J. Nonlinear Sci. Appl., 15 (2022), 61–66

ISSN: 2008-1898

Journal of Nonlinear Sciences and Applications

Journal Homepage: www.isr-publications.com/jnsa

Local and global existence of a nonlocal equation with a singular integral drift term

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Abstract

We study an initial value problem with fractional Laplacian and a singular integral drift term. This equation quantifies fractal interfaces in statistical mechanics. The singularity of the drift term is a generalization of existing results. Making use of some important boundedness properties of Calderón-Zygmund operator in L_p and Lipschitz spaces, we obtain local and global existence theorems.

Keywords: Singular integration, nonlocal equations.

2020 MSC: 35Q82, 35C15.

1. Introduction

Consider the following initial value problem,

$$\begin{cases} \partial_t u(t,x) = -(-\Delta)^{\alpha/2} u - \nabla \cdot (uB(u)), \\ u(0,x) = u_0(x), \end{cases} \tag{1.1}$$

where $u : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ for positive integer d and $\alpha \in (1,2)$. The operator $-(-\Delta)^{\alpha/2}$ is the fractional power of the Laplacian Δ, analytically, it can be defined as,

$$-(-\Delta)^{\alpha/2} v(x) = \mathcal{F}^{-1}(|\xi|^\alpha \mathcal{F}(v)(\xi))(x),$$

for any Schwartz function $v \in \mathcal{S}$, with $\mathcal{F}$ denoting the Fourier operator. Probabilistically, it can be also viewed as a Markov jump process operator, thus, has the following equivalent form,

$$-(-\Delta)^{\alpha/2} v(x) = K \int_{\mathbb{R}^d} (v(x+y) - v(x) - \nabla v(x) \cdot y \mathbf{1}_{|y| \leqslant 1}) \frac{dy}{|y|^{d+\alpha}},$$

where $K = K_{\alpha,d}$ is a constant. $B(u)$ is a singular integral operator defined by,

$$B(u)(x) = \int_{\mathbb{R}^d} b(x,y) u(y) dy,$$

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doi: 10.22436/jnsa.015.01.05

Received: 2021-04-03 Revised: 2021-04-16 Accepted: 2021-04-24

with a Calderón-Zygmund singular integral kernel $b(x,y)$. Recall the conditions that a Calderón-Zygmund kernel has to satisfy: there are constants C and $\delta > 0$, such that for any $x, y \in \mathbb{R}^d$,

$$
\begin{aligned}
|b(x,y)| &\leqslant \frac{C}{|x-y|^d}, \\
|b(x,y)-b(x',y)| &\leqslant \frac{C|x-x'|}{(|x-y|+|x'-y|)^{\delta}}, \quad \text{whenever } |x-x'| \leqslant \frac{1}{2}\max(|x-y|,|x'-y|), \\
|b(x,y')-b(x,y)| &\leqslant \frac{C|y-y'|}{(|x-y|+|x'-y|)^{\delta}}, \quad \text{whenever } |y-y'| \leqslant \frac{1}{2}\max(|x-y|,|x'-y|).
\end{aligned}
$$

For more detailed analysis on integration with respect to Calderón-Zygmund kernel, see, e.g., [8–11]. We also denote the singular integral operator as $\mathcal{K}_b$, hence, write $B(u)(x) = \mathcal{K}_b u(x)$.

Equation (1.1), in various forms, has been studied in both mathematical and physics literature. In [6], one form of this equation characterizes fractal interfaces in statistical mechanics in the presence of self-similar hopping surface diffusion, and generalizes the classical Kardar-Parisi-Zhang (KPZ) model. Regularity and conservation laws for (1.1) with different drift $B(u)$, are considered in [1, 2, 5]. When $B(u)$ is a more regular operator, where $b(x,y)$ is a convolutional kernel satisfy necessary bound on value and derivative such that the integral operator is (p,∞) ((p,q) refers to bounded operator from L_p to L_q) for some $p > d/\beta$, the local and global existences of (1.1) are obtained in [5]. In this paper, we deal with the case where $B(u)$ is represented by a general Calderón-Zygmund operator. The integration is singular, the boundedness of the operator is weaker. Our results consists of identifying the function spaces in which local and global existences results of (1.1) can be derived.

In Sec. 2, we will provide necessary background material and notations; in Sec. 3, we will present the local existence results; and the global existence results will be presented in Sec. 4.

2. Notation and basic formulas

2.1. *Function spaces and norms*

For $0 < p < \infty$, the Calderón-Zygmund operator $\mathcal{K}_b$ is known to be (p,p), i.e., bounded operator maps L^p functions to L^p functions. More precisely, there exists a constant A_p, such that $\|\mathcal{K}_b f\|_p \leqslant A_p \|f\|_p$ for any function $f \in L^p(\mathbb{R}^d)$ with the L^p norm $\|\cdot\|_p$ defined as $\|f\|_p := (\int_{\mathbb{R}^d} |f(x)|^p\, dx)^{1/r}$, and the space $L^p(\mathbb{R}^d)$ includes all the measurable functions that has finite L^p norm. In addition, the operator $\mathcal{K}_b$ is bounded on Lipschitz space $\mathrm{Lip}(\epsilon)$ for any $\epsilon \leqslant \delta$ with δ being the parameter in the Calderón-Zygmund kernel $K1 = 0$ by the main theorem (Theorem 1.6) in [11]. Recall that $\mathrm{Lip}(\alpha)$ refers the space of functions that satisfy $|f(x) - f(y)| \leqslant C d(x,y)^{\alpha}$.

The function space in which we will derive the local existence theorem is defined to be $L^p_x L^\infty_t(\mathbb{R}^d \times [0,t])$ the space of all functions whose $L^p_x L^\infty_t$ norm is finite, with

$$
\|f(x,t)\|_{L^p_x L^\infty_t} := \sup_{0\leqslant s\leqslant t} \left(\int_{x\in\mathbb{R}^d} |f(x,s)|^p\, dx \right)^{1/p}.
$$

2.2. *Weak solution*

For any test function $\psi(x,s)$ in $C_0^\infty(\mathbb{R}^d \times \mathbb{R}_+)$, a function $u_t(x)$ in a suitable function space is a weak solution if the following is satisfied,

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \psi(x,t) u_t(x)\, dx - \int_{\mathbb{R}^d} \psi(x,0) u_0(x)\, dx \\
&\quad = \int_0^t \int_{\mathbb{R}^d} \left[\frac{\partial}{\partial s}\psi(x,s) - (-\Delta)^{\alpha/2}\psi(s,x) + B(u_s)(x).\nabla\psi(x,s) \right] u_s(x)\, dx\, ds.
\end{aligned} \tag{2.1}
$$

2.3. Semigroup and generator

The generator for semi-group $\exp(-t(-\Delta)^{\alpha/2})$ is denoted as p_t^α. For any smooth function ϕ, then $\Psi(s,x) = p_{t-s}^\alpha \star \phi(x)$ satisfies

$$\frac{\partial}{\partial s}\Psi(s,x) - (-\Delta)^{\alpha/2}\Psi(\sigma,x) = 0.$$

It is known that, see, e.g., [5].

Lemma 2.1. *If* $m \geqslant q \geqslant 1$, *and* $f \in L^q$, *then,*

$$\|p_t^\alpha \star f\|_m \leqslant Ct^{-\frac{d}{\alpha}(\frac{1}{q}-\frac{1}{m})}\|f\|_q, \qquad \|\nabla p_t^\alpha \star f\|_m \leqslant Ct^{-\frac{d}{\alpha}(\frac{1}{q}-\frac{1}{m})-\frac{1}{\alpha}}\|f\|_q. \tag{2.2}$$

3. Local existence

The following version of the Banach fixed point theorem, can be found in e.g., [3].

Lemma 3.1. *Suppose that* $B : X \times X \to X$ *is a bilinear mapping for a Banach space* $(X, \|\cdot\|_X)$, *and it satisfies,*

$$\|B(x_1,x_2)\|_X \leqslant \eta\|x_1\|_X\|x_2\|_X,$$

for some $\eta > 0$. *Then, for each* $y \in X$ *satisfying* $4\eta\|y\| < 1$, *equation,*

$$x = y + B(x,x)$$

admits a unique solution x *in the ball* $\{z \in X, \|z\| \leqslant R\}$ *with* $R = \frac{1-\sqrt{1-4\eta\|y\|}}{2\eta}$. *Moreover, the solution satisfies inequality* $\|x\|_X \leqslant 2\|y\|_X$.

Previous results assume that the kernel $|b(x,y)| \leqslant C|x|^{\beta-d}$ for some $\beta > 0$. This will lead to $\|B(u)\|_\infty \lesssim \|u\|$. In order to apply the fixed point theorem in Lemma 3.1, define the following bilinear map,

$$B(u,v)(t,x) = \int_0^t \nabla p_{t-s}^\alpha \star (B(v_s)u_s)ds. \tag{3.1}$$

Lemma 3.2. *For any* $p \geqslant 2$, *there exists a constant* $C > 0$, *such that,*

$$\left\|\int_0^t \nabla p_{t-s}^\alpha \star (B(v_s)u_s)ds\right\|_{L_x^pL_T^\infty} \leqslant CT^{1-\frac{1}{\alpha}}\|v\|_{L_x^pL_T^\infty}\|u\|_{L_x^pL_T^\infty}.$$

Proof.

$$\begin{aligned}
\left\|\int_0^t \nabla p_{t-s}^\alpha \star (B(v_s)u_s)ds\right\|_{L_x^pL_T^\infty} &\overset{(a)}{\leqslant} \int_0^T \left\|\nabla p_{t-s}^\alpha \star (B(v_s)u_s)\right\|_{L_x^p} ds \\
&\overset{(b)}{\leqslant} C\int_0^T (t-s)^{-\frac{d}{\alpha}(\frac{1}{p})-\frac{1}{\alpha}} \|(B(v_s)u_s)\|_{L_x^{\frac{p}{2}}} ds \\
&\overset{(c)}{\leqslant} C\int_0^T (t-s)^{-\frac{d}{\alpha}(\frac{1}{p})-\frac{1}{\alpha}} \|(B(v_s)\|_{L_x^p}\|u_s\|_{L_x^p} ds \\
&\overset{(d)}{\leqslant} C'\int_0^T (t-s)^{-\frac{d}{\alpha}(\frac{1}{p})-\frac{1}{\alpha}} \|v_s\|_{L_x^p}\|u_s\|_{L_x^p} ds \\
&\overset{(e)}{\leqslant} C'T^{1-\frac{1}{\alpha}}\|v\|_{L_x^pL_T^\infty}\|u\|_{L_x^pL_T^\infty},
\end{aligned}$$

where (a) is due to the nonnegativity of the integrand on the right hand side; (b) comes from the property (2.2) in Lemma 2.1 for the operator p_t^α with $m = p, q = \frac{p}{2}$; (c) is an application of the Hölder's inequality applied to $\|(B(v_s)u_s)\|_{L_x^{\frac{p}{2}}}$; (d) follows from the boundedness of the Calderón-Zygmund operator, more specifically, $\|(B(v_s)\|_{L_x^p} \leqslant C'\|v_s\|_{L_x^p}$; and (e) is the result of a simple integration calculation. □

Then, applying the fixed point theorem of Lemma 3.1, the following local existence theorem can be established.

Theorem 3.3. *For any* $p > 2$, *and* $u_0 \in L^p_x$, *there exist a a constant* $T^* > 0$ *and function* $u(x,t) \in L^p_x L^\infty_{T^*}$ *such that* $u(x,t)$ *is a weak solution, in the sense of* (2.1), *to* (1.1).

Proof. Lemma 3.2 indicates that the bilinear form defined in (3.1) satisfies the condition for the fixed point theorem, Lemma 3.1. Hence, we can conclude that, for any function $u_0 \in L^p_x L^\infty_{T^*}$, there exist a constant $T^* > 0$ and a weak solution $u(x,t) \in L^p_x L^\infty_{T^*}$ in the sense that is defined in (2.1). □

4. Global existence

The goal of this section is to establish the global existence of (1.1) via a combined probabilistic and analytic argument. Precisely, global existence means that for any given time horizon $T < \infty$, the solution $u(t,x)$ solves the equation in a weak sense. The basic approach is a probabilistic one, we will construct a solution to a stochastic differential equation, (4.1), that will be defined below. It has been demonstrated, see, e.g., [5] that the density function of this solution solves (1.1). The solution of (4.1) will be constructed through an iterative procedure, with key steps obtained utilizing the properties of the Calderón-Zygmund operator.

We start with the following lemma on the existence and uniqueness of a class of stochastic differential equations defined by a stable process.

Lemma 4.1. *Given initial condition* X_0, α*-stable process* S_t, *and a bounded function* $a_t : \mathbb{R} \to \mathbb{R}^d$, *and* a_t *is Lipschitz for any* $t \in [0,T]$, *the following stochastic differential equation,*

$$X_t = X_0 + S_t + \int_0^t a_s(X_s)ds,$$

has a unique (pathwise and in law) solution in the Skorohod space $D([0,T], \mathbb{R}^d)$.

Proof. When a_s is constant over time, this lemma is stated and proved in [5]. Similarly, here, the pathwise existence and uniqueness are the result of a fixed point argument. Then the Yamada-Watanabe Theorem, see, e.g., [4] ensures the uniqueness in probability law. □

Note that we are studying the probability measure of the paths, so let us first introduce the Skorohod space $D([0,T], \mathbb{R}^d)$, the set of cádlád functions (functions that are right-continuous and have left limits everywhere) from $[0,T]$ to $\mathbb{R}^d$. Let $\mathcal{P}_T$ denote the set of all probability measures on $D([0,T], \mathbb{R}^d)$ that are absolutely continuous with respect to the Lebesgue measure, and

$$\tilde{\mathcal{P}}_T = \left\{ P \in \mathcal{P}_T, P_0 = \frac{|u_0|}{\|u_0\|_1} \right\},$$

i.e., the subset of $\mathcal{P}_T$ with the initial condition fixed. Define the following metric on $\tilde{\mathcal{P}}_T$, for any $p > 0$,

$$d_{T,p}(m_1, m_2) = \max\{\rho_{p,T}(m_1, m_2), \|f_1 - f_2\|_p\},$$

where $\rho_{p,T}$ denotes the p-Wasserstein distance on $\tilde{\mathcal{P}}_T$, i.e.,

$$\rho_{p,T}(P,Q) = \left\{ \inf \int_{D_T \times D_T} \left[\sup_{t \leqslant T} |x(t) - y(t)| \wedge 1 \right]^p R(dx, dy) \right\}^{\frac{1}{p}},$$

with the infimum is taken over measures R with marginals P and Q, and $R(x(0) = y(0)) = 1$. And f_i denotes the density function of m_i for $i = 1, 2$. It is worth noting that the metric space $(\mathcal{P}_T, d_{T,p})$ is complete. The stochastic stochastic differential equation is the following one,

$$X_t = X_0 + S_t + \int_0^t \int_{\mathbb{R}^d} b(X_s, y)\tilde{P}_t(dy)ds, \tag{4.1}$$

with $\tilde{P}_t(dy)$ denotes the probability law of X_t.

4.1. An operator defined on the space of probability measures

Define an operator Ψ^Y on $\tilde{\mathcal{P}}_T$, indexed by a process $Y(t)$ as follows. For each $m \in \tilde{\mathcal{P}}_T$, denote m_t^Y as the Y_t-weighted version of m, more specifically,

$$m_t^Y(A) = \|u_0\|_1 \mathsf{E}[\mathbf{1}_A(Y_t)\mathrm{sgn}(u_0(Y_0))].$$

Thus, $\Psi_t^Y(m)$ is the probability measure induced by the solution to the following stochastic differential equation,

$$X_t = X_0 + S_t + \int_0^t \int_{R^d} b(X_s, y) m_t^Y(dy) ds.$$

The following result is the key estimation for the global existence.

Lemma 4.2. *With the condition that* $\delta > 1$ *and* $\mathcal{K}_b 1 = 0$*, we can conclude that, There exists a constant* $C > 0$*, such that, for any* $m_1, m_2 \in \hat{\mathcal{P}}_T$*,* $d_{t,p}(\Psi_t^Y(m^1), \Psi_t^Y(m^2)) \leqslant C \int_0^t d_{s,p}(m^1, m^2) ds$.

Proof. First of all, we know that, for $p > 1$,

$$\rho_{p,t}(\Psi_t^Y(m^1), \Psi_t^Y(m^2)) \leqslant \left(\mathsf{E}\left[\sup_{s \leqslant t} |X_s^1 - X_s^2|^p \right] \right)^{\frac{1}{p}}.$$

Similarly, from relation between density and the process, see, e.g., [7], we can conclude that,

$$\|f_1 - f_2\|_p \leqslant \left(\mathsf{E}\left[\sup_{s \leqslant t} |X_s^1 - X_s^2|^p \right] \right)^{\frac{1}{p}}.$$

Hence, we have,

$$d_{p,t}(\Psi_t^Y(m^1), \Psi_t^Y(m^2)) \leqslant \left(\mathsf{E}\left[\sup_{s \leqslant t} |X_s^1 - X_s^2|^p \right] \right)^{\frac{1}{p}} \leqslant \left(\mathsf{E}\left[\int_0^t \left| \int_{R^d} b(X_s^1, y) m^1(dy) - b(X_s^2, y) m^2(dy) \right|^p \right] \right)^{\frac{1}{p}}.$$

By triangular inequality, it suffices to establish the following two inequalities,

$$\left(\mathsf{E}_\Pi \left| \int_0^t \int_{\mathbb{R}^d} b(X_s^1, y) m_t^1(dy) ds - \int_0^t \int_{\mathbb{R}^d} b(X_s^2, y) m_t^1(dy) ds \right|^p \right)^{1/p} \leqslant C \int_0^t d_{s,p}(m^1, m^2) ds, \tag{4.2}$$

and

$$\left(\mathsf{E}_\Pi \left| \int_0^t \int_{\mathbb{R}^d} b(X_s^2, y) m_t^1(dy) ds - \int_0^t \int_{\mathbb{R}^d} b(X_s^2, y) m_t^2(dy) ds \right|^p \right)^{1/p} \leqslant C \int_0^t d_{s,p}(m^1, m^2) ds. \tag{4.3}$$

The inequality (4.2) follows from the Lipschitz property of the Calderón-Zygmund operator (Theorem 1.6 in [11]). The inequality (4.3) holds because of the boundedness of the Calderón-Zygmund operator. More specifically,

$$\begin{aligned}
&\left(\mathsf{E}_\Pi \left| \int_0^t \int_{\mathbb{R}^d} b(X_s^2, y) m_t^1(dy) ds - \int_0^t \int_{\mathbb{R}^d} b(X_s^2, y) m_t^2(dy) ds \right|^p \right)^{1/p} \\
&\leqslant \left(\mathsf{E}_\Pi \int_0^t \left| \int_{\mathbb{R}^d} b(X_s^2, y) f^1 - f_2 dy \right|^p ds \right)^{1/p} \\
&\leqslant \int_0^t \left(\mathsf{E}_\Pi \left| \int_{\mathbb{R}^d} b(X_s^2, y) f^1 - f_2 dy \right|^p \right)^{1/p} ds \leqslant C \int_0^t d_{s,p}(m^1, m^2) ds.
\end{aligned}$$

□

4.2. Global existence

Theorem 4.3. *Under the following conditions, we will have the global existence of the solution to equation* (1.1)*:*

i. the Calderón-Zygmund operator with $\delta > 1$*, and* $\mathcal{K}_b 1 = 0$*;*

ii. the initial condition u_0 *is a Lipschitz function, in other word,* $u_0 \in \mathrm{Lip}(1)$.

Proof. Start with an arbitrary stochastic process $Y_t^0 \in D([0,T], \mathbb{R}^d)$, for any $n \geqslant 1$, define, $Y_t^n := \Psi^{Y_t^{n-1}}$. Then, iterating the inequality in Lemma 4.2, we get,

$$d_{T,p}(\Psi_T^n(m^1), \Psi_T^n(m^2)) \leqslant \frac{C^n}{n!} d_{T,p}(\Psi_T^0(m^1), \Psi_T^0(m^2)).$$

This means that Y_n^t is a Cauchy sequence in $d_{T,p}$, hence there exists a limit, denoted as Y_∞^t. It can be verified that Y_∞^t is a strong solution to the stochastic differential equation (4.1). Proposition 4.4 in [5] confirms that this solution will produce a density function that is a solution to equation (1.1). □

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