha$ for each $\alpha \in \mathcal{A}$. It is known that every projection map is a continuous open surjection. Also, it is well known that a function f from an arbitrary space X into the Cartesian product Y of the family of spaces $\{Y_\alpha : \alpha \in \mathcal{A}\}$ with the Tychonoff topology is continuous if and only if each coordinate function $p_\alpha \circ f$ is continuous, where p_α is the α-th coordinate projection map.

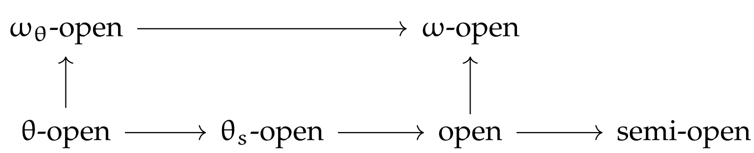
In this paper, we define a new type of topology that is strictly finer than $\mathcal{T}_\theta$, strictly coarser than $\mathcal{T}$, and is not comparable with $\mathcal{T}_{\omega_\theta}$.

2. θ_s-open sets and some functions

In this section, we shall define the concept of θ_s-open set and determine its connection to the classical open, θ-open, and ω_θ-open sets. We shall also define and characterize the concepts of θ_s-open and θ_s-closed functions.

Definition 2.1. Let X be a topological space. $A \subseteq X$ is said to be θ_s-open if for every $x \in A$, there exists an open set U containing x such that $\mathrm{sCl}(U) \subseteq A$. A subset F of X is called a θ_s-closed if its complement $X \setminus F$ is θ_s-open.

Remark 2.2. The following diagram holds for a subset of a topological space.



We also remark that the above diagram is also true for their respective closed sets. We note that since ω_θ-open and open are two independent notions [15, Example 5], ω_θ-open does not imply θ_s-open.

Remark 2.3. The implications in the above diagram (with respect to θ_s-open set) are not reversible. To see this, consider the following examples.

(i) Let $X = \{a, b, c, d\}$ with topology $\mathcal{T} = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$. Then $\{a\}$ is θ_s-open but not θ-open.

(ii) Let $X = \{a, b, c\}$ with topology $\mathcal{T} = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}\}$. Then $\{a, b\}$ is open but not θ_s-open.

(iii) Consider $\mathbb{R}$ as the real number line with topology $\mathcal{T} = \{\varnothing, \mathbb{R}, \mathbb{N}, \mathbb{R} \setminus \mathbb{Z}, \mathbb{N} \cup (\mathbb{R} \setminus \mathbb{Z})\}$. Then $\mathbb{R} \setminus \mathbb{Z}$ is θ_s-open but not ω_θ-open.

Before showing that the collection of θ_s-open sets forms a topology, we shall consider first the following remark.

Remark 2.4. Let $A \subseteq X$. Then $sCl(A)$ is the smallest semi-closed set containing A, that is, $sCl(A)=\bigcap\{F : F$ is semi-closed and $A \subseteq F\}$. Moreover, for $B, C \subseteq X$, we have

(i) if $B \subseteq C$, then $sCl(B) \subseteq sCl(C)$;

(ii) $sCl(sCl(B)) = sCl(B)$;

(iii) $sCl(B) \cup sCl(C) \subseteq sCl(B \cup C)$;

(iv) if A is closed, then $sCl(A) = A$.

Theorem 2.5. *Let $\mathcal{T}_{\theta_s}$ be a family of all θ_s-open subsets of a topological space X. Then $\mathcal{T}_{\theta_s}$ forms a topology on X.*

Proof. Obviously, $\varnothing, X \in \mathcal{T}_{\theta_s}$. Now, let $\{A_\alpha : \alpha \in I\}$ be a collection of a θ_s-open subsets of X. Let $x \in \cup\{A_\alpha : \alpha \in I\}$. Then $x \in A_\alpha$ for some $\alpha \in I$. Since A_α is θ_s-open, there exists an open set U containing x such that $sCl(U) \subseteq A_\alpha \subseteq \cup\{A_\alpha : \alpha \in I\}$. Thus, $\cup\{A_\alpha : \alpha \in I\}$ is θ_s-open.

Next, $x \in A_1 \cap A_2$, where $A_1, A_2 \in \mathcal{T}_{\theta_s}$. Then there exist open sets U_1 and U_2 both containing x such that $sCl(U_1) \subseteq A_1$ and $sCl(U_1) \subseteq A_2$. Note that $U_1 \cap U_2$ is an open set containing x. By Remark 2.4, $sCl(U_1 \cap U_2) \subseteq sCl(U_1) \cap sCl(U_2) \subseteq A_1 \cap A_2$. Hence, $A_1 \cap A_2$ is θ_s-open. Consequently, $\mathcal{T}_{\theta_s}$ forms a topology on X. □

Definition 2.6. Let X be a topological space and $A \subseteq X$. Then the θ_s-interior of A is denoted and defined by $Int_{\theta_s}(A)=\cup\{U : U$ is θ_s-open and $U \subseteq A\}$. We note that by Theorem 2.5, $Int_{\theta_s}(A)$ is the largest θ_s-open set contained in A. Moreover, $x \in Int_{\theta_s}(A)$ if and only if there exists a θ_s-open set U containing x such that $U \subseteq A$.

Definition 2.7. Let X be a topological space and $A \subseteq X$. Then the θ_s-closure of A is denoted and defined by $Cl_{\theta_s}(A)=\cap\{F : F$ is θ_s-closed and $A \subseteq F\}$. We note that by Theorem 2.5, $Cl_{\theta_s}(A)$ is the smallest θ_s-closed set containing A.

Remark 2.8. Let X be a topological space and $A, B \subseteq X$. Then

(i) if $A \subseteq B$, then $Int_{\theta_s}(A) \subseteq Int_{\theta_s}(B)$;

(ii) A is θ_s-open if and only if $A = Int_{\theta_s}(A)$;

(iii) $Int_{\theta_s}(Int_{\theta_s}(A)) = Int_{\theta_s}(A)$;

(iv) $Int_{\theta_s}(A \cap B) = Int_{\theta_s}(A) \cap Int_{\theta_s}(B)$;

(v) $x \in Cl_{\theta_s}(A)$ if and only if for every θ_s-open subset U containing x, $U \cap A \neq \varnothing$;

(v) $A \subseteq B$ implies that $Cl_{\theta_s}(A) \subseteq Cl_{\theta_s}(B)$;

(vi) A is θ_s-closed if and only if $Cl_{\theta_s}(A) = A$;

(vii) $Cl_{\theta_s}(Cl_{\theta_s}(A)) = Cl_{\theta_s}(A)$;

(viii) $Cl_{\theta_s}(A) \cup Cl_{\theta_s}(B) = Cl_{\theta_s}(A \cup B)$;

(ix) $Int_{\theta_s}(X \setminus A) = X \setminus Cl_{\theta_s}(A)$;

(x) $Cl_{\theta_s}(X \setminus A) = X \setminus Int_{\theta_s}(A)$;

(xi) A is θ_s-open if and only if for every $x \in A$, there exists a basic open set B containing x such that $sCl(B) \subseteq A$;

(xii) $x \in \text{Int}_{\theta_s}(A)$ if and only if there exists an open set O containing x such that $\text{sCl}(O) \subseteq A$;

(xiii) $x \in \text{Cl}_{\theta_s}(A)$ if and only if for every open set U containing x, $\text{sCl}(U) \cap A \neq \varnothing$.

Next, we introduce and characterize the concepts of θ_s-open and θ_s-closed functions.

Definition 2.9. Let X and Y be topological spaces. A function $f : X \to Y$ is θ_s-open (resp., θ_s-closed) on X if $f(G)$ is θ_s-open (resp., θ_s-closed) in Y for every open (resp., closed) set G in X.

Theorem 2.10. *Let X and Y be topological spaces and $f : X \to Y$ be a function. Then the following statements are equivalent.*

(i) f *is* θ_s*-open on* X.

(ii) $f(\text{Int}(A)) \subseteq \text{Int}_{\theta_s}(f(A))$ *for every* $A \subseteq X$.

(iii) $f(B)$ *is* θ_s*-open for every basic open set* B *in* X.

(iv) *For each* $x \in X$ *and for every open set* U *in* X *containing* x, *there exists an open set* V *in* Y *containing* $f(x)$ *such that* $\text{sCl}(V) \subseteq f(U)$.

Proof.

(i)⇒(ii): Let $A \subseteq X$. Note that $f(\text{Int}(A)) \subseteq f(A)$ and $f(\text{Int}(A))$ is θ_s-open. In view of Definition 2.6, $f(\text{Int}(A)) \subseteq \text{Int}_{\theta_s}(f(A))$.

(ii)⇒(iii): Let B be a basic open set in X. Then $f(B) = f(\text{Int}(B)) \subseteq \text{Int}_{\theta_s}(f(B)) \subseteq f(B)$. By Remark 2.8, $f(B)$ is θ_s-open.

(iii)⇒(iv): Let $x \in X$ and let U be an open set containing x. Then there exists a basic open set B containing x such that $B \subseteq U$, which implies that $f(x) \in f(B) \subseteq f(U)$. By assumption, there exists an open set V in Y containing $f(x)$ such that $\text{sCl}(V) \subseteq f(B) \subseteq f(U)$.

(iv)⇒(i): Let U be an open set in X and let $y \in f(U)$. Then there exists $x \in U$ such that $f(x) = y$. By assumption, there exists an open set V in Y containing y such that $\text{sCl}(V) \subseteq f(U)$, hence $f(U)$ is θ_s-open in Y. Thus, f is θ_s-open on X. □

Theorem 2.11. *Let X and Y be topological spaces and $f : X \to Y$ be a function. Then the following statements are equivalent:*

(i) f *is* θ_s*-closed on* X;

(ii) $\text{Cl}_{\theta_s}(f(A)) \subseteq f(\text{Cl}(A))$ *for every* $A \subseteq X$.

Proof.

(i)⇒ (ii): Let $A \subseteq X$. Note that $f(A) \subseteq f(\text{Cl}(A))$ and $f(\text{Cl}(A))$ is θ_s-closed. In view of Definition 2.7, $\text{Cl}_{\theta_s}(f(A)) \subseteq f(\text{Cl}(A))$.

(ii)⇒(i): Let F be a closed set in X. By assumption, $f(F) \subseteq \text{Cl}_{\theta_s}(f(F)) \subseteq f(\text{Cl}(F)) = f(F)$. Thus, $f(F)$ is θ_s-closed. Therefore, f is θ_s-closed on X. □

Remark 2.12. Let X and Y be topological spaces and $f : X \to Y$ be a bijective function. Then f is θ_s-open on X if and only if f is θ_s-closed on X.

3. θ_s-continuity of functions in the product space

In this section, we define the concept of θ_s-continuous function and then give its characterization from an arbitrary topological space into the product space.

Definition 3.1. A function $f : X \to Y$ is θ_s-continuous if $f^{-1}(G)$ is θ_s-open for every open set G of Y.

The proofs of the following results are standard, hence omitted.

Theorem 3.2. *Let* X *and* Y *be topological spaces and* $f : X \to Y$ *be a function. Then the following statements are equivalent:*

(i) f *is* θ_s*-continuous on* X*;*
(ii) $f^{-1}(F)$ *is* θ_s*-closed in* X *for each closed subset* F *of* Y*;*
(iii) $f^{-1}(B)$ *is* θ_s*-open for each (subbasic) basic open set* B *in* Y*;*
(iv) *for every* $p \in X$ *and every open set* V *of* Y *containing* $f(p)$*, there exists a* θ_s*-open set* U *containing* p *such that* $f(U) \subseteq V$*;*
(v) $f(Cl_{\theta_s}(A) \subseteq Cl(f(A))$ *for each* $A \subseteq X$*;*
(vi) $Cl_{\theta_s}(f^{-1}(B)) \subseteq f^{-1}(Cl(B))$.

Theorem 3.3. *Let* X *and* Y *be topological spaces and* $f_A : X \to \mathcal{D}$ *the characteristic function of a subset* A *of* X*, where* $\mathcal{D}$ *is the set* $\{0, 1\}$ *with the discrete topology. Then* f_A *is* θ_s*-continuous if and only if* A *is both* θ_s*-open and* θ_s*-closed.*

In the following results, if $Y = \Pi\{Y_\alpha : \alpha \in \mathcal{A}\}$ is a product space and $A_\alpha \subseteq Y_\alpha$ for each $\alpha \in \mathcal{A}$, we denote $A_{\alpha_1} \times \cdots \times A_{\alpha_n} \times \Pi\{Y_\alpha : \alpha \notin K\}$ by $\langle A_{\alpha_1}, \cdots, A_{\alpha_n}\rangle$, where $K = \{\alpha_1, \cdots, \alpha_n\}$.

If $Y = \Pi\{Y_{\alpha_i} : 1 \leqslant i \leqslant n\}$ is a finite product, denote $A_{\alpha_1} \times \cdots \times A_{\alpha_n}$ by $\langle A_{\alpha_1}, \cdots, A_{\alpha_n}\rangle$.

Theorem 3.4. *Let* $Y = \Pi\{Y_\alpha : \alpha \in \mathcal{A}\}$ *be a product space,* $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subseteq \mathcal{A}$ *and* $\varnothing \neq O_{\alpha_i} \subseteq Y_{\alpha_i}$ *for each* $\alpha_i \in S$*. If each* O_{a_i} *is semi-open in* Y_{α_i}*, then* $O = \langle O_{\alpha_1}, O_{\alpha_2}, \ldots, O_{\alpha_n}\rangle$ *is semi-open in* Y*.*

Proof. Suppose that O_{α_i} is semi-open in Y_{α_i}. Then for each $\alpha_i \in S$, there exists an open set G_{α_i} such that $G_{\alpha_i} \subseteq O_{\alpha_i} \subseteq Cl(G_{\alpha_i})$. Let $G = \langle G_{\alpha_1}, G_{\alpha_2}, \ldots, G_{\alpha_n}\rangle$ which is open in Y. Hence,

$$G \subseteq O \subseteq \langle Cl(G_{\alpha_1}), Cl(G_{\alpha_2}), \ldots, Cl(G_{\alpha_n})\rangle = Cl(\langle G\rangle).$$

Thus, O is semi-open in Y. □

Theorem 3.5. *Let* $Y = \Pi\{Y_\alpha : \alpha \in \mathcal{A}\}$ *be a product space and* $A_\alpha \subseteq Y_\alpha$ *for each* $\alpha \in \mathcal{A}$*. Then* $sCl(\Pi\{A_\alpha : \alpha \in \mathcal{A}\}) \subseteq \Pi\{sCl(A_\alpha) : \alpha \in \mathcal{A}\}$*.*

Proof. Let $x = \langle a_\alpha\rangle \notin \Pi\{sCl(A_\alpha) : \alpha \in \mathcal{A}\}$. Then $a_\beta \notin sCl(A_\beta)$ for some $\beta \in \mathcal{A}$. This means that there exists a semi-open set G_β containing a_β such that $G_\beta \cap A_\beta = \varnothing$. By Theorem 3.4, $\langle G_\beta\rangle$ is semi-open containing x. Hence $\langle G_\beta\rangle \cap \Pi\{A_\alpha : \alpha \in \mathcal{A}\} = \varnothing$. Thus, $x \notin sCl(\Pi\{A_\alpha : \alpha \in \mathcal{A}\})$. □

Theorem 3.6. *Let* $Y = \Pi\{Y_\alpha : \alpha \in \mathcal{A}\}$ *be a product space and* $A_\alpha \subseteq Y_\alpha$ *for each* $\alpha \in \mathcal{A}$*. Then* $Cl_{\theta_s}(\Pi\{A_\alpha : \alpha \in \mathcal{A}\}) \subseteq \Pi\{Cl_{\theta_s}(A_\alpha) : \alpha \in \mathcal{A}\}$*.*

Proof. Let $x = \langle a_\alpha\rangle \in Cl_{\theta_s}(\Pi\{A_\alpha : \alpha \in \mathcal{A}\})$. Then for every open set O containing x, $sCl(O) \cap \Pi\{A_\alpha : \alpha \in \mathcal{A}\} \neq \varnothing$. Suppose that there exists $\beta \in \mathcal{A}$ such that $a_\beta \notin Cl_{\theta_s}(A_\beta)$. Then there exists an open set U_β containing a_β such that $sCl(U_\beta) \cap A_\beta = \varnothing$. By Theorem 3.5, $sCl(\langle G_\beta\rangle) \subseteq \langle sCl(G_\beta)\rangle$. It follows that $x = \langle a_\alpha\rangle \in \langle U_\beta\rangle$ and $sCl(\langle G_\beta\rangle) \cap \Pi\{A_\alpha : \alpha \in \mathcal{A}\} = \varnothing$, a contradiction. Thus, $x \in \Pi\{Cl_{\theta_s}(A_\alpha) : \alpha \in \mathcal{A}\}$. □

Theorem 3.7. *Let* $Y = \Pi\{Y_i : 1 \leqslant i \leqslant n\}$ *be a (finite) product space and* $A_i \subseteq Y_i$ *for each* $i = 1, \ldots, n$*. Then*

$$\Pi\{Int_{\theta_s}(A_i) : 1 \leqslant i \leqslant n\} \subseteq Int_{\theta_s}(\Pi\{A_i : 1 \leqslant i \leqslant n\}).$$

Proof. Let $x = \langle a_1, a_2, \ldots, a_n\rangle \in \Pi\{Int_{\theta_s}(A_i) : 1 \leqslant i \leqslant n\}$. Then $a_i \in Int_{\theta_s}(A_i)$ for all $i = 1, 2, \ldots, n$. This means that for all $i = 1, 2, \ldots, n$, there esists an open set O_i containing a_i such that $sCl(O_i) \subseteq A_i$. Let $O = \langle O_1, O_2, \ldots, O_n\rangle$, which is an open set in Y containing x. By Theorem 3.5, $sCl(O) = sCl(\langle O_1, O_2, \ldots, O_n\rangle) \subseteq \langle sCl(O_1), sCl(O_2), \ldots, sCl(O_n)\rangle \subseteq \langle A_1, A_2, \ldots A_n\rangle$. Hence, $x \in Int(\Pi\{A_i : 1 \leqslant i \leqslant n\})$. □

Theorem 3.8. *Let* $Y = \Pi\{Y_\alpha : \alpha \in \mathcal{A}\}$ *be a product space,* $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subseteq \mathcal{A}$ *and* $\varnothing \neq O_{\alpha_i} \subseteq Y_{\alpha_i}$ *for each* $\alpha_i \in S$*. If each* O_{a_i} *is* θ_s*-open in* Y_{α_i}*, then* $O = \langle O_{\alpha_1}, O_{\alpha_2}, \ldots, O_{\alpha_n}\rangle$ *is* θ_s*-open in* Y*.*

Proof. Let $x = \langle a_\alpha \rangle \in O$. Then, $a_{\alpha_i} \in O_{\alpha_i}$ for every $\alpha_i \in S$. This means that for every $\alpha_i \in S$, there exists an open set U_{α_i} containing a_{α_i} such that $sCl(U_{\alpha_i}) \subseteq O_{\alpha_i}$. Let $U = \langle U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_n} \rangle$. Then $x \in U$ and by Theorem 3.5, $sCl(U) = \langle sCl(U_{\alpha_1}), sCl(U_{\alpha_2}), \ldots, sCl(U_{\alpha_n}) \rangle \subseteq O$. Thus, O is θ_s-open in Y. □

Theorem 3.9. *Let $X = \Pi\{X_\alpha : \alpha \in \mathcal{A}\}$ and $Y = \Pi\{Y_\alpha : \alpha \in \mathcal{A}\}$ be product spaces and for each $\alpha \in \mathcal{A}$, let $f_\alpha : X_\alpha \to Y_\alpha$ be a function. If each f_α is θ_s-continuous on X_α, then the function $f : X \to Y$ defined by $f(\langle x_\alpha \rangle) = \langle f_\alpha(x_\alpha) \rangle$ is θ_s-continuous on X.*

Proof. Let $\langle V_\alpha \rangle$ be a subbasic open set in Y. Then $f^{-1}(\langle V_\alpha \rangle) = \langle f_\alpha^{-1}(V_\alpha) \rangle$. Since each f_α is θ_s-continuous, $f_\alpha^{-1}(V_\alpha)$ is θ_s-open in X_α. Let $x = \langle x_\beta \rangle \in f^{-1}(\langle V_\alpha \rangle) = \langle f_\alpha^{-1}(V_\alpha) \rangle$. Then $x_\alpha \in f_\alpha^{-1}(V_\alpha)$. Hence, there exists an open set U_α containing x_α such that $sCl(U_\alpha) \subseteq f_\alpha^{-1}(V_\alpha)$. Note that $\langle U_\alpha \rangle$ is open in X containing x. By Theorem 3.5, $sCl(U) \subseteq \langle sCl(U\alpha) \rangle \subseteq \langle f_\alpha^{-1}(V_\alpha) \rangle = f^{-1}(\langle V_\alpha \rangle)$. This means that $f^{-1}(\langle V_\alpha \rangle)$ is θ_s-open in X. Thus, f is θ_s-continuous on X. □

Theorem 3.10. *Let X be a topological space and $Y = \Pi\{Y_\alpha : \alpha \in \mathcal{A}\}$ be a product space. A function $f : X \to Y$ is θ_s-continuous on X if and only if $p_\alpha \circ f$ is θ_s-continuous on X for every $\alpha \in \mathcal{A}$.*

Proof. Suppose that f is θ_s-continuous on X. Let $\alpha \in \mathcal{A}$, and O_α be open in Y_α. Since p_α is continuous, $p_\alpha^{-1}(O_\alpha)$ is open in Y. Hence, $f^{-1}(p_\alpha^{-1}(O_\alpha)) = (p_\alpha \circ f)^{-1}(O_\alpha)$ is θ_s-open in X. Thus, $p_\alpha \circ f$ is θ_s-continuous for every $\alpha \in \mathcal{A}$.

Conversely, suppose that each coordinate function $p_\alpha \circ f$ is θ_s-continuous. Let G_α be open in Y_α. Then $\langle G_\alpha \rangle$ is a subbasic open set in Y and $(p_\alpha \circ f)^{-1}(G_\alpha) = f^{-1}(p_\alpha^{-1}(G_\alpha)) = f^{-1}(\langle G_\alpha \rangle)$ is θ_s-open in X. Therefore, f is θ_s-continuous on X. □

Corollary 3.11. *Let X be a topological space, $Y = \Pi\{Y_\alpha : \alpha \in \mathcal{A}\}$ be a product space, and $f_\alpha : X \to Y_\alpha$ be a function for each $\alpha \in \mathcal{A}$. Let $f : X \to Y$ be the function defined by $f(x) = \langle f_\alpha(x) \rangle$. Then, f is θ_s-continuous on X if and only if each f_α is θ_s-continuous on X for each $\alpha \in \mathcal{A}$.*

4. θ_s-connected spaces and some versions of separation axioms

In this section, we define and characterize the concepts of θ_s-connected, θ_s-Hausdorff, θ_s-regular, and θ_s-normal spaces

Definition 4.1. A topological space X is said to be θ_s-connected if it is not the union of two nonempty disjoint θ_s-open sets. Otherwise, X is θ_s-disconnected. A subset B of X is θ_s-connected if it is θ_s-connected as a subspace of X.

Theorem 4.2. *Let X be a topological space. Then the following statements are equivalent:*

(i) *X is θ_s-connected;*
(ii) *the only subsets of X that are both θ_s-open and θ_s-closed are $\varnothing$ and X;*
(iii) *no θ_s-continuous function $f : X \to \mathcal{D}$ is surjective.*

Theorem 4.3 ([20]). *A topological space X is connected if and only if it is θ-connected.*

In view of Remark 2.2 and Theorem 4.3, we have the following result.

Theorem 4.4. *A topological space X is θ_s-connected if and only if it is θ-connected.*

Remark 4.5. The following diagram holds for a subset of a topological space.

$$\begin{array}{ccccccc} \omega_\theta\text{-connected} & \longleftarrow & & & \omega\text{-connected} & & \\ \downarrow & & & & \downarrow & & \\ \theta\text{-connected} & \longleftrightarrow & \theta_s\text{-connected} & \longleftrightarrow & \text{connected} & \longleftarrow & \text{semi-connected} \end{array}$$

Definition 4.6. A topological space X is said to be

(i) θ_s-*Hausdorff* if given any pair of distinct points p, q in X there exist disjoint θ_s-open sets U and V such that $p \in U$ and $q \in V$;

(ii) θ_s-*regular* if for each closed set F and each point $x \notin F$, there exist disjoint θ_s-open sets U and V such that $x \in U$ and $F \subseteq V$;

(iii) θ_s-*normal* if for every pair of disjoint closed sets E and F of X, there exist disjoint θ_s-open sets U and V such that $E \subseteq U$ and $F \subseteq V$.

In view of Remark 2.2, every θ_s-Hausdorff (resp., θ_s-regular, θ_s-normal) space is Hausdorff (resp., regular, normal).

Theorem 4.7. *Let X be a topological space. Then the following statements are equivalent:*

(i) *X is θ_s-Hausdorff;*

(ii) *let $x \in X$, for $y \neq x$, there exists a θ_s-open set U containing x such that $y \notin Cl_{\theta_s}(U)$;*

(iii) *for each $x \in X$, $C = \cap\{Cl_{\theta_s}(U) : U$ is θ_s-open containing $x\}=\{x\}$.*

Proof.

(i)⇒(ii): For every distinct points $x, y \in X$, there exist disjoint θ_s-open sets U and V such that $x \in U$ and $y \in V$. By Remark 2.8, $y \notin Cl_{\theta_s}(U)$.

(ii) ⇒ (iii): Note that $x \in C$. By assumption, for every $y \neq x$, there exists a θ_s-open set U containing x such that $y \notin Cl_{\theta_s}(U)$. Thus, $y \notin C$. Since y is arbitrary, $C = \{x\}$.

(iii)⇒(i): Let $x, y \in X$ such that $x \neq y$. By assumption, there exists a θ_s-open set U containing x such that $y \notin Cl_{\theta_s}(U)$. By Remark 2.8, there exists a θ_s-open set V containing y such that $U \cap V = \varnothing$. Hence, X is a θ_s-Hausdorff. □

Theorem 4.8. *Let X be a topological space. Then the following statements are equivalent:*

(i) *X is θ_s-regular;*

(ii) *for each $x \in X$ and an open set U containing x, there exists θ_s-open set V such that $x \in V \subseteq Cl_{\theta_s}(V) \subseteq U$;*

(iii) *for each $x \in X$ and closed set F with $x \notin F$, there exists a θ_s-open set V containing x such that $F \cap Cl_{\theta_s}(V) = \varnothing$.*

Proof.

(i)⇒(ii): Let $x \in X$ and U be an open set containing x. By assumption, there exist disjoint θ_s-open sets V and W such that $x \in V$ and $X \setminus U \subseteq W$. By Remark 2.8, $Cl_{\theta_s}(V) \subseteq Cl_{\theta_s}(X \setminus W) = X \setminus W$. Moreover, $Cl_{\theta_s}(V) \cap (X \setminus U) \subseteq Cl_{\theta_s}(V) \cap W = \varnothing$. Hence, $Cl_{\theta_s}(V) \subseteq U$. Therefore, $x \in V \subseteq Cl_{\theta_s}(V) \subseteq U$.

(ii)⇒(iii): Let $x \in X$ and F be a closed set in X with $x \notin F$. By assumption, there exists a θ_s-open set V containing x such that $V \subseteq Cl_{\theta_s}(V) \subseteq X \setminus F$. This means that $Cl_{\theta_s}(V) \cap F = \varnothing$.

(iii)⇒(i): Let $x \in X$ and F be a closed set with $x \notin F$. By assumption, there exists a θ_s-open set V containing x such that $Cl_{\theta_s}(V) \cap F = \varnothing$. Note that $X \setminus Cl_{\theta_s}(V)$ is a θ_s-open set and $F \subseteq X \setminus Cl_{\theta_s}(V)$. Furthermore, $V \cap X \setminus Cl_{\theta_s}(V) = \varnothing$. Hence, X is a θ_s-regular. □

Theorem 4.9. *Let X be a topological space. Then the following statements are equivalent:*

(i) *X is θ_s-normal;*

(ii) *for each closed set A and an open set $U \supseteq A$, there exists a θ_s-open set V containing A such that $Cl_{\theta_s}(V) \subseteq U$;*

(iii) *for each pair of disjoint closed sets A and B, there exists a θ_s-open set V containing A such that $Cl_{\theta_s}(V) \cap B = \varnothing$.*

Proof.

(i)⇒(ii): Let A be a closed set and U be an open set in X containing A. Then A and $X \setminus U$ are disjoint closed sets in X. By assumption, there exist disjoint θ_s-open sets V and W such that $A \subseteq V$ and $X \setminus U \subseteq W$. By Remark 2.8, $Cl_{\theta_s}(V) \subseteq Cl_{\theta_s}(X \setminus W) = X \setminus W$. Hence, $Cl_{\theta_s}(V) \subseteq X \setminus W \subseteq U$.

(ii)⇒(iii): Let A and B be a disjoint closed sets in X. Then $A \subseteq X \setminus B$ and $X \setminus B$ is open. By assumption, there exists a θ_s-open set V containing A such that $Cl_{\theta_s}(V) \subseteq X \setminus B$. This means that $Cl_{\theta_s}(V) \cap B = \varnothing$.

(iii)⇒(i): Let A and B be disjoint closed sets in X. By assumption, there exists a θ_s-open set V containing A such that $Cl_{\theta_s}(V) \cap B = \varnothing$. Then $B \subseteq X \setminus Cl_{\theta_s}(V)$. Since V and $X \setminus Cl_{\theta_s}(V)$ are disjoint θ_s-open sets, X is θ_s-normal. □

A topological space X is said to be a T_1-space if for each $p, q \in X$ with $p \neq q$, there exist an open sets U and V such that $p \in U$, $q \notin U$ and $q \in V$, $p \notin V$.

Theorem 4.10. *Let X be a T_1 space. Then*

(i) *if X is θ_s-regular, then X is θ_s-Hausdorff;*
(ii) *if X is θ_s-normal, then X is θ_s-regular.*

Proof.

(i): Suppose that X is θ_s-regular. Since X is a T_1-space, for each $x, y \in X$ with $x \neq y$, there exist disjoint open sets U and V such that $x \in U$ and $y \notin U$, and $y \in V$ and $x \notin V$. This implies that $x \notin X \setminus U$ and $y \notin X \setminus V$. Since X is θ_s-regular, there exist disjoint θ_s-open sets A and B such that $x \in A$ and $X \setminus U \subseteq B$. Note that $y \in X \setminus U$. Hence, $y \in B$. Thus, X is θ_s-Hausdorff.

We can prove (ii) by following the same argument used in (i). □

5. Conclusion and recommendations

The paper has introduced the concept of θ_s-open set and described its connection to the other well-known concepts such as the classical open, θ-open, and ω_θ-open sets. The paper has also defined and characterized the concepts of θ_s-open and θ_s-closed functions, θ_s-continuous function, and θ_s-connected, θ_s-Hausdorff, θ_s-regular, and θ_s-normal spaces. Moreover, the paper has formulated a necessary and sufficient condition for a θ_s-continuous function from an arbitrary space into the product space. A worthwhile direction for further investigation is to establish versions of Urysohn's Lemma and Tietze Extension Theorem with respect to θ_s-open sets. One may also try to investigate θ_s-open sets in a generalized topological space.

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