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New coupled and common coupled fixed point results with generalized c-distance on cone b-metric spaces

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Abstract

In this paper, we prove the existence and uniqueness of common coupled fixed point and coupled fixed point in cone b-metric spaces with generalized c-distance. Our results extend and generalize several well-known comparable results in literature. We provide one example to support our obtained results.

Keywords: Cone b-metric spaces, coupled fixed points, coupled coincidence points, common coupled fixed points, generalized c-distance.

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1. Introduction

In 2011, Hussain and Shah [26] introduced a cone b-metric space as a generalization of b-metric spaces and cone metric spaces of Bakhtin [3] (for more information about b-metric space see [32]) and Huang and Zhang [24], respectively. They provided and build up some topological properties which will be needed to upgrade and prove some results in literature to cone b-metric space. This work, opened a new area in analysis which stimulated many authors to generalized several well-known comparable results in literature under many type of contractive conditions to cone b-metric spaces (see [7, 13, 20, 23, 25, 33, 35–37] and the references therein).

On the other hand for a cone b-metric space in 2015, Bao et al. [4] introduced the concept of a generalized c-distance on a cone b-metric space which is a generalization of c-distance of Cho et al. [6] in cone metric see (for more details about c-distance in cone metric spaces and abstract metric spaces see [8–12, 14, 15, 17–19, 28, 31, 34, 38] and the references contained therein). He proved some fixed and common fixed point results in ordered cone b-metric spaces using this distance. Bao et al. [4] have done a beginning work on generalized c-distance then, many authors have been studied and proved some fixed point and common fixed points results in cone b-metric space under generalized c-distance see for example ([16, 21, 22, 30]).

Fadail and Ahmad [13] proved the following Coupled coincidence point and common coupled fixed point results in cone b-metric spaces for w-compatible mappings.

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Theorem 1.1. *Let (X, d) be a cone b-metric space with the coefficient $s \geqslant 1$ relative to a solid cone P. Let $F : X^2 \longrightarrow X$ and $g : X \longrightarrow X$ be two mappings and suppose that there exist nonnegative constants $a_i \in [0, 1), i = 1, 2, \dots, 10$ with $(s+1)(a_1 + a_2 + a_3 + a_4) + s(s+1)(a_5 + a_6 + a_7 + a_8) + 2s(a_9 + a_{10}) < 2$ and $\sum_{i=1}^{10} a_i < 1$ such that the following contractive condition hold for all $x, y, u, v \in X$:*

$$\begin{aligned} d(F(x,y), F(u,v)) \preceq & \left[a_1 d(gx, F(x,y)) + a_2 d(gy, F(y,x))\right] + \left[a_3 d(gu, F(u,v)) + a_4 d(gv, F(v,u))\right] \\ & + \left[a_5 d(gx, F(u,v)) + a_6 d(gy, F(v,u))\right] + \left[a_7 d(gu, F(x,y)) + a_8 d(gv, F(y,x))\right] \\ & + \left[a_9 d(gx, gu) + a_{10} d(gy, gv)\right]. \end{aligned}$$

If $F(X^2) \subseteq g(X)$ and $g(X)$ is a complete subspace of X, then F and g have a coupled coincidence point $(x^*, y^*) \in X^2$.

Theorem 1.2. *In addition to the hypotheses of Theorem 1.1, if F and g are w-compatible, then F and g have a unique common coupled fixed point. Moreover, a common coupled fixed point of F and g is of the form (u, u) for some $u \in X$.*

In this paper, we extend the results of Fadail and Ahmad [13] and prove it on generalized c-distance in cone b-metric spaces for w-compatible mappings with out condition of normality for cones and continuity for mappings, but the only assumption is that the cone P is solid, that is $\text{int}(P) \neq \emptyset$.

2. Preliminaries

Let E be a real Banach space and θ denote to the zero element in E. A cone P is called normal if there exists a number K such that:

$$\theta \preceq x \preceq y \quad \text{implies} \quad \|x\| \leqslant K\|y\| \tag{2.1}$$

for all $x, y \in E$. Equivalently, the cone P is normal if for all n:

$$x_n \preceq y_n \preceq z_n \text{ and } \lim_{n\to+\infty} x_n = \lim_{n\to+\infty} z_n = x \quad \text{imply} \quad \lim_{n\to+\infty} y_n = x. \tag{2.2}$$

The least positive number K satisfying condition (2.1) is called the normal constant of P.

Example 2.1 ([2]). Let $E = C^1_{\mathbb{R}}[0, 1]$ with $\|x\| = \|x\|_\infty + \|x'\|_\infty$ and $P = \{x \in E : x(t) \geqslant 0\}$. This cone is nonnormal. Consider, for example, $x_n(t) = \frac{t^n}{n}$ and $y_n(t) = \frac{1}{n}$. Then $\theta \preceq x_n \preceq y_n$, and $\lim_{n\to\infty} y_n = \theta$, but $\|x_n\| = \max_{t\in[0,1]} |\frac{t^n}{n}| + \max_{t\in[0,1]} |t^{n-1}| = \frac{1}{n} + 1 > 1$; hence x_n does not converge to zero. It follows by condition (2.2) that P is a nonnormal cone.

Definition 2.2 ([26]). Let X be a nonempty set and E be a real Banach space equipped with the partial ordering $\preceq$ with respect to the cone P. A vector-valued function $d : X \times X \longrightarrow E$ is said to be a cone b-metric function on X with the constant $s \geqslant 1$ if the following conditions are satisfied:

1. $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$ for all $x, y \in X$;
3. $d(x, y) \preceq s(d(x, y) + d(y, z))$ for all $x, y, z \in X$.

Then pair (X, d) is called a cone b-metric space (or a cone metric type space), we will use the first mentioned term.

Definition 2.3 ([26]). Let (X, d) be a cone b-metric space, $\{x_n\}$ be a sequence in X, and $x \in X$.

1. For all $c \in E$ with $\theta \ll c$, if there exists a positive integer N such that $d(x_n, x) \ll c$ for all $n > N$, then x_n is said to be convergent and x is the limit of $\{x_n\}$. We denote this by $x_n \longrightarrow x$.
2. For all $c \in E$ with $\theta \ll c$, if there exists a positive integer N such that $d(x_n, x_m) \ll c$ for all $n, m > N$, then $\{x_n\}$ is called a Cauchy sequence in X.
3. A cone b-metric space (X, d) is called complete if every Cauchy sequence in X is convergent.

Lemma 2.4 ([27]).

1. *If* E *be a real Banach space with a cone* P *and* $a \preceq \lambda a$*, where* $a \in P$ *and* $0 \leqslant \lambda < 1$*, then* $a = \theta$.
2. *If* $c \in intP$, $\theta \preceq a_n$ *and* $a_n \longrightarrow \theta$*, then there exists a positive integer* N *such that* $a_n \ll c$ *for all* $n \geqslant N$.
3. *If* $a \preceq b$ *and* $b \ll c$*, then* $a \ll c$.
4. *If* $\theta \preceq u \ll c$ *for each* $\theta \ll c$*, then* $u = \theta$.

Recall the following definitions.

Definition 2.5 ([5]). An element $(x, y) \in X^2$ is said to be a coupled fixed point of the mapping $F : X^2 \longrightarrow X$ if $F(x, y) = x$ and $F(y, x) = y$.

Definition 2.6 ([29]). An element $(x, y) \in X^2$ is called

1. a coupled coincidence point of mappings $F : X^2 \longrightarrow X$ and $g : X \longrightarrow X$ if $gx = F(x, y)$ and $gy = F(y, x)$, and (gx, gy) is called coupled point of coincidence;
2. a common coupled fixed point of mappings $F : X^2 \longrightarrow X$ and $g : X \longrightarrow X$ if $x = gx = F(x, y)$ and $y = gy = F(y, x)$.

Definition 2.7 ([1]). The mappings $F : X^2 \longrightarrow X$ and $g : X \longrightarrow X$ are called *w*-compatible if $g(F(x, y)) = F(gx, gy)$ whenever $gx = F(x, y)$ and $gy = F(y, x)$.

Definition 2.8 ([4]). Let (X, d) be a cone b-metric space with the coefficient $s \geqslant 1$ relative to a solid cone P. A function $q : X \times X \longrightarrow E$ is called a generalized c-distance on X if the following conditions hold:

(q1) $\theta \preceq q(x, y)$ for all $x, y \in X$;
(q2) $q(x, z) \preceq s(q(x, y) + q(y, z))$ for all $x, y, z \in X$;
(q3) for each $x \in X$ and $n \geqslant 1$, if $q(x, y_n) \preceq u$ for some $u = u_x \in P$, then $q(x, y) \preceq su$ whenever $\{y_n\}$ is a sequence in X converging to a point $y \in X$;
(q4) for all $c \in E$ with $\theta \ll c$, there exists $e \in E$ with $\theta \ll e$ such that $q(z, x) \ll e$ and $q(z, y) \ll e$ imply $d(x, y) \ll c$.

Example 2.9. Let $X = [0, 1]$ and $E = C^1_{\mathbb{R}}[0, 1]$ with $\|u\| = \|u\|_\infty + \|u'\|_\infty, u \in E$ and let $P = \{u \in E : u(t) \geqslant 0$ on $[0, 1]\}$. It is well known that this cone is solid but it is not normal (see Example 2.1). Define a cone b-metric $d : X \times X \longrightarrow E$ by $d(x, y)(t) = |x - y|^2 e^t$. Then (X, d) is a complete cone b-metric space with the coefficient $s = 2$. Define a mapping $q : X \times X \longrightarrow E$ by $q(x, y)(t) := y^2 \cdot e^t$ for all $x, y \in X$. Then q is a generalized c-distance on X. In fact, (q1), (q2), and (q3) are immediate. Let $c \in E$ with $\theta \ll c$ be given and put $e = \frac{c}{4}$. Suppose that $q(z, x) \ll e$ and $q(z, y) \ll e$, then we have

$$d(x, y)(t) = |x - y|^2 e^t \preceq 2x^2 e^t + 2y^2 e^t = 2q(z, x)(t) + 2q(z, y) \ll 2\frac{c}{4} + 2\frac{c}{4} = c.$$

This shows that q satisfies (q4) and hence q is a generalized c-distance.

Lemma 2.10. Let (X, d) be a cone b-metric space with the coefficient $s \geqslant 1$ relative to a solid cone P and q is a generalized c-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X and $x, y, z \in X$. Suppose that u_n is a sequence in P converging to θ. Then the following hold.

1. If $q(x_n, y) \preceq u_n$ and $q(x_n, z) \preceq u_n$, then $y = z$.
2. If $q(x_n, y_n) \preceq u_n$ and $q(x_n, z) \preceq u_n$, then $\{y_n\}$ converges to z.
3. If $q(x_n, x_m) \preceq u_n$ for $m > n$, then $\{x_n\}$ is a Cauchy sequence in X.
4. If $q(y, x_n) \preceq u_n$, then $\{x_n\}$ is a Cauchy sequence in X.

Remark 2.11.

1. $q(x, y) = q(y, x)$ does not necessarily for all $x, y \in X$.
2. $q(x, y) = \theta$ is not necessarily equivalent to $x = y$ for all $x, y \in X$.

3. Common coupled fixed point results

In this section, we prove some common coupled fixed point results in cone b-metric spaces with generalized c-distance.

Theorem 3.1. *Let* (X, d) *be a cone b-metric space with the coefficient* $s \geqslant 1$ *relative to a solid cone* P *and* q *is a generalized* c*-distance on* X*. Let* $F : X^2 \longrightarrow X$ *and* $g : X \longrightarrow X$ *be two mappings and suppose that there exist nonnegative constants* $a_i \in [0, 1), i = 1, 2, \ldots, 10$ *with* $s(a_1+a_2+a_7+a_8)+s(s+1)(a_5+a_6)+2s(a_3+a_4) < 1$ *and* $\sum_{i=1}^{8} a_i < 1$ *such that the following contractive condition hold for all* $x, y, u, v \in X$*:*

$$q(F(x,y),F(u,v)) \preceq \big[a_1q(gx,F(x,y)) + a_2q(gy,F(y,x))\big] + \big[a_3q(gu,F(u,v)) + a_4q(gv,F(v,u))\big] + \big[a_5q(gx,F(u,v)) + a_6q(gy,F(v,u))\big] + \big[a_7q(gx,gu) + a_8q(gy,gv)\big].$$

If $F(X^2) \subseteq g(X)$ *and* $g(X)$ *is a complete subspace of* X*, then* F *and* g *have a coupled coincidence point* $(x^*, y^*) \in X^2$*. Further, if* $u_1 = gx_1 = F(x_1, y_1)$ *and* $v_1 = gy_1 = F(y_1, x_1)$ *then* $q(u_1, u_1) = \theta$ *and* $q(v_1, v_1) = \theta$*. In addition, if* F *and* g *are w-compatible, then* F *and* g *have a unique common coupled fixed point. Moreover, a common coupled fixed point of* F *and* g *is of the form* (u, u) *for some* $u \in X$*.*

Proof. Choose $x_0, y_0 \in X$. Set $gx_1 = F(x_0, y_0), gy_1 = F(y_0, x_0)$, this can be done because $F(X^2) \subseteq g(X)$. Continuing this process we obtain two sequences $\{x_n\},\{y_n\}$ such that $gx_{n+1} = F(x_n, y_n), gy_{n+1} = F(y_n, x_n)$. Then we have

$$\begin{aligned} q(gx_n, gx_{n+1}) &= q(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \\ &\preceq \big[a_1q(gx_{n-1}, F(x_{n-1}, y_{n-1})) + a_2q(gy_{n-1}, F(y_{n-1}, x_{n-1}))\big] \\ &\quad + \big[a_3q(gx_n, F(x_n, y_n)) + a_4q(gy_n, F(y_n, x_n))\big] \\ &\quad + \big[a_5q(gx_{n-1}, F(x_n, y_n)) + a_6q(gy_{n-1}, F(y_n, x_n))\big] \\ &\quad + \big[a_7q(gx_{n-1}, gx_n) + a_8q(gy_{n-1}, gy_n)\big]. \end{aligned}$$

So that,

$$\begin{aligned} q(gx_n, gx_{n+1}) &= q(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \\ &\preceq \big[a_1q(gx_{n-1}, gx_n) + a_2q(gy_{n-1}, gy_n)\big] + \big[a_3q(gx_n, gx_{n+1}) + a_4q(gy_n, gy_{n+1})\big] \\ &\quad + \big[a_5q(gx_{n-1}, gx_{n+1}) + a_6q(gy_{n-1}, gy_{n+1})\big] + \big[a_7q(gx_{n-1}, gx_n) + a_8q(gy_{n-1}, gy_n)\big]. \end{aligned}$$

Then, we have

$$\begin{aligned} q(gx_n, gx_{n+1}) &= q(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \\ &\preceq \big[a_1q(gx_{n-1}, gx_n) + a_2q(gy_{n-1}, gy_n)\big] + \big[a_3q(gx_n, gx_{n+1}) + a_4q(gy_n, gy_{n+1})\big] \\ &\quad + \big[sa5(q(gx_{n-1}, gx_n) + q(gx_n, gx_{n+1})) + sa_6(q(gy_{n-1}, gy_n) + q(gy_n, gy_{n+1}))\big] \\ &\quad + \big[a_7q(gx_{n-1}, gx_n) + a_8q(gy_{n-1}, gy_n)\big]. \end{aligned}$$

Hence

$$\begin{aligned} q(gx_n, gx_{n+1}) \preceq &\big[(a_1 + sa_5 + a_7)q(gx_{n-1}, gx_n) + (a_2 + sa_6 + a_8)q(gy_{n-1}, gy_n)\big] \\ &+ \big[(a_3 + sa_5)q(gx_n, gx_{n+1}) + (a_4 + sa_6)q(gy_n, gy_{n+1})\big]. \end{aligned} \tag{3.1}$$

Similarly, we can prove that

$$\begin{aligned} q(gy_n, gy_{n+1}) \preceq &\big[(a_1 + sa_5 + a_7)q(gy_{n-1}, gy_n) + (a_2 + sa_6 + a_8)q(gx_{n-1}, gx_n)\big] \\ &+ \big[(a_3 + sa_5)q(gy_n, gy_{n+1}) + (a_4 + sa_6)q(gx_n, gx_{n+1})\big]. \end{aligned} \tag{3.2}$$

Put $q_n = q(gx_n, gx_{n+1}) + q(gy_n, gy_{n+1})$. Adding inequalities (3.1) and (3.2), one can assert that

$$q_n \preceq (a_1 + a_2 + s(a_5 + a_6) + a_7 + a_8)q_{n-1} + (a_3 + a_4 + s(a_5 + a_6))q_n.$$

Consequently, we have

$$q_n \preceq \frac{(a_1 + a_2 + s(a_5 + a_6) + a_7 + a_8)}{1 - (a_3 + a_4 + s(a_5 + a_6))} q_{n-1} = hq_{n-1} \preceq h^2 q_{n-2} \preceq h^3 q_{n-3} \preceq \cdots \preceq h^n q_0, \tag{3.3}$$

where $h = \frac{(a_1+a_2+s(a_5+a_6)+a_7+a_8)}{1-(a_3+a_4+s(a_5+a_6))}$. Note that, $s(a_1 + a_2 + a_7 + a_8) + s(s + 1)(a_5 + a_6) + 2s(a_3 + a_4) < 1$ means that $h = \frac{(a_1+a_2+s(a_5+a_6)+a_7+a_8)}{1-(a_3+a_4+s(a_5+a_6))} < \frac{1}{s}$ and $sh < 1$. Let $m > n \geqslant 1$. It follows that

$$q(gx_n, gx_m) \preceq sq(gx_n, gx_{n+1}) + s^2 q(gx_{n+1}, gx_{n+2}) + \cdots + s^{m-n} q(gx_{m-1}, gx_m),$$

and

$$q(gy_n, gy_m) \preceq sq(gy_n, gy_{n+1}) + s^2 q(gy_{n+1}, gx_{n+2}) + \cdots + s^{m-n} q(gy_{m-1}, gy_m).$$

Now, (3.3) and $sh < 1$ imply that

$$\begin{aligned} q(gx_n, gx_m) + q(gy_n, gy_m) &\preceq sq_n + s^2 q_{n+1} + \cdots + s^{m-n} q_{m-1} \\ &\preceq sh^n q_0 + s^2 h^{n+1} q_0 + \cdots + s^{m-n} h^{m-1} q_0 \\ &= (sh^n + s^2 h^{n+1} + \cdots + s^{m-n} h^{m-1}) q_0 \\ &= sh^n (1 + sh + (sh)^2 + \cdots + (sh)^{m-n-1}) q_0 \\ &\preceq \frac{sh^n}{1-h} q_0. \end{aligned} \tag{3.4}$$

From (3.4) we have

$$q(gx_n, gx_m) \preceq \frac{sh^n}{1-h} q_0 \longrightarrow \theta \quad \text{as} \quad (n \longrightarrow +\infty),$$

and

$$q(gy_n, gy_m) \preceq \frac{sh^n}{1-h} q_0 \longrightarrow \theta \quad \text{as} \quad (n \longrightarrow +\infty).$$

Thus, Lemma 2.10 (3) shows that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in $g(X)$. Since $g(X)$ is complete, there exist x^* and $y^* \in X$ such that $gx_n \longrightarrow gx^*$ and $gy_n \longrightarrow gy^*$ as $n \longrightarrow +\infty$. By (q3) we have:

$$q(gx_n, gx^*) \preceq \frac{s^2 h^n}{1-h} q_0, \tag{3.5}$$

and

$$q(gy_n, gy^*) \preceq \frac{s^2 h^n}{1-h} q_0. \tag{3.6}$$

On the other hand, from (3.3) we have:

$$\begin{aligned} q(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) &= q(gx_n, gx_{n+1}) \\ &\preceq q(gx_n, gx_{n+1}) + q(gy_n, gy_{n+1}) \preceq h(q(gx_{n-1}, gx_n) + q(gy_{n-1}, gy_n)). \end{aligned}$$

Hence

$$q(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \preceq h(q(gx_{n-1}, gx_n) + q(gy_n, gy_{n-1})).$$

Then we have

$$q(F(x_{n-1}, y_{n-1}), F(x^*, y^*)) \preceq h(q(gx_{n-1}, gx^*) + q(gy_{n-1}, gy^*)).$$

By using (3.5) and (3.6), we get

$$\begin{aligned} q(gx_n, F(x^*,y^*)) &= q(F(x_{n-1},y_{n-1}), F(x^*,y^*)) \\ &\preceq h(q(gx_{n-1},gx^*) + q(gy_{n-1},gy^*)) \\ &\preceq h(\frac{s^2h^{n-1}}{1-h}q_0 + \frac{s^2h^{n-1}}{1-h}q_0) = \frac{2s^2h^n}{1-h}q_0. \end{aligned} \tag{3.7}$$

Also, from (3.5), we have

$$q(gx_n, gx^*) \preceq \frac{s^2h^n}{1-h}q_0 \preceq \frac{2s^2h^n}{1-h}q_0. \tag{3.8}$$

By Lemma 2.10 (1), (3.7), and (3.8), we have $gx^* = F(x^*,y^*)$. By similar way, we can prove that $gy^* = F(y^*,x^*)$. Therefore (x^*,y^*) is a coupled coincidence point of F and g. Suppose that $u_1 = gx_1 = F(x_1,y_1)$ and $v_1 = gy_1 = F(y_1,x_1)$. Then we have

$$\begin{aligned} q(u_1,u_1) &= q(gx_1,gx_1) \\ &= q(F(x_1,y_1),F(x_1,y_1)) \\ &\preceq \left[a_1q(gx_1,F(x_1,y_1)) + a_2q(gy_1,F(y_1,x_1))\right] + \left[a_3q(gx_1,F(x_1,y_1)) + a_4q(gy_1,F(y_1,x_1))\right] \\ &\quad + \left[a_5q(gx_1,F(x_1,y_1)) + a_6q(gy_1,F(y_1,x_1))\right] + \left[a_7q(gx_1,gx_1) + a_8q(gy_1,gy_1)\right] \\ &= \left[a_1q(gx_1,gx_1) + a_2q(gy_1,gy_1)\right] + \left[a_3q(gx_1,gx_1) + a_4q(gy_1,gy_1)\right] \\ &\quad + \left[a_5q(gx_1,gx_1) + a_6q(gy_1,gy_1)\right] + \left[a_7q(gx_1,gx_1) + a_8q(gy_1,gy_1)\right] \\ &= \left[a_1q(u_1,u_1) + a_2q(v_1,v_1)\right] + \left[a_3q(u_1,u_1) + a_4q(v_1,v_1)\right] \\ &\quad + \left[a_5q(u_1,u_1) + a_6q(v_1,v_1)\right] + \left[a_7q(u_1,u_1) + a_8q(v_1,v_1)\right]. \end{aligned}$$

Hence,

$$q(u_1,u_1) \preceq (a_1+a_3+a_5+a_7)q(u_1,u_1) + (a_2+a_4+a_6+a_8)q(v_1,v_1). \tag{3.9}$$

By similar way we can show that

$$q(v_1,v_1) \preceq (a_1+a_3+a_5+a_7)q(v_1,v_1) + (a_2+a_4+a_6+a_8)q(u_1,u_1). \tag{3.10}$$

By adding inequalities (3.9) and (3.10), we get

$$q(u_1,u_1) + q(v_1,v_1) \preceq \left(\sum_{i=1}^{8} a_i\right)(q(u_1,u_1) + q(v_1,v_1)).$$

Since $\sum_{i=1}^{8} a_i < 1$, Lemma 2.4 (1) shows that $q(u_1,u_1) + q(v_1,v_1) = \theta$. But $q(u_1,u_1) \succeq \theta$, and $q(v_1,v_1) \succeq \theta$. Hence, $q(u_1,u_1) = \theta$ and $q(v_1,v_1) = \theta$. Finally, since F and g have a coupled coincidence point $(x^*,y^*) \in X^2$, then, (gx^*,gy^*) is a coupled point of coincidence of F and g such that $gx^* = F(x^*,y^*)$ and $gy^* = F(y^*,x^*)$ with $q(gx^*,gx^*) = \theta$, and $q(gy^*,gy^*) = \theta$. First, we will show that the coupled point of coincidence is unique. Suppose that F and g have another coupled point of coincidence (gx',gy') such that $gx' = F(x',y')$, and $gy' = F(y',x')$, where $x',y' \in X$. Then we have

$$\begin{aligned} q(gx^*,gx') &= q(F(x^*,y^*),F(x',y')) \\ &\preceq \left[a_1q(x^*,F(x^*,y^*)) + a_2q(gy^*,F(y^*,x^*))\right] + \left[a_3q(gx',F(x',y')) + a_4q(gy',F(y',x'))\right] \\ &\quad + \left[a_5q(gx^*,F(x',y')) + a_6q(gy^*,F(y',x'))\right] + \left[a_7q(gx^*,gx') + a_8q(gy^*,gy')\right] \\ &= \left[a_1q(gx^*,gx^*) + a_2q(gy^*,gy^*)\right] + \left[a_3q(gx',gx') + a_4q(gy',gy')\right] \\ &\quad + \left[a_5q(gx^*,gx') + a_6q(gy^*,gy')\right] + \left[a_7q(gx^*,gx') + a_8q(gy^*,gy')\right] \\ &= \left[a_5q(gx^*,gx') + a_6qg(y^*,gy')\right] + \left[a_7q(gx^*,gx') + a_8q(gy^*,gy')\right]. \end{aligned}$$

Hence,
$$q(gx^*, gx') \preceq (a_5 + a_7)q(gx^*, gx') + (a_6 + a_8)q(gy^*, gy'). \tag{3.11}$$
By similar way, we can show that
$$q(gy^*, gy') \preceq (a_5 + a_7)q(gy^*, gy') + (a_6 + a_8)q(gx^*, gx'). \tag{3.12}$$
By adding inequalities (3.11) and (3.12), we get
$$q(gx^*, gx') + q(gy^*, gy') \preceq (a_5 + a_6 + a_7 + a_8)(q(gx^*, gx') + q(gy^*, gy')).$$
Since $(a_5 + a_6 + a_7 + a_8) < 1$, Lemma 2.4 (1) shows that $q(gx^*, gx') + q(gy^*, gy') = \theta$. But $q(gx^*, gx') \succeq \theta$ and $q(gy^*, gy') \succeq \theta$. Hence, $q(gx^*, gx') = \theta$ and $q(gy^*, gy') = \theta$. Also, we have from Theorem 3.1, $q(gx^*, gx^*) = \theta$ and $q(gy^*, gy^*) = \theta$. Hence, Lemma 2.10 (1) shows that
$$gx^* = gx' \quad \text{and} \quad gy^* = gy', \tag{3.13}$$
which implies the uniqueness of the coupled point of coincidence of F and g, that is, (gx^*, gy^*). Note that
$$\begin{aligned} q(gx^*, gy') &= q(F(x^*, y^*), F(y', x')) \\ &\preceq \left[a_1 q(x^*, F(x^*, y^*)) + a_2 q(gy^*, F(y^*, x^*))\right] + \left[a_3 q(gy', F(y', x')) + a_4 q(gx', F(x', y'))\right] \\ &\quad + \left[a_5 q(gx^*, F(y', x')) + a_6 q(gy^*, F(x', y'))\right] + \left[a_7 q(gx^*, gy') + a_8 q(gy^*, gx')\right] \\ &= \left[a_1 q(gx^*, gx^*) + a_2 q(gy^*, gy^*)\right] + \left[a_3 q(gy', gy') + a_4 q(gx', gx')\right] \\ &\quad + \left[a_5 q(gx^*, gy') + a_6 q(gy^*, gx')\right] + \left[a_7 q(gx^*, gy') + a_8 q(gy^*, gx')\right] \\ &= \left[a_5 q(gx^*, gy') + a_6 q(gy^*, gx')\right] + \left[a_7 q(gx^*, gy') + a_8 q(gy^*, gx')\right]. \end{aligned}$$
Hence,
$$q(gx^*, gy') \preceq (a_5 + a_7)q(gx^*, gy') + (a_6 + a_8)q(gy^*, gx'). \tag{3.14}$$
By similar way, we can show that
$$q(gy^*, gx') \preceq (a_5 + a_7)q(gy^*, gx') + (a_6 + a_8)q(gx^*, gy'). \tag{3.15}$$
By adding inequalities (3.14) and (3.15), we get
$$q(gx^*, gy') + q(gy^*, gx') \preceq (a_5 + a_6 + a_7 + a_8)(q(gx^*, gy') + q(gy^*, gx')).$$
Since $(a_5 + a_6 + a_7 + a_8) < 1$, Lemma 2.4 (1) shows that $q(gx^*, gy') + q(gy^*, gx') = \theta$. But $q(gx^*, gy') \succeq \theta$ and $q(gy^*, gx') \succeq \theta$. Hence, $q(gx^*, gy') = \theta$ and $q(gy^*, gx') = \theta$. Also, we have $q(gx^*, gx^*) = \theta$ and $q(gy^*, gy^*) = \theta$. Hence, Lemma 2.10 (1) shows that
$$gx^* = gy' \quad \text{and} \quad gy^* = gx'. \tag{3.16}$$
In view of (3.13) and (3.16), one can assert that
$$gx^* = gy^*.$$
That is, the unique coupled point of coincidence of F and g is (gx^*, gx^*). Now, let $u = gx^* = F(x^*, y^*)$. Since F and g are w-compatible, then we have
$$gu = g(gx^*) = gF(x^*, y^*) = F(gx^*, gy^*) = F(gx^*, gx^*) = F(u, u).$$
Then (gu, gu) is a coupled point of coincidence and also we have (u, u) is a coupled point of coincidence. The uniqueness of the coupled point of coincidence implies that $gu = u$. Therefore $u = gu = F(u, u)$. Hence (u, u) is the unique common coupled fixed point of F and g. This completes the proof. □

Now, we give one example to explain our results. The conditions of Theorem 3.1 is fulfilled, but Theorems 1.1 and 1.2 of Fadail and Ahmad [13] are not applicable.

Example 3.2 (The case of a nonnormal cone)**.** Consider Example 2.9. Define the mappings $F : X \times X \longrightarrow X$ by $F(x, y) = \frac{(x+y)^2}{16}$ and $g : X \longrightarrow X$ by $gx = \frac{x}{2}$ for all $x \in X$. Clear that $F(X^2) \subseteq g(X)$ and $g(X)$ is a complete subset of X. We have

$$\begin{aligned} d(F(x,y),F(u,v))(t) &= \left|\frac{(x+y)^2}{8} - \frac{(u+v)^2}{8}\right|^2 e^t \\ &= \frac{1}{16^2}\left|(x+y-u-v)(x+y+u+v)\right|^2 e^t \\ &= \frac{1}{16^2}\left|((x-u)+(y-v))(x+y+u+v)\right|^2 e^t \\ &\preceq \frac{4^2}{16^2}\left|(x-u)+(y-v)\right|^2 e^t \\ &\preceq \frac{32}{16^2}|x-u|^2 e^t + \frac{3}{16^2}|y-v|^2 e^t \\ &= \frac{32}{16^2}\left(4\left|\frac{x}{2}-\frac{u}{2}\right|^2\right) e^t + \frac{32}{16^2}\left(4\left|\frac{y}{2}-\frac{v}{2}\right|^2\right) e^t \\ &= \frac{1}{2}\left|\frac{x}{2}-\frac{u}{2}\right|^2 e^t + \frac{1}{2}\left|\frac{y}{2}-\frac{v}{2}\right|^2 e^t \\ &= \frac{1}{2}d(gx,gu)(t) + \frac{1}{2}d(gy,gv)(t), \end{aligned}$$

where $a_9 = \frac{1}{2}, a_{10} = \frac{1}{2}, a_i = 0, i = 1, 2, \ldots, 8$. Note that, $2s(a_9 + a_{10}) = 4(\frac{1}{2} + \frac{1}{2}) = 4 \nless 2$. Then, we can not use Theorems 1.1 and 1.2 of Fadail and Ahmad [13] for this example on a cone b-metric space. To check this example on generalized c-distance, we have:

$$\begin{aligned} q(F(x,y),F(u,v))(t) &= (F(u,v))^2 \cdot e^t \\ &= (\frac{(u+v)^2}{16})^2 \cdot e^t \\ &= \frac{1}{16^2}(u+v)^4 \cdot e^t \\ &\preceq \frac{4}{16^2}(u+v)^2 \cdot e^t \\ &\preceq \frac{8}{16^2}u^2 \cdot e^t + \frac{8}{16^2}v^2 \cdot e^t \\ &= \frac{32}{16^2}\frac{u^2}{4} \cdot e^t + \frac{32}{16^2}\frac{v^2}{4} \cdot e^t \\ &= \frac{1}{8}\frac{u^2}{4} \cdot e^t + \frac{1}{8}\frac{v^2}{4} \cdot e^t \\ &= \frac{1}{8}q(gx,gu)(t) + \frac{1}{8}q(gy,gv)(t), \end{aligned}$$

where $a_7 = \frac{1}{8}, a_8 = \frac{1}{8}, a_i = 0, i = 1, 2, \ldots, 6$. Note that, $s(a_7 + a_8) = 2(\frac{1}{8} + \frac{1}{8}) = \frac{1}{2} < 2$. Hence, the conditions of Theorem 3.1 are satisfied, that is, F and g have a coupled coincidence point $(0, 0)$. Also, F and g are w-compatible at $(0, 0)$. Again, Theorem 3.1 shows that, $(0, 0)$ is the unique common coupled fixed point of F and g.

Finally, we have the following coupled fixed point theorem.

Theorem 3.3. *Let* (X, d) *be a cone b-metric space with the coefficient* $s \geqslant 1$ *relative to a solid cone* P *and* q *is a generalized* c*-distance on* X. *Let* $F : X^2 \longrightarrow X$ *be a mapping and suppose that there exist nonnegative constants* $a_i \in [0, 1), i = 1, 2, \dots, 10$ *with* $s(a_1 + a_2 + a_7 + a_8) + s(s + 1)(a_5 + a_6) + 2s(a_3 + a_4) < 1$ *and* $\sum_{i=1}^{8} a_i < 1$ *such that the following contractive condition holds for all* $x, y, u, v \in X$:

$$q(F(x,y),F(u,v)) \preceq \left[a_1 q(x,F(x,y)) + a_2 q(y,F(y,x))\right] + \left[a_3 q(u,F(u,v)) + a_4 q(v,F(v,u))\right] + \left[a_5 q(x,F(u,v)) + a_6 q(y,F(v,u))\right] + \left[a_7 q(x,u) + a_8 q(y,v)\right].$$

Then F *has a coupled fixed point* $(x^*, y^*) \in X^2$. *Further, if* $x_1 = F(x_1, y_1)$ *and* $y_1 = F(y_1, x_1)$, *then* $q(x_1, x_1) = \theta$, *and* $q(y_1, y_1) = \theta$. *Moreover, the coupled fixed point is unique and of the form* (x^*, x^*) *for some* $x^* \in X$.

Proof. Put $g(x) = x$ in Theorem 3.1. The proof is complete. □

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