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Controllability of impulsive stochastic functional integrodifferential equations driven by Rosenblatt process and Lévy noise

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Abstract

In this paper, we develop controllability findings for impulsive neutral stochastic delay partial integrodifferential equations in Hilbert spaces driven by Rosenblatt process and Lévy noise. A novel set of adequate requirements is obtained by utilizing a fixed point method without imposing a stringent compactness constraint on the semigroup. The observed results represent a generalization and continuation of previous findings on this topic. Finally, an example is given to demonstrate how the acquired findings may be used.

Keywords: Stochastic functional integrodifferential equations, resolvent operator, rosenblatt process, Lévy noise, controllability.

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1. Introduction

Neutral integrodifferential equations have been the subject of extensive study by several authors over the last many years (see[7, 12, 16, 17, 22, 23, 28, 52] and references therein). Integrodifferential equations of the neutral type are used to represent a wide variety of physical processes that occur in fluid dynamics, electronics, chemical kinetics, and other domains. As a result of their inherent properties, neutral differential equations can be found naturally in the mathematical modeling of a wide range of real phenomena in fields as diverse as mechanics, electronics, control theory, engineering, economics, and statistics. The heat equation developed by Lunardi in [35] and the one used to perform a quantitative assessment of unemployment, wage bills, and income policies in [9] are both good instances of this type of equation. Chukwu [9], provides a number of excellent examples, most of them related to economics. For deterministic neutral functional differential equations, Hale and Mayer [20] were the first to examine them.

Stochastic differential equations have been extensively studied as a mathematical model for describing the dynamical behavior of a real-world phenomenon. The inclusion of environmental disturbances and

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time delay is critical when constructing realistic models in the domains of engineering, biology, and other sciences. In recent years, there has been a significant amount of interest in the examination of qualitative aspects of neutral stochastic differential equations, such as existence, uniqueness, and stability (see [11, 21, 30, 42, 45] and references therein). Lakhel [32] has shown the existence and uniqueness of mild solutions for a class of neutral stochastic functional differential evolution equations driven by a Rosenblatt process with changing time delays using a fixed point theorem.

The underlying notions in current mathematical theory known as controllability play a significant role both in deterministic and stochastic control problems, such as stabilization of unstable systems by feedback control. In recent years, the controllability problems for various linear and nonlinear deterministic and stochastic dynamical systems have been studied by employing diverse methodologies (see for example [4, 13, 23, 28, 39, 40, 43, 47, 54, 55] and the reference therein). For solving nonlinear problems (differential equations, stochastic differential equations, integrodifferential equations, and . . .), it is generally recognized that fixed point theory is a valuable tool. Several scholars have contributed to the development of this theory by solving stochastic differential equations across multiple fixed point theorems. However, research on the controllability of impulsive neutral stochastic PDEs with delays and fractional Brownian motion (fBm) is scarce. Ahmed examined approximate controllability of impulsive neutral stochastic functional differential equations with finite delay and fBm in Hilbert space in [2]. Cui and Yan [10] investigated controllability for fBm-driven neutral stochastic evolution equations with the Hurst parameter. Controllability of impulsive neutral stochastic functional differential equations with infinite delay driven by fBm in a real separable Hilbert space was studied by Boudaoui and Lakhel [5]. Chen [8] also addressed approximation control liability for semilinear stochastic equations driven by fBm using the Banach fixed point theorem.

Furthermore, stochastic functional differential equations with Poisson jumps have been increasingly popular in the modeling of phenomena that arise in a range of domains such as finance, economics, medicine, biology, and other related fields. In real life, it is usual for a stochastic system to transition from a normal state or a good state to a bad state, with the strength of the system being random. It is then obvious and required to incorporate a jump element in the stochastic differential equations. As a result, when analyzing the controllability of stochastic differential equations, it is crucial to evaluate the implications of Poisson leaps. We are pleased by the vast number of results on the controllability of stochastic differential equations with Poisson jumps that have recently been reported in the literature. So far, these themes have garnered a significant amount of attention, and there are a wealth of materials available on them. Lakhel et al. [33] showed the existence, uniqueness, and asymptotic behavior of mild solutions for a family of neutral functional stochastic differential equations with fBm and Poisson jumps. Huan and Agarwal [25] found attractive and quasi-invariant sets of the mild solution for impulsive neutral stochastic PDEs driven by Levy noise. Sakthivel and Ren [56], addressed the complete controllability of stochastic evolution equations with jumps in a separable Hilbert space, while in, Ren et al. [53] studied the approximate controllability of stochastic differential systems driven by Teugels martingales coupled with a Levy process. Huan and Gao [26] have expanded the conclusions of the study [24] for a class of nonlocal second-order impulsive neutral stochastic integrodifferential equations with indefinite delay and Poisson jumps. For further details concerning the stochastic PDEs with Poisson jumps, one can check the recent monograph [48] and the references therein.

In this paper, we consider the controllability results for impulsive neutral stochastic delay integrodifferential controls systems with delay driven by Rosenblatt process and Lévy noise of the form

$$
\begin{cases}
d\left[\vartheta(t)-h(t,\vartheta(t-\delta(t)))\right]=\Big[A\left[\vartheta(t)-h(t,\vartheta(t-\delta(t)))\right]+\int_0^t \Upsilon(t-s)\left[\vartheta(s)-h(s,\vartheta(t-\delta(s)))\right]ds \\
\qquad\qquad +f\left(t,\vartheta(t-\rho(t))\right)+\mathbb{B}u(t)\Big]dt+\int_{\mathcal{U}} q\left(t,\vartheta(t-\gamma(t)),z\right)\widehat{\eta}(dt,dz) \\
\qquad\qquad +\sigma(t)dZ_Q^H(t), \quad t\neq t_k\in J:=[0,T], \\
\Delta(t_k)=\vartheta(t_k^+)-\vartheta(t_k^-)=I_k(t_k^-), \quad k=1,2,\ldots, \\
\vartheta_0(t)=\varphi(\cdot)\in\mathcal{C}^0_{\mathcal{F}_0}\left([-r,0],\mathcal{L}^2(\Omega,\mathbb{H})\right), \quad t\in[-r,0],
\end{cases}
\tag{1.1}
$$

where $A : D(A) \subset \mathbb{H} \to \mathbb{H}$ is the infinitesimal generator of a C_0-semigroup $(S(t))_{t\geqslant 0}$ on Hilbert space

$\mathbb{H}$, $\Upsilon(t)$ a closed linear operator on $\mathbb{H}$ with the domain $D(A) \subset D(\Upsilon)$, which is independent of t, $t \geqslant 0$. The functions $h, f : \mathbb{R}^+ \times \mathbb{H} \to \mathbb{H}$, $\sigma : [0, \infty) \to \mathcal{L}_2^0(\mathbb{Y}, \mathbb{H})$ and $q : \mathbb{R}^+ \times \mathbb{H} \times \mathcal{U} \to \mathbb{H}$ are Borel measurable. The functions δ, $\rho, \gamma : \mathbb{R}^+ \to [0, r]$ are continuous. Z_Q^H is a Rosenblatt process with parameter $H \in (1/2, 1)$ in a real and separable Hilbert space $\mathbb{Y}$. The control function u takes values in $L^2(J, \mathbb{U})$, the Hilbert space of admissible control functions for a separable Hilbert space $\mathbb{U}$ and $\mathbb{B}$ is a bounded linear operator from $\mathbb{U}$ into $\mathbb{H}$, $I_k : \mathbb{H} \to \mathbb{H}$, $k = 1, 2, \ldots$ are appropriate functions. Furthermore, let $0 = t_0 < t_1 < \cdots < t_k < \cdots$ be prefixed points, where $\vartheta(t_k^+)$ and $\vartheta(t_k^-)$ represent the right and left limits of $\vartheta(t)$ at $t = t_k$, respectively, and $\Delta(t_k) = \vartheta(t_k^+) - \vartheta(t_k^-)$, represents the jump of the function ϑ at time t_k with I_k determining the size of the jump.

However, to the best of authors knowledge the controllability of impulsive neutral stochastic delay integrodifferential controls systems with delay driven by Rosenblatt process and Lévy noise has not been investigated yet. Several researchers express the controllability results by the semigroup approach. The proposed work on the controllability of impulsive neutral stochastic delay integrodifferential controls systems with delay driven by Rosenblatt process and Lévy noise is new to the literature and more general result than the existing literature. The following are the most significant contributions and advantages of this article.

- Nonlinear impulsive neutral stochastic delay integrodifferential controls systems with Rosenblatt process and Lévy noise are developed.
- The fundamental advantage of the targeted technique is that it is based on resolvent operator theory in the sense of the Grimmer and Banach fixed point theorem, together with appropriate hypotheses.
- An example is provided in order to validate the theoretical conclusions that have been suggested.

The following is the overall structure of this study. In Section,2, we present a high-level overview of several fundamental notations, preliminaries, and assumptions. The results in Section 3 are devoted to the research of the controllability of the system (1.1), as well as their verification. Section 4 presents an illustration of the idea through the use of an example.

2. Preliminaries

Let $(\mathbb{H}, \|\cdot\|_{\mathbb{H}}, \langle\cdot,\cdot\rangle)$, $(\mathbb{K}, \|\cdot\|_{\mathbb{K}}, \langle\cdot,\cdot\rangle)$ denote two real separable Hilbert spaces, with their vectors norms and their inner products, respectively. We denote by $\mathcal{L}(\mathbb{K}, \mathbb{H})$ the set of all linear bounded operators from $\mathbb{K}$ into $\mathbb{H}$, which is equipped with the usual operator norm $\|\cdot\|$. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in J}, \mathbb{P})$ be a complete filtered probability space satisfying the usual condition (i.e., it is right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$ null sets). Let $\mathcal{L}(\mathbb{K}, \mathbb{H})$ represents the space of all bounded linear operators from $\mathbb{K}$ to $\mathbb{H}$ and $Q \in \mathcal{L}(\mathbb{K}, \mathbb{K})$ represents a non-negative self-adjoint operator. Let $\mathcal{L}_2^0 = L^2(Q^{\frac{1}{2}}\mathbb{K}, \mathbb{H})$ be the space of all Hilbert-Schmidt operators from $Q^{\frac{1}{2}}\mathbb{K}$ into $\mathbb{H}$, where $\mathcal{L}_2^0$ is a separable Hilbert space, equipped with the norm $\| \Psi \|_{\mathcal{L}_2^0}^2 = \| \Psi Q^{\frac{1}{2}} \|^2 = Tr(\Psi Q \psi^*)$. Suppose that $p(t)$, $t \geqslant 0$ is σ-finite stationary $\mathcal{F}_t$-adapted Poisson point process taking values in a measurable space $(\mathcal{U}, \mathcal{B}(\mathcal{U}))$.

2.1. *Rosenblatt process*

In this subsection, we recall some basic concepts on the Rosenblatt process as well as the Wiener integral with respect to it. Consider $(\xi_n)_{n \in Z}$ a stationary Gaussian sequence with mean zero and variance 1 such that its correlation function satisfies that $R(n) := \mathbb{E}(\xi_0 \xi_n) = n^{\frac{2H-2}{k}} L(n)$, with $H \in (\frac{1}{2}, 1)$ and L is a slowly varying function at infinity. Let g be a function of Hermite rank k, that is, if g admits the following expansion in Hermite polynomials

$$g(x) = \sum_{j \geqslant 0} c_j H_j(x), \quad c_j = \frac{1}{j!} \mathbb{E}(g(\xi_0 H_j(\xi_0))),$$

then $k = \min\{j|c_j \neq 0\} \geqslant 1$, where $H_j(x)$ is the Hermite polynomial of degree j given by $H_j(x) = (-1)^j e^{\frac{x^2}{2}} \frac{d^j}{dx^j} e^{-\frac{x^2}{2}}$. Then, the Non-Central Limit Theorem (see, for example, Dobrushin and Major [15]) says $\frac{1}{n^H}\sum_{j=1}^{[nt]} g(\xi_j)$ converges as $n \to \infty$, in the sense of finite dimensional distributions, to the process

$$Z_H^k(t) = c(H,k)\int_{\mathbb{R}^k}\int_0^t \left(\prod_{j=1}^k (s-y_j)_+^{\left(-\frac{1}{2}+\frac{1-H}{k}\right)}\right) ds dB(y_1)\cdots dB(y_k), \tag{2.1}$$

where the above integral is a Wiener-Itô multiple integral of order k with respect to the standard Brownian motion $(B(y))_{y\in\mathbb{R}}$ and $c(H,k)$ is a positive normalization constant depending only on H and k. The process $(Z_H^k(t))_{t\geqslant 0}$ is called as the Hermite process and it is H self-similar in the sense that for any $c > 0$, $(Z_H^k(ct)) \stackrel{d}{=} (c^H Z_H^k(t))$ and it has stationary increments.

The the fractional Brownian motion (which is obtained from (2.1) when $k = 1$) is the most used Hermite process for study evolution equations due to its large range of applications. When $k = 2$ in (2.1), Taqqu [57] named the process as the Rosenblatt process. The stationarity of increments, self-similarity and long range dependence (see Tindel et al. [58]) were made that the Rosenblatt process is very important in practical applications. However, it is noted that Rosenblatt process is not Gaussian. In fact, due to their proprieties (long range dependence, self-similarity), the fractional Brownian motion process has large utilization in practical models, for instance in telecommunications and hydrology. So, many researchers prefer to use fractional Brownian motion than other processes because it is Gaussian and it facilitate calculations. However in concrete situations when the Gaussianity is not plausible for the model, one can use the Rosenblatt process. In recent years, there exist many works that investigated on diverse theoretical aspects of the Rosenblatt process. For example, Leonenko and Ahn [34] gave the rate of convergence to the Rosenblatt process in the non-central limit theorem and the wavelet-type expansion has been presented by Abry and Pipiras [1]. Tudor [60] established, the representation as a Wiener-Itô multiple integral with respect to the Brownian motion on a finite interval and developed the stochastic calculus with respect to it by using both pathwise type calculus and Malliavin calculus (see also Maejima and Tudor [36]). For more details for Rosenblatt process, we refer the reader to Maejima and Tudor [37, 38]), Pipiras and Taqqu [49] and the references therein.

Consider a time interval $[0,T]$ with arbitrary fixed horizon T and let $\{Z_H(t), t \in [0,T]\}$ be a one-dimensional Rosenblatt process with parameter $H \in (\frac{1}{2}, 1)$. According to the work of Tudor [60], the Rosenblatt process with parameter $H > \frac{1}{2}$ can be written as

$$Z_H(t) = d(H)\int_0^t\int_0^t \left[\int_{y_1\vee y_2}^t \frac{\partial K^{H'}}{\partial u}(u,y_1)\frac{\partial K^{H'}}{\partial u}(u,y_2)du\right] dB(y_1)dB(y_2), \tag{2.2}$$

where $K^H(t,s)$ is given by

$$K^H(t,s) = c_H s^{\frac{1}{2}-H}\int_s^t (u-s)^{H-3/2}u^{H-1/2}du \text{ for } t > s, \quad \text{with } \ c_H = \sqrt{\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})}},$$

$\beta(.,.)$ denotes the Beta function, $K^H(t,s) = 0$ when $t \leqslant s$, $(B(t), t \in [0,T])$ is a Brownian motion, $H' = \frac{H+1}{2}$ and $d(H) = \frac{1}{H+1}\sqrt{\frac{H}{2(2H-1)}}$ is a normalizing constant. The covariance of the Rosenblatt process $\{Z_H(t), t \in [0,T]\}$ satisfies

$$\mathbf{E}(Z_H(t)Z_H(s)) = \frac{1}{2}\left(s^{2H} + t^{2H} - |s-t|^{2H}\right).$$

The covariance structure of the Rosenblatt process allows to construct Wiener integral with respect to it. We refer to Maejima and Tudor [36] for the definition of Wiener integral with respect to general Hermite processes and to Kruk et al. [31] for a more general context (see also Tudor [60]). Note that

$$Z_H(t) = \int_0^T\int_0^T I(1_{[0,t]})(y_1,y_2)dB(y_1)dB(y_2),$$

where the operator I is defined on the set of functions $f : [0,T] \to \mathbb{R}$, which takes its values in the set of functions $g : [0,T]^2 \to \mathbb{R}^2$ and is given by

$$I(f)(y_1,y_2) = d(H)\int_{y_1\vee y_2}^{T} f(u)\frac{\partial K^{H'}}{\partial u}(u,y_1)\frac{\partial K^{H'}}{\partial u}(u,y_2)du.$$

Let f be an element of the set $\mathcal{E}$ of step functions on $[0,T]$ of the form

$$f = \sum_{i=0}^{n-1} a_i 1_{(t_i,t_{i+1}]},\ t_i \in [0,T].$$

Then, it is natural to define its Wiener integral with respect to Z_H as

$$\int_0^T f(u)dZ_H(u) := \sum_{i=0}^{n-1} a_i(Z_H(t_{i+1}) - Z_H(t_i)) = \int_0^T\int_0^T I(f)(y_1,y_2)dB(y_1)dB(y_2).$$

Let $\mathcal{H}$ be the set of functions f such that

$$\|f\|^2_{\mathcal{H}} := 2\int_0^T\int_0^T (I(f)(y_1,y_2))^2 dy_1dy_2 < \infty.$$

It follows that (see Tudor[60])

$$\|f\|^2_{\mathcal{H}} = H(2H-1)\int_0^T\int_0^T f(u)f(v)|u-v|^{2H-2}dudv.$$

It has been proved in Maejima and Tudor [36] that the mapping $f \longmapsto \int_0^T f(u)dZ_H(u)$ defines an isometry from $\mathcal{E}$ to $L^2(\Omega)$ and it can be extended continuously to an isometry from $\mathcal{H}$ to $L^2(\Omega)$ because $\mathcal{E}$ is dense in $\mathcal{H}$. We call this extension as the Wiener integral of $f \in \mathcal{H}$ with respect to Z_H. It is noted that the space $\mathcal{H}$ contains not only functions but its elements could be also distributions. Therefore it is suitable to know subspaces $|\mathcal{H}|$ of $\mathcal{H}$: $|\mathcal{H}| = \left\{f : [0,T] \to \mathbb{R} | \int_0^T\int_0^T |f(u)||f(v)||u-v|^{2H-2}dudv < \infty\right\}$. The space $|\mathcal{H}|$ is not complete with respect to the norm $\|.\|_{\mathcal{H}}$ but it is a Banach space with respect to the norm

$$\|f\|^2_{|\mathcal{H}|} = H(2H-1)\int_0^T\int_0^T |f(u)||f(v)||u-v|^{2H-2}dudv.$$

As a consequence, we have

$$L^2([0,T]) \subset L^{1/H}([0,T]) \subset |\mathcal{H}| \subset \mathcal{H}.$$

For any $f \in L^2([0,T])$, we have

$$\|f\|^2_{|\mathcal{H}|} \leqslant 2HT^{2H-1}\int_0^T |f(s)|^2 ds \quad\text{and}\quad \|f\|^2_{|\mathcal{H}|} \leqslant C(H)\|f\|^2_{L^{1/H}([0,T])},$$

for some constant $C(H) > 0$. Let $C(H) > 0$ stands for a positive constant depending only on $\mathcal{H}$ and its value may be different in different appearances. Define the linear operator K^*_H from $\mathcal{E}$ to $L^2([0,T])$ by

$$(K^*_H f)(y_1,y_2) = \int_{y1\vee y_2}^T f(t)\frac{\partial \mathcal{K}}{\partial t}(t,y_1,y_2)dt,$$

where $\mathcal{K}$ is the kernel of Rosenblatt process in representation (2.2)

$$\mathcal{K}(t,y_1,y_2) = 1_{[0,t]}(y_1)1_{[0,t]}(y_2)\int_{y_1\vee y_2}^t \frac{\partial K^{H'}}{\partial u}(u,y_1)\frac{\partial K^{H'}}{\partial u}(u,y_2)du.$$

Note that $(K_H^* 1_{[0,t]})(y_1, y_2) = \mathcal{K}(t, y_1, y_2) 1_{[0,t]}(y_1) 1_{[0,t]}(y_2)$. The operator K_H^* is an isometry between $\mathcal{E}$ to $L^2([0,T])$, which can be extended to the Hilbert space $\mathcal{H}$. In fact, for any $s, t \in [0,T]$ we have

$$\begin{aligned}
\left\langle K_H^* 1_{[0,t]}, K_H^* 1_{[0,s]} \right\rangle_{L^2([0,T])} &= \left\langle \mathcal{K}(t,.,.) 1_{[0,t]}, \mathcal{K}(s,.,.) 1_{[0,s]} \right\rangle_{L^2([0,T])} \\
&= \int_0^{t\wedge s} \int_0^{t\wedge s} \mathcal{K}(t, y_1, y_2) \mathcal{K}(s, y_1, y_2) dy_1 dy_2 \\
&= H(2H-1) \int_0^t \int_0^s |u - v|^{2H-2} du dv \\
&= \left\langle 1_{[0,t]}, 1_{[0,s]} \right\rangle_{\mathcal{H}}.
\end{aligned}$$

Moreover, for $f \in \mathcal{H}$, we have

$$Z_H(f) = \int_0^T \int_0^T (K_H^* f)(y_1, y_2) dB(y_1) dB(y_2).$$

Let $\{Z_n(t)\}_{n\in\mathbb{N}}$ be a sequence of two-sided one dimensional Rosenblatt process mutually independent on $(\Omega, \mathcal{F}, \mathbb{P})$. We consider a K-valued stochastic process $Z_Q(t)$ given by the following series

$$Z_Q(t) = \sum_{n=1}^{\infty} z_n(t) Q^{1/2} e_n, \quad t \geqslant 0.$$

Moreover, if Q is a non-negative self-adjoint trace class operator, then this series converges in the space K, that is, it holds that $Z_Q(t) \in L^2(\Omega, K)$. Then, we say that the above $Z_Q(t)$ is a K-valued Q-Rosenblatt process with covariance operator Q. For instance, if $\{\sigma_n\}_{n\in\mathbb{N}}$ is a bounded sequence of non-negative real numbers such that $Qe_n = \sigma_n e_n$, by assuming that Q is a nuclear operator in K, then the stochastic process

$$Z_Q(t) = \sum_{n=1}^{\infty} z_n(t) Q^{1/2} e_n = \sum_{n=1}^{\infty} \sqrt{\sigma_n} z_n(t) e_n, \quad t \geqslant 0,$$

is well-defined as a K-valued Q-Rosenblatt process.

Definition 2.1 (Tudor [60])**.** Let $\varphi : [0,T] \to \mathcal{L}_2^0$ such that $\sum_{n=1}^{\infty} \|K_H^*(\varphi Q^{1/2} e_n)\|_{L^2([0,T];\mathcal{H})} < \infty$. Then, its stochastic integral with respect to the Rosenblatt process $Z_Q(t)$ is defined, for $t \geqslant 0$, as follows:

$$\int_0^t \varphi(s) dZ_Q(s) := \sum_{n=1}^{\infty} \int_0^t \varphi(s) Q^{1/2} e_n dz_n(s) = \sum_{n=1}^{\infty} \int_0^t \int_0^t (K_H^*(\varphi Q^{1/2} e_n))(y_1, y_2) dB(y_1) dB(y_2). \tag{2.3}$$

Lemma 2.2. *For $\psi : [0,T] \to \mathcal{L}_2^0$ such that $\sum_{n=1}^{\infty} \|\psi Q^{1/2} e_n\|_{L^{1/H}([0,T];U)} < \infty$ holds, and for any $a, b \in [0,T]$ with $b > a$, we have*

$$\mathbf{E} \left\| \int_a^b \psi(s) dZ_Q(s) \right\|^2 \leqslant c(H)(b-a)^{2H-1} \sum_{n=1}^{\infty} \int_a^b \|\psi(s) Q^{1/2} e_n\|^2 ds.$$

If, in addition,

$$\sum_{n=1}^{\infty} \|\psi(t) Q^{1/2} e_n\| \text{ is uniformly convergent for } t \in [0,T],$$

then, it holds that

$$\mathbf{E} \left\| \int_a^b \psi(s) dZ_Q(s) \right\|^2 \leqslant C(H)(b-a)^{2H-1} \int_a^b \|\psi(s)\|_{\mathcal{L}_2^0}^2 ds.$$

Proof. Let $\{e_n\}_{n\in\mathbb{N}}$ be the complete orthogonal basis of $\mathbb{K}$ introduced above. Applying (2.3) and Hölder inequality, we have

$$\begin{aligned}
\mathbf{E}\left\|\int_a^b \psi(s)dZ_Q(s)\right\|^2 &= \mathbf{E}\left\|\sum_{n=1}^{\infty}\int_a^b \psi(s)Q^{1/2}e_n dz_n(s)\right\|^2\\
&= \sum_{n=1}^{\infty}\mathbf{E}\left\|\int_a^b \psi(s)Q^{1/2}e_n dz_n(s)\right\|^2\\
&= \sum_{n=1}^{\infty} H(2H-1)\int_a^b\int_a^b \|\psi(s)Q^{1/2}e_n\|\|\psi(t)Q^{1/2}e_n\||t-s|^{2H-2}dsdt\\
&\leqslant C(H)\sum_{n=1}^{\infty}\left(\int_a^b \|\psi(s)Q^{1/2}e_n\|^{1/H}ds\right)^{2H}\\
&\leqslant C(H)(b-a)^{2H-1}\sum_{n=1}^{\infty}\int_a^b \|\psi(s)Q^{1/2}e_n\|^2 ds.
\end{aligned}$$

□

Let $\mathbb{Y}$ be a separable Hilbert space. Let $p(t)$, $t \geqslant 0$ be σ-finite stationary $\mathcal{F}_t$-adapted $\mathbb{Y}$-valued Poisson point process. Then for any $F \in \mathcal{B}(\mathbb{Y}-\{0\})$, which denotes the Borel σ-field of $(\mathbb{Y}-\{0\})$, where $0 \notin \overline{F}$, we get a counting Poisson random measure η on $(\mathbb{Y}-\{0\})$:

$$\eta\left((0,t)\times F\right) := \sum_{0<s\leqslant t} 1_F(p(s)) = \#\{0<s\leqslant t,\, p(s)\in F\},$$
$$\eta\left((t_1,t_2]\times F\right) := \eta\left((0,t_2)\times F\right) - \eta\left((0,t_1)\times F\right).$$

We shall denote $\eta\,(t,F) := \eta\left((0,t)\times F\right)$. Then, it is known that there exists a σ-finite measure λ such that

$$\mathbf{E}\left[\eta(t,F)\right] = \lambda(F)t, \qquad \mathbb{P}\left[\eta(t,F)=n\right] = \frac{\exp\left(-t\lambda(F)(\lambda(F)t)^n\right)}{n!}.$$

The measure λ is called the Lévy measure. Then, $t>0$, the measure $\widehat{\eta}$ is defined by

$$\widehat{\eta}\left([0,t],F\right) = \eta\left([0,t],F\right) - t\lambda(F).$$

The measure $\widehat{\eta}(dt,dy)$ is called the compensated Poisson random measure and $\lambda(F)$ is called the compensator.

Let $\mathcal{U} \in \mathcal{B}(\mathbb{Y}-\{0\})$, where $0 \notin$ the closure of $\mathcal{U}$. Let $\lambda_{\mathcal{U}}$ denotes the restriction of the measure λ to $\mathcal{U}$ still denoted by λ, such that λ is finite on $\mathcal{U}$. Denote by $\mathcal{P}^2\left([0,t]\times\mathcal{U};\mathbb{H}\right)$ the space of all predictable mappings $\kappa : [0,t]\times\mathcal{U} \to \mathbb{H}$ for which

$$\int_0^t\int_{\mathcal{U}} \mathbf{E}\|\kappa(s,y)\|^2_{\mathbb{H}}\lambda(dy)ds < \infty.$$

We may then define the $\mathbb{H}$-valued stochastic integral

$$\int_0^t\int_{\mathcal{U}}\kappa(s,y)\widehat{\eta}(ds,dy) := \int_0^t\int_{\mathcal{U}}\kappa(s,y)\eta(ds,dy) - \int_0^t\int_{\mathcal{U}}\kappa(s,y)\lambda(dy)ds,$$

where

$$\int_0^t\int_{\mathcal{U}}\kappa(s,y)\eta(ds,dy) := \sum_{0<s\leqslant t}\kappa(s,p(s))1_{\mathcal{U}}(p(s)).$$

Furthermore, we can see that $\int_0^t \int_{\mathcal{U}} \kappa(s,y)\widehat{\eta}(ds,dy)$ is an $\mathbb{H}$-valued centered square integrable martingale such that

$$\mathbf{E}\left(\left\|\int_0^t \int_{\mathcal{U}} \kappa(s,y)\widehat{\eta}(ds,dy)\right\|_{\mathbb{H}}^2\right) = \int_0^t \int_{\mathcal{U}} \mathbf{E}\|\kappa(s,y)\|_{\mathbb{H}}^2 \lambda(dy)ds.$$

We can refer to Protter [50] for a systematic theory about stochastic integrals of this kind.

2.2. *Partial integrodifferential equations in Banach spaces*

In this section, we recall some fundamental results needed to establish our main results. For the theory of resolvent operators we refer the reader to [19]. Throughout this paper, $\mathbb{X}$ is a Banach space, A and $\Upsilon(t)$ are closed linear operators on $\mathbb{X}$. Y represents the Banach space $D(A)$ equipped with the graph norm defined by

$$|y|_Y := |Ay| + |y| \quad \text{for } y \in Y.$$

The notations $C([0,+\infty);Y), \mathcal{B}(Y,\mathbb{X})$ stand for the space of all continuous functions from $[0,+\infty)$ into Y, the set of all bounded linear operators from Y into $\mathbb{X}$, respectively. We consider the following Cauchy problem

$$\begin{cases} \vartheta'(t) = A\vartheta(t) + \int_0^t \Upsilon(t-s)\vartheta(s)ds, & \text{for } t \geqslant 0, \\ \vartheta(0) = \vartheta_0 \in \mathbb{X}. \end{cases} \tag{2.4}$$

Definition 2.3 ([19]). A resolvent operator for Eq. (2.4) is a bounded linear operator valued function $R(t) \in \mathcal{L}(\mathbb{X})$ for $t \geqslant 0$, satisfying the following properties:

(i) $R(0) = I$ and $|R(t)| \leqslant Me^{\beta t}$ for some constants M and β;
(ii) for each $x \in \mathbb{X}$, $R(t)x$ is strongly continuous for $t \geqslant 0$;
(iii) $R(t) \in \mathcal{L}(Y)$ for $t \geqslant 0$. For $\vartheta \in Y$, $R(\cdot)\vartheta \in C^1([0,+\infty);\mathbb{X}) \cap C([0,+\infty);Y)$ and

$$R'(t)\vartheta = AR(t)\vartheta + \int_0^t \Upsilon(t-s)R(s)\vartheta ds = R(t)A\vartheta + \int_0^t R(t-s)\Upsilon(s)\vartheta ds \quad \text{for } t \geqslant 0.$$

For additional details on resolvent operators, we refer the reader to [19]. To deal with the existence of a resolvent operator we introduce the following assumptions.

($\mathbf{H}_1$) A is the infinitesimal generator of a strongly continuous semigroup $\{S(t)\}_{t\geqslant 0}$ on $\mathbb{X}$.
($\mathbf{H}_2$) For all $t \geqslant 0$, $\Upsilon(t)$ is a closed linear operator from $D(A)$ to $\mathbb{X}$, and $\Upsilon(t) \in \mathcal{B}(Y,\mathbb{X})$. For any $y \in Y$, the map $t \to \Upsilon(t)y$ is bounded, differentiable and the derivative $t \to \Upsilon'(t)y$ is bounded and uniformly continuous on $\mathbb{R}^+$.

Theorem 2.4 ([19, Theorem 3.7]). *Assume that* ($\mathbf{H}_1$)-($\mathbf{H}_2$) *hold. Then there exists a unique resolvent operator for the Cauchy problem* (2.4).

We denote by $\mathcal{C}_T := \mathcal{C}\left([-r,T], \mathcal{L}^2(\Omega,\mathbb{H})\right)$ the Banach space of all continuous functions from from $[-r,T]$ into $\mathcal{L}^2(\Omega,\mathbb{H})$ such that for all $\vartheta \in \mathcal{C}_T$,

$$\|\vartheta\|_{\mathcal{C}_T} := \sup_{t\in[-r,T]} \left(\mathbf{E}\|\vartheta(t)\|^2\right)^{\frac{1}{2}},$$

and let us consider the set $\mathcal{B}_T = \{\vartheta \in \mathcal{C}_T : \vartheta(s) = \varphi(s), \forall s \in [-r,0]\}$. The set $\mathcal{B}_T$ is a closed subset of $\mathcal{C}_T$ endowed with norm $\|\cdot\|_{\mathcal{C}_T}$. Then,$\mathcal{C}_T$ and $\mathcal{B}_T$ with the above norm are Banach spaces. Let $\mathcal{C}^0_{\mathcal{F}_0}\left([-r,0], \mathcal{L}^2(\Omega,\mathbb{H})\right)$ denote the family of all bounded $\mathcal{F}_0(\mathcal{F}_t)$-measurable, $\mathcal{C}_0 := \mathcal{C}\left([-r,0], \mathcal{L}^2(\Omega,\mathbb{H})\right)$-valued random variables φ, satisfying

$$\|\varphi\|_{\mathcal{C}_0} := \sup_{t\in[-r,0]} \left(\mathbf{E}\|\varphi(t)\|^2\right)^{\frac{1}{2}} < \infty.$$

Now, we give the definition of mild solution for (1.1).

Definition 2.5. A càdlàg stochastic process $\vartheta : [-r, T] \to \mathbb{H}, 0 \leqslant T < \infty$ is called a mild solution of (1.1) on $[-r, T]$ if $\vartheta_0(\cdot) = \varphi \in \mathcal{C}^0_{\mathcal{F}_0}\left([-r, 0], \mathcal{L}^2(\Omega, \mathbb{H})\right)$ on $[-r, 0]$ a.s., and for each $t \geqslant 0$ the following conditions hold:

(i) $\vartheta(t)$ is $\mathcal{F}_t$-adapted;
(ii) $\vartheta(t)$ satisfies the following integral equation:

$$\begin{aligned}\vartheta(t) = R(t)\left[\varphi(0) - h\left(0, \varphi(-\delta(0))\right)\right] + h\left(t, \vartheta(t-\delta(t))\right) + \int_0^t R(t-s) f\left(s, \vartheta(s-\rho(s))\right) ds \\ + \int_0^t R(t-s)\mathbb{B}\mathbb{U}(s) ds + \int_0^t \int_{\mathcal{U}} R(t-s) q\left(s, \vartheta(s-\gamma(s)), z\right) \widehat{\eta}(ds, dz) \\ + \int_0^t R(t-s)\sigma(s) dZ_Q^H(s) + \sum_{0<t_k<t} R(t-t_k) I_k(\vartheta(t_k^-)), \ \mathbb{P}, \text{ a.s.}\end{aligned}$$

Definition 2.6. The stochastic integrodifferential equations (1.1) is said to be controllable on the interval $[-r, T]$, if for every initial stochastic process $\vartheta(\cdot) = \varphi$ defined on $[-r, 0]$, there exists a stochastic control $u \in \mathcal{L}^2(J, \mathbb{U})$ which is adapted to the filtration $\{\mathcal{F}_t\}_{t \in J}$ such that the solution $\vartheta(\cdot)$ of the system (1.1) satisfies $\vartheta(T) = \vartheta_1$, where ϑ_1 and T are preassigned the terminal state and time, respectively.

3. Main results

In order to prove the existence and controllability results, one need to assume the following hypotheses hold.

($\mathbf{H}_3$) The resolvent operator associated with equation (2.4) is exponentially stable. That is: there exit some constants $\mu_0 > 0$, $\tilde{M} > 0$, such that
$$\|R(t)\| \leqslant \tilde{M} e^{-\mu_0 t}.$$

($\mathbf{H}_4$) The function $f : J \times \mathbb{H} \to \mathbb{H}$ satisfies : there exist positive constants $C_h > 0$ and $M_h > 0$ such that for all $t \geqslant 0$ and $x, y \in \mathbb{H}$
$$\|h(t, x) - h(t, y)\| \leqslant C_h \|x - y\|, \ \text{ and } \ \|h(t, x)\|^2 \leqslant M_h \left(1 + \|x\|^2\right).$$

($\mathbf{H}_5$) For all $t \geqslant 0$, $x, y \in \mathbb{H}$, $z \in \mathcal{U}$, the functions f, q satisfy: there exist constants $C_f > 0$, $C_q > 0$, $M_f > 0$, $M_q > 0$ such that
$$\|f(t, x) - f(t, y)\| \leqslant C_f \|x - y\|\ , \quad \|f(t, x)\|^2 \leqslant M_f \left(1 + \|x\|^2\right),$$
and
$$\int_{\mathcal{U}} \|q(t, x, z) - q(t, y, z)\|^2 \nu(dz) \leqslant C_q^2 \|x - y\|^2, \quad \int_{\mathcal{U}} \|q(t, y, z)\|^2 \nu(dz) \leqslant M_q \left(1 + \|y\|^2\right).$$

($\mathbf{H}_6$) The functions $I_k \in \mathcal{C}(\mathbb{H}, \mathbb{H})$, $k = 1, 2 \ldots$, satisfy the following conditions: there exist some positive constants C_{I_k}, M_{I_k} such that for all x, $y \in \mathbb{H}$,
$$\|I_k(x) - I_k(y)\| \leqslant C_{I_k} \|x - y\|, \quad \|I_k(x)\|^2 \leqslant d_k \left(1 + \|x\|^2\right), \quad \sum_{k=1}^{\infty} C_{I_k} < \infty, \quad \sum_{k=1}^{\infty} d_k < \infty.$$

($\mathbf{H}_7$) The function h is continuous in the quadratic mean sense: for all $\vartheta \in \mathcal{C}\left(J, \mathcal{L}^2(\Omega, \mathbb{H})\right)$,
$$\lim_{t \to s} \mathbf{E}\|h(t, \vartheta(t)) - h(s, \vartheta(s))\|^2 = 0.$$

$(\mathbf{H}_8)$ The function $\sigma : [0,\infty) \to \mathcal{L}_2^0(\mathbb{Y},\mathbb{H})$ satisfies the following condition:

$$\int_0^t e^{\mu_0 s}\|\sigma(s)\|^2_{\mathcal{L}_2^0} ds < \infty, \ \forall\, t > 0.$$

$(\mathbf{H}_9)$ $\mathbb{B} : \mathbb{U} \to \mathbb{H}$ is bounded linear operator and the operator $\Gamma : L^2(J,\mathbb{U}) \to L^2(\Omega,\mathbb{H})$ defined by

$$\Gamma u = \int_0^b R(t-s)\mathbb{B}u(s)ds,$$

has an inverse operator Γ^{-1} which takes values in $L^2(J,\mathbb{U}) \setminus \mathrm{Ker}\Gamma$, where the kernel space of Γ is defined by $\mathrm{Ker}\,\Gamma = \{x \in L^2(J,\mathbb{U}) \,:\, \Gamma x = 0\}$ (see [51, 59]) and there exist two positive constants $M_{\mathbb{B}}$ and M_Γ such that

$$\|\mathbb{B}\|^2 \leqslant M_{\mathbb{B}}, \ \|\Gamma^{-1}\|^2 \leqslant M_\Gamma.$$

Theorem 3.1. *If hypotheses* $(\mathbf{H}_1)$-$(\mathbf{H}_9)$ *hold and* $\vartheta_0 \in \mathbb{H}$, *then, the impulsive stochastic integrodifferential system* (1.1) *is controllable on* $[-r, T]$ *provided that*

$$C_h^2 + \tilde{M}^2\left(\sum_{k=1}^{+\infty} C_{I_k}\right)^2 < \frac{1}{5}. \tag{3.1}$$

Proof. Using the hypothesis $(\mathbf{H}_9)$, we define the control $u(\cdot)$ for an arbitrary $\vartheta(\cdot)$ by

$$\begin{aligned} u(t) = \Gamma^{-1}\Big\{&\vartheta_1 - R(T)\left[\varphi(0) - h\left(0,\varphi(-\delta(0))\right)\right] - h\left(T,\vartheta(T-\delta(T))\right) - \int_0^T R(T-s)f\left(s,\vartheta(s-\rho(s))\right)ds \\ &- \int_0^T\int_{\mathcal{U}} R(T-s)q\left(s,\vartheta(s-\gamma(s)),z\right)\widehat{\eta}(ds,dz) - \int_0^T R(T-s)\sigma(s)dZ_Q^H(s) \\ &- \sum_{0<t_k<T} R(T-t_k)I_k(\vartheta(t_k^-))\Big\}(t). \end{aligned}$$

We transform (1.1) into a fixed point problem. By using the above control, we show that the operator $\Psi : \mathcal{B}_T \to \mathcal{B}_T$ defined by $(\Psi\vartheta)(t) = \varphi(t)$, $t \in [-r, 0]$ and $\forall t \in J$,

$$\begin{aligned} (\Psi\vartheta)(t) = {}& R(t)\left[\varphi(0) - h\left(0,\varphi(-\delta(0))\right)\right] + h\left(t,\vartheta(t-\delta(t))\right) + \int_0^t R(t-s)f\left(s,\vartheta(s-\rho(s))\right)ds \\ &+ \int_0^t\int_{\mathcal{U}} R(t-s)q\left(s,\vartheta(s-\gamma(s)),z\right)\widehat{\eta}(ds,dz) + \int_0^t R(t-s)\sigma(s)dZ_Q^H(s) \\ &+ \sum_{0<t_k<t} R(t-t_k)I_k(\vartheta(t_k^-)) + \int_0^t R(t-u)\mathbb{B}\Gamma^{-1}\Big\{\vartheta_1 - R(b)\left[\varphi(0) - h\left(0,\varphi(-\delta(0))\right)\right] \\ &- h\left(b,\vartheta(T-\delta(T))\right) - \int_0^T R(T-s)f\left(s,\vartheta(s-\rho(s))\right)ds - \sum_{0<t_k<T} R(T-t_k)I_k(\vartheta(t_k^-)) \\ &- \int_0^T\int_{\mathcal{U}} R(T-s)q\left(s,\vartheta(s-\gamma(s)),z\right)\widehat{\eta}(ds,dz) - \int_0^T R(T-s)\sigma(s)dZ_Q^H(s)\Big\}(u)du. \end{aligned} \tag{3.2}$$

has a fixed point, which is then a mild solution for the stochastic impulsive integrodifferential system (1.1). Clearly, $\Psi\vartheta(T) = \vartheta_1$, which implies that the stochastic control u steers the system from the initial state φ to ϑ_1 in time b, provided we can find a fixed point of the operator Ψ which means that the system is controllable on $[-r, T]$. The proof is given in the following two steps.

Step 1. Ψ is well defined. Let $\vartheta \in \mathcal{B}_T$ and $t \in J$, we are going to show that each function $\Psi(\vartheta)(\cdot)$ is continuous on J in the $\mathcal{L}^2(\Omega, \mathbb{H})$-sense. Let $t \in (0, T)$ and $|\epsilon|$ be sufficiently small. Then for any fixed $\vartheta \in \mathcal{B}_T$, we have

$$\begin{aligned}
&\mathbf{E}\|\left(\Psi\vartheta\right)(t+\epsilon)-\left(\Psi\vartheta\right)(t)\|^2_{\mathcal{B}_T}\\
&\leqslant 7\mathbf{E}\|\left[R(t+\epsilon)-R(t)\right]\left[\varphi(0)-h\left(0,\varphi(-\delta(0))\right)\right]\|^2+7\mathbf{E}\|h\left(t+\epsilon,\vartheta(t+\epsilon-\delta(t+\epsilon))\right)-h\left(t,\vartheta(t-\delta(t))\right)\|^2\\
&\quad+7\mathbf{E}\left\|\int_0^{t+\epsilon}R(t+\epsilon-s)f\left(s,\vartheta(s-\rho(s))\right)ds-\int_0^t R(t-s)f\left(s,\vartheta(s-\rho(s))\right)ds\right\|^2\\
&\quad+7\mathbf{E}\left\|\int_0^{t+\epsilon}\int_{\mathcal{U}}R(t+\epsilon-s)q\left(s,\vartheta(s-\gamma(s)),z\right)\widehat{\eta}(ds,dz)-\int_0^t\int_{\mathcal{U}}R(t-s)q\left(s,\vartheta(s-\gamma(s)),z\right)\widehat{\eta}(ds,dz)\right\|^2\\
&\quad+7\mathbf{E}\left\|\int_0^{t+\epsilon}R(t+\epsilon-s)\sigma(s)dZ_Q^H(s)-\int_0^t R(t-s)\sigma(s)dZ_Q^H(s)\right\|^2\\
&\quad+7\mathbf{E}\left\|\sum_{0<t_k<T}\left[R(t+\epsilon-t_k)-R(t-t_k)\right]I_k(\vartheta(t_k^-))\right\|^2\\
&\quad+7\mathbf{E}\left\|\int_0^{t+\epsilon}R(t+\epsilon-u)\mathbb{B}\Gamma^{-1}\Big\{\vartheta_1-R(T)\left[\varphi(0)-h\left(0,\varphi(-\delta(0))\right)\right]\right.\\
&\quad-h\left(T,\vartheta(T-\delta(T))\right)-\int_0^T R(T-s)f\left(s,\vartheta(s-\rho(s))\right)ds-\sum_{0<t_k<T}R(T-t_k)I_k(\vartheta(t_k^-))\\
&\quad-\int_0^T\int_{\mathcal{U}}R(T-s)q\left(s,\vartheta(s-\gamma(s)),z\right)\widehat{\eta}(ds,dz)-\int_0^T R(T-s)\sigma(s)dZ_Q^H(s)\Big\}(u)du\\
&\quad-\int_0^t R(t-u)\mathbb{B}\Gamma^{-1}\Big\{\vartheta_1-R(T)\left[\varphi(0)-h\left(0,\varphi(-\delta(0))\right)\right]\\
&\quad-h\left(T,\vartheta(T-\delta(T))\right)-\int_0^T R(T-s)f\left(s,\vartheta(s-\rho(s))\right)ds-\sum_{0<t_k<T}R(T-t_k)I_k(\vartheta(t_k^-))\\
&\quad\left.-\int_0^T\int_{\mathcal{U}}R(T-s)q\left(s,\vartheta(s-\gamma(s)),z\right)\widehat{\eta}(ds,dz)-\int_0^T R(T-s)\sigma(s)dZ_Q^H(s)\Big\}(u)du\right\|^2\\
&:=7\sum_{i=1}^7\mathcal{J}_i(\epsilon).
\end{aligned}\tag{3.3}$$

Strong's continuity of $R(t)$ permits us to deduce

$$\lim_{\epsilon\to 0}\left[R(t+\epsilon)-R(t)\right]\left[\varphi(0)-h\left(0,\varphi(-\delta(0))\right)\right]=0.$$

One can infer from Definition 2.3 (i) that

$$\begin{aligned}
&\|\left[R(t+\epsilon)-R(t)\right]\left[\varphi(0)-h\left(0,\varphi(-\delta(0))\right)\right]\|^2\\
&\quad\leqslant 2M^2\left[e^{-2\mu_0(t+\epsilon)}+e^{-2\mu_0 t}\right]\|\varphi(0)-h\left(0,\varphi(-\delta(0))\right)\|^2\in\mathcal{L}^2(\Omega).
\end{aligned}$$

As a result, the Lebesgue dominated theorem implies that

$$\lim_{\epsilon\to 0}\mathcal{J}_1(\epsilon)=0. \tag{3.4}$$

On the basis of ($\mathbf{H}_7$), we reach the conclusion that

$$\lim_{\epsilon\to 0}\mathbf{E}\|h\left(t+\epsilon,\vartheta(t+\epsilon-\delta(t+\epsilon))\right)-h\left(t,\vartheta(t-\delta(t))\right)\|^2=0. \tag{3.5}$$

Thus
$$\lim_{\epsilon\to 0}\mathcal{J}_2(\epsilon)=0. \tag{3.6}$$
Consider the case of $\mathcal{J}_3(\epsilon)$. Without loss of generality, we suppose that $\epsilon>0$ (the case $\epsilon<0$ is similar). We have
$$\begin{aligned}
\mathcal{J}_3(\epsilon) &\leqslant \mathbf{E}\Big\|\int_0^{t+\epsilon} R(t+\epsilon-s)f(s,\vartheta(s-\rho(s)))\,ds-\int_0^t R(t-s)f(s,\vartheta(s-\rho(s)))\,ds\Big\|^2\\
&\leqslant 2\mathbf{E}\Big\|\int_0^t [R(t+\epsilon-s)-R(t-s)]\,f(s,\vartheta(s-\rho(s)))\,ds\Big\|^2+2\mathbf{E}\Big\|\int_t^{t+\epsilon} R(t+\epsilon-s)f(s,\vartheta(s-\rho(s)))\,ds\Big\|^2\\
&:=2\mathcal{J}_{31}(\epsilon)+2\mathcal{J}_{32}(\epsilon).
\end{aligned}$$
When we apply the Hölder inequality on $\mathcal{J}_{31}(\epsilon)$, we obtain
$$\mathcal{J}_{31}(\epsilon)\leqslant t\mathbf{E}\int_0^t\|\,[R(t+\epsilon-s)-R(t-s)]\,f(s,\vartheta(s-\rho(s)))\,\|^2 ds.$$
Using Definition 2.3 (ii), for each $s\in[0,t]$, we have
$$\lim_{\epsilon\to 0}[R(t+\epsilon-s)-R(t-s)]\,f(s,\vartheta(s-\rho(s)))=0.$$
In light of assumption $(\mathbf{H}_5)$ and Definition 2.3 (i), we can conclude that
$$\begin{aligned}
&\|\,[R(t+\epsilon-s)-R(t-s)]\,f(s,\vartheta(s-\rho(s)))\,\|^2\\
&\qquad\leqslant 2M^2\left[e^{-2\mu_0(t+\epsilon-s)}+e^{-2\mu_0(t-s)}\right]\|f(s,\vartheta(s-\rho(s)))\,\|^2\in\mathcal{L}^2([0,t]\times\Omega).
\end{aligned}$$
In this case, too, the Lebesgue dominated convergence theorem implies
$$\lim_{\epsilon\to 0}\mathcal{J}_{31}(\epsilon)=0. \tag{3.7}$$
Furthermore, using the Hölder inequality and assumptions $(\mathbf{H}_3)$, $(\mathbf{H}_5)$, we can derive
$$\begin{aligned}
\mathcal{J}_{32}(\epsilon) &\leqslant \int_t^{t+\epsilon}\|R(t+\epsilon-s)\|^2 ds\,\mathbf{E}\int_t^{t+\epsilon}\|f(s,\vartheta(s-\rho(s)))\,\|^2 ds\\
&\leqslant \tilde{M}^2(2\mu_0)^{-1}M_f\left[1-e^{-2\mu_0\epsilon}\right]\int_t^{t+\epsilon}\left(1+\mathbf{E}\|\vartheta(s-\rho(s))\|^2\right)ds\overset{\epsilon\to 0}{\longrightarrow}0\,.
\end{aligned}$$
Thus, we obtain
$$\lim_{\epsilon\to 0}\mathcal{J}_3(\epsilon)=0. \tag{3.8}$$
For the term $\mathcal{J}_4(\epsilon)$, we have by assumption $(\mathbf{H}_3)$:
$$\begin{aligned}
\mathcal{J}_4(\epsilon) &\leqslant 2\mathbf{E}\Big\|\int_0^t\int_{\mathcal{U}}[R(t+\epsilon-s)-R(t-s)]\,q(s,\vartheta(s-\gamma(s)),z)\,\widehat{\eta}(ds,dz)\Big\|^2\\
&\quad+2\mathbf{E}\Big\|\int_t^{t+\epsilon}\int_{\mathcal{U}}R(t+\epsilon-s)q(s,\vartheta(s-\gamma(s)),z)\,\widehat{\eta}(ds,dz)\Big\|^2\\
&\leqslant 2\int_0^t\int_{\mathcal{U}}\mathbf{E}\Big\|\,[R(t+\epsilon-s)-R(t-s)]\,q(s,\vartheta(s-\gamma(s)),z)\,\|^2\lambda(dz)ds\\
&\quad+2\tilde{M}^2\int_t^{t+\epsilon}\int_{\mathcal{U}}e^{-\mu_0(t+\epsilon-s)}(\|q(s,\vartheta(s-\gamma(s)),z)\,\|^2\lambda(dz)ds\\
&:=2\mathcal{J}_{41}(\epsilon)+2\mathcal{J}_{42}(\epsilon).
\end{aligned}$$

By $(\mathbf{H}_3)$, the strong continuity of $R(t)$ and Lebesgue dominated convergence theorem, we get $\lim\limits_{\epsilon\to 0} \mathcal{J}_{41}(\epsilon) = 0$. Therefore

$$\lim_{\epsilon\to 0} \mathcal{J}_4(\epsilon) = 0. \tag{3.9}$$

For the term $\mathcal{J}_5(\epsilon)$, we have

$$\begin{aligned}\mathcal{J}_5(\epsilon) &\leqslant 2\mathbf{E}\Big\|\int_0^t [R(t+\epsilon-s)-R(t-s)]\,\sigma(s)dZ_Q^H(s)\Big\|^2 + 2\mathbf{E}\Big\|\int_t^{t+\epsilon} R(t+\epsilon-s)\sigma(s)dZ_Q^H(s)\Big\|^2\\ &:= 2\mathcal{J}_{51}(\epsilon)+2\mathcal{J}_{52}(\epsilon).\end{aligned}$$

Lemma 2.2 implies that

$$\mathcal{J}_{51}(\epsilon) \leqslant C(H)T^{2H-1}\int_0^t \big\|\,[R(t+\epsilon-s)-R(t-s)]\,\sigma(s)\big\|^2_{\mathcal{L}_2^0}ds.$$

Based on strong continuity of $R(t)$, for each $s \in [0,t]$ the following limit holds:

$$\lim_{\epsilon\to 0}\Big\|\,[R(t+\epsilon-s)-R(t-s)]\,\sigma(s)\Big\|^2_{\mathcal{L}_2^0} = 0.$$

By $(\mathbf{H}_3)$ and Lebesgue dominated theorem, we have

$$\Big\|\,[R(t+\epsilon-s)-R(t-s)]\,\sigma(s)\Big\|^2 \leqslant 2M^2\left[e^{-2\mu_0(t+\epsilon-s)}+e^{-2\mu_0(t-s)}\right]\|\sigma(s)\|^2_{\mathcal{L}_2^0} \in \mathcal{L}^1(J, ds)$$

and

$$\lim_{\epsilon\to 0} \mathcal{J}_{51}(\epsilon) = 0. \tag{3.10}$$

Applying Lemma 2.2 to $\mathcal{J}_{52}(\epsilon)$ we obtain

$$\mathcal{J}_{52}(\epsilon) \leqslant 2HM^2\epsilon^{2H-1}\int_t^{t+\epsilon} e^{-2\mu_0 s}\|\sigma(s)\|^2_{\mathcal{L}_2^0}ds \overset{\epsilon\to 0}{\longrightarrow} 0\,.$$

Thus,

$$\lim_{\epsilon\to 0} \mathcal{J}_{52}(\epsilon) = 0. \tag{3.11}$$

Therefore

$$\lim_{\epsilon\to 0} \mathcal{J}_{5}(\epsilon) = 0. \tag{3.12}$$

Now, we have

$$\mathcal{J}_6(\epsilon) \leqslant \sum_{0<t_k<T}\left[\Big\|R(t+\epsilon-t_k)-R(t-t_k)\Big\|^2\right]\mathbf{E}\Big\|I_k(\vartheta(t_k^-))\Big\|^2.$$

By the assumptions $(\mathbf{H}_1)$, $(\mathbf{H}_6)$, and the strong continuity of $R(t)$, one has that

$$\lim_{\epsilon\to 0} \mathcal{J}_{6}(\epsilon) = 0. \tag{3.13}$$

For the estimation of term $\mathcal{J}_7(\epsilon)$, we have

$$\begin{aligned}\mathcal{J}_7(\epsilon) \leqslant 2\mathbf{E}\Big\|\int_0^t [R(t+\epsilon-u)-R(t-u)]\,\mathbb{B}\Gamma^{-1}\Big\{&\vartheta_1 - R(T)\,[\varphi(0)-h\,(0,\varphi(-\delta(0)))]\\ &-h\,(T,\vartheta(T-\delta(T))) - \int_0^T R(T-s)f\,(s,\vartheta(s-\rho(s)))\,ds - \sum_{0<t_k<T} R(T-t_k)I_k(\vartheta(t_k^-))\end{aligned}$$

$$
\begin{aligned}
&-\int_0^T\int_{\mathcal{U}} R(T-s)q\left(s,\vartheta(s-\gamma(s)),z\right)\widehat{\eta}(ds,dz)-\int_0^T R(T-s)\sigma(s)dZ_Q^H(s)\Big\}(u)du\Big\|^2\\
&+2\mathbf{E}\Big\|\int_t^{t+\epsilon} R(t+\epsilon-u)\mathbb{B}\Gamma^{-1}\Big\{\vartheta_1-R(T)\left[\varphi(0)-h\left(0,\varphi(-\delta(0))\right)\right]\\
&-h\left(T,\vartheta(T-\delta(T))\right)-\int_0^T R(T-s)f\left(s,\vartheta(s-\rho(s))\right)ds-\sum_{0<t_k<T}R(T-t_k)I_k(\vartheta(t_k^-))\\
&-\int_0^T\int_{\mathcal{U}} R(T-s)q\left(s,\vartheta(s-\gamma(s)),z\right)\widehat{\eta}(ds,dz)-\int_0^T R(T-s)\sigma(s)dZ_Q^H(s)\Big\}(u)du\Big\|^2\\
:=\;&\mathcal{J}_{71}(\epsilon)+\mathcal{J}_{72}(\epsilon).
\end{aligned}
$$

Since

$$
\begin{aligned}
\lim_{\epsilon\to 0}\Big\|&\left[R(t+\epsilon-u)-R(t-u)\right]\mathbb{B}\Gamma^{-1}\Big\{\vartheta_1-R(T)\left[\varphi(0)-h\left(0,\varphi(-\delta(0))\right)\right]\\
&-h\left(T,\vartheta(T-\delta(T))\right)-\int_0^T R(T-s)f\left(s,\vartheta(s-\rho(s))\right)ds-\sum_{0<t_k<T}R(T-t_k)I_k(\vartheta(t_k^-))\\
&-\int_0^T\int_{\mathcal{U}} R(T-s)q\left(s,\vartheta(s-\gamma(s)),z\right)\widehat{\eta}(ds,dz)-\int_0^T R(b-s)\sigma(s)dZ_Q^H(s)\Big\}\Big\|^2=0
\end{aligned}
$$

and

$$
\begin{aligned}
\Big\|&\left[R(t+\epsilon-u)-R(t-u)\right]\mathbb{B}\Gamma^{-1}\Big\{\vartheta_1-R(T)\left[\varphi(0)-h\left(0,\varphi(-\delta(0))\right)\right]\\
&-h\left(T,\vartheta(T-\delta(T))\right)-\int_0^T R(T-s)f\left(s,\vartheta(s-\rho(s))\right)ds-\sum_{0<t_k<T}R(T-t_k)I_k(\vartheta(t_k^-))\\
&-\int_0^T\int_{\mathcal{U}} R(T-s)q\left(s,\vartheta(s-\gamma(s)),z\right)\widehat{\eta}(ds,dz)-\int_0^T R(T-s)\sigma(s)dZ_Q^H(s)\Big\}(u)\Big\|^2\\
\leqslant\;&14M^2\left(e^{-2\mu_0(t+\epsilon-u)}+e^{-2\mu_0(t-u)}\right)M_{\mathbb{B}}M_{\Gamma}\Big\{\mathbf{E}\|\vartheta_1\|^2+\tilde{M}^2e^{-2\mu_0T}\mathbf{E}\|\varphi(0)-h\left(0,\varphi(-\delta(0))\right)\|^2\\
&+M_h\left(1+\sup_{s\in[-r,T]}\mathbf{E}\|\vartheta(s)\|^2\right)+T\tilde{M}^2\left(2\mu_0\right)^{-1}M_f\left(1+\sup_{s\in[-r,T]}\mathbf{E}\|\vartheta(s)\|^2\right)\\
&+\tilde{M}^2\left(\sum_{k=1}^{\infty}d_k\right)^2\left(1+\sup_{s\in[-r,T]}\mathbf{E}\|\vartheta(s)\|^2\right)+T\tilde{M}^2M_q^2\left(2\mu_0\right)^{-1}\left(1+\sup_{s\in[-r,T]}\mathbf{E}\|\vartheta(s)\|^2\right)\\
&+\tilde{M}^2\left(2\mu_0\right)^{-1}c(H)T^{2H-1}\int_0^T\|\sigma(s)\|^2_{\mathcal{L}_2^0}ds\Big\}\in\mathcal{L}^1,
\end{aligned}
$$

we conclude, by the dominated convergence theorem that

$$
\lim_{\epsilon\to 0}\mathcal{J}_{71}(\epsilon)=0. \tag{3.14}
$$

From the assumptions ($\mathbf{H}_3$)-($\mathbf{H}_9$) and applying Lemma 2.2 to $\mathcal{J}_{71}(\epsilon)$ we obtain

$$\begin{aligned}
\mathcal{J}_{72}(\epsilon) \leqslant{}& \tilde{M}^2 (2\mu_0)^{-1} \left[1-e^{-2\mu_0\epsilon}\right] M_{\mathbb{B}} M_{\Gamma} \int_t^{t+\epsilon} \Bigg\{ \mathbf{E}\|\vartheta_1\|^2 \\
&+ \tilde{M}^2 e^{-2\mu_0 T}\mathbf{E}\|\varphi(0) - h(0,\varphi(-\delta(0)))\|^2 + M_h\left(1+\sup_{s\in[-r,T]}\mathbf{E}\|\vartheta(s)\|^2\right) \\
&+ b\tilde{M}^2 (2\mu_0)^{-1} M_f \left(1+\sup_{s\in[-r,T]}\mathbf{E}\|\vartheta(s)\|^2\right) + \tilde{M}^2\left(\sum_{k=1}^{\infty} d_k\right)^2\left(1+\sup_{s\in[-r,T]}\mathbf{E}\|\vartheta(s)\|^2\right) \\
&+ T\tilde{M}^2 M_q^2 (2\mu_0)^{-1}\left(1+\sup_{s\in[-r,T]}\mathbf{E}\|\vartheta(s)\|^2\right) \\
&+ \tilde{M}^2 (2\mu_0)^{-1} c(H) T^{2H-1}\int_0^b \|\sigma(s)\|^2_{\mathcal{L}_2^0} ds \Bigg\} du \to 0 \text{ as } \epsilon \to 0.
\end{aligned} \tag{3.15}$$

Replacing (3.4)-(3.15) in (3.3), we deduce

$$\lim_{\epsilon\to 0}\mathbf{E}\|(\Psi\vartheta)(t+\epsilon) - (\Psi\vartheta)(t)\|^2 = 0.$$

Thus, the function $t \to (\Psi\vartheta)(t)$ is continuous on J.

Step 2. In this part of the proof, we will prove that Ψ is a contraction mapping in $\mathcal{B}_T$ with some $T_1 < T$ to be specified later. Let $\vartheta, \chi \in \mathcal{B}_{T_1}$ and $t \in [0,T]$. We have

$$\begin{aligned}
&\mathbf{E}\|(\Psi\vartheta)(t) - (\Psi\chi)(t)\|^2 \\
&\leqslant 5\mathbf{E}\|h(t,\vartheta(t-\delta(t))) - h(t,\chi(t-\delta(t)))\|^2 + 5\mathbf{E}\left\|\int_0^t R(t-s)\left[f(s,\vartheta(s-\rho(s))) - f(s,\chi(s-\rho(s)))\right]ds\right\|^2 \\
&\quad + 5\mathbf{E}\left\|\int_0^t\int_{\mathcal{U}} R(t-s)\left[q(s,\vartheta(s-\gamma(s)),z) - q(s,\chi(s-\gamma(s)),z)\right]\widehat{\eta}(ds,dz)\right\|^2 \\
&\quad + 5\mathbf{E}\left\|\sum_{0<t_k<T} R(t-t_k)\left[I_k(\vartheta(t_k^-)) - I_k(\chi(t_k^-))\right]\right\|^2 \\
&\quad + 5\mathbf{E}\Bigg\|\int_0^t R(t-u)\mathbb{B}\Gamma^{-1}\Big\{-\left[h(T,\vartheta(T-\delta(T))) - h(T,\chi(T-\delta(T)))\right] \\
&\quad - \int_0^T R(T-s)\left[f(s,\vartheta(s-\rho(s))) - f(s,\chi(s-\rho(s)))\right]ds - \sum_{0<t_k<T} R(T-t_k)\left[I_k(\vartheta(t_k^-)) - I_k(\chi(t_k^-))\right] \\
&\quad - \int_0^T\int_{\mathcal{U}} R(T-s)\left[q(s,\vartheta(s-\gamma(s)),z) - q(s,\chi(s-\gamma(s)),z)\right]\widehat{\eta}(ds,dz)\Big\}(u)du\Bigg\|^2.
\end{aligned}$$

Using (3.2), we have

$$\begin{aligned}
&\mathbf{E}\|(\Psi\vartheta)(t) - (\Psi\chi)(t)\|^2 \\
&\leqslant 5\mathbf{E}\|h(t,\vartheta(t-\delta(t))) - h(t,\chi(t-\delta(t)))\|^2 + 5\mathbf{E}\left\|\int_0^t R(t-s)\left[f(s,\vartheta(s-\rho(s))) - f(s,\chi(s-\rho(s)))\right]ds\right\|^2 \\
&\quad + 5\mathbf{E}\left\|\int_0^t\int_{\mathcal{U}} R(t-s)\left[q(s,\vartheta(s-\gamma(s)),z) - q(s,\chi(s-\gamma(s)),z)\right]\widehat{\eta}(ds,dz)\right\|^2 \\
&\quad + 5\mathbf{E}\left\|\sum_{0<t_k<T} R(t-t_k)\left[I_k(\vartheta(t_k^-)) - I_k(\chi(t_k^-))\right]\right\|^2
\end{aligned}$$

$$
\begin{aligned}
&+5\mathbf{E}\Big\| \int_0^t R(t-u)\mathbb{B}\Gamma^{-1}\Big\{-[h(T,\vartheta(T-\delta(T)))-h(T,\chi(T-\delta(T)))]\\
&-\int_0^T R(T-s)\,[f(s,\vartheta(s-\rho(s)))-f(s,\chi(s-\rho(s)))]\,ds-\sum_{0<t_k<T} R(T-t_k)\left[I_k(\vartheta(t_k^-))-I_k(\chi(t_k^-))\right]\\
&-\int_0^T\int_{\mathcal{U}} R(b-s)\,[q(s,\vartheta(s-\gamma(s)),z)-q(s,\chi(s-\gamma(s)),z)]\,\widehat{\eta}(ds,dz)\Big\}(u)du\Big\|^2.
\end{aligned}
$$

By using Definition 2.3, Hölder inequality and assumptions $(\mathbf{H}_3)$-$(\mathbf{H}_7)$, $(\mathbf{H}_9)$, we get

$$
\begin{aligned}
&\mathbf{E}\|(\Psi\vartheta)(t)-(\Psi\chi)(t)\|^2\\
&\quad\leqslant 5C_h^2\sup_{s\in[-r,t]}\mathbf{E}\|\vartheta(s-\delta(s))-\chi(s-\delta(s))\|^2\\
&\quad\quad+5\tilde{M}^2(2\mu_0)^{-1}tC_f^2\sup_{s\in[-r,t]}\mathbf{E}\|\vartheta(s-\rho(s))-\chi(s-\rho(s))\|^2+5\tilde{M}^2\left(\sum_{k=1}^{\infty}C_{I_k}\right)^2\sup_{s\in[-r,t]}\mathbf{E}\|\vartheta(s)-\chi(s)\|^2\\
&\quad\quad+5\tilde{M}^2(2\mu_0)^{-1}tC_q^2\sup_{s\in[-r,t]}\mathbf{E}\|\vartheta(s-\gamma(s))-\chi(s-\gamma(s))\|^2\\
&\quad\quad+20(2\mu_0)^{-1}\tilde{M}^2\left(1-e^{-\mu_0 t}\right)M_{\mathbb{B}}M_{\Gamma}\Big\{C_h^2\sup_{s\in[-r,T]}\mathbf{E}\|\vartheta(s-\delta(s))-\chi(s-\delta(s))\|^2\\
&\quad\quad+\tilde{M}^2(2\mu_0)^{-1}TC_f^2\sup_{s\in[-r,T]}\mathbf{E}\|\vartheta(s-\rho(s))-\chi(s-\rho(s))\|^2+\tilde{M}^2\left(\sum_{k=1}^{\infty}C_{I_k}\right)^2\sup_{s\in[-r,T]}\mathbf{E}\|\vartheta(s)-\chi(s)\|^2\\
&\quad\quad+\tilde{M}^2(2\mu_0)^{-1}TC_q^2\sup_{s\in[-r,T]}\mathbf{E}\|\vartheta(s-\gamma(s))-\chi(s-\gamma(s))\|^2\Big\}.
\end{aligned}
$$

Hence, we have

$$
\sup_{s\in[-r,t]}\mathbf{E}\|(\Psi\vartheta)(s)-(\Psi\chi)(s)\|^2\leqslant\alpha(t)\sup_{s\in[-r,t]}\mathbf{E}\|(\Psi\vartheta)(s)-(\Psi\chi)(s)\|^2,
$$

where

$$
\begin{aligned}
\alpha(t)\leqslant\,&5C_h^2+5\tilde{M}^2(2\mu_0)^{-1}tC_f^2+5\tilde{M}^2\left(\sum_{k=1}^{\infty}C_{I_k}\right)^2+5\tilde{M}^2(2\mu_0)^{-1}tC_q^2\\
&+20(2\mu_0)^{-1}\tilde{M}^2\left(1-e^{-\mu_0 t}\right)M_{\mathbb{B}}M_{\Gamma}\left\{C_h^2+\tilde{M}^2(2\mu_0)^{-1}TC_f^2+\tilde{M}^2\left(\sum_{k=1}^{\infty}C_{I_k}\right)^2+\tilde{M}^2(2\mu_0)^{-1}bC_q^2\right\}.
\end{aligned}
$$

By inequality (3.1), we have

$$
\alpha(0)=5C_h^2+5\tilde{M}^2\left(\sum_{k=1}^{\infty}C_{I_k}\right)^2<1.
$$

Then there exists $0<T_1\leqslant T$ such that $0<\alpha(0)<1$ and the operator Ψ is a contraction on $\mathcal{B}_{T_1}$ and hence it has a unique fixed point on $[-r,T_1]$, which is a mild solution of system (1.1) on the interval $[-r,T_1]$. By repeating a similar process the solution can be extended to the entire interval $[-r,b]$. Therefore, the system (1.1) is controllable on $[-r,T]$.
The proof is complete. □

4. Example

We consider the following nonlocal stochastic integrodifferential system to illustrate the previous theoretical results.

$$\begin{cases} d\left[w(t,\xi)-\beta_1 w(t-\rho(t),\xi)\right]=\left[\dfrac{\partial^2}{\xi^2}\left[w(t,\xi)-\beta_1 w(t-\rho(t),\xi)\right]\right.\\ \left.+\displaystyle\int_0^t g(t-s)\frac{\partial^2}{\xi^2}\left[w(s,\xi)-\beta_1 w(s-\rho(s),\xi)\right]ds+\beta_2 w(t-\delta(t),\xi)+k(\xi)u(t)\right]dt\\ +\displaystyle\int_{\mathcal{U}}\beta_3 y\left(w(t-r(t),\xi)\right)\tilde{N}(dt,dy)+e^{-t}dZ_Q^H(t),\quad 0\leqslant\xi\leqslant\pi,\quad t\neq t_k,\quad t\in[0,T],\\ \Delta w(t_k,\cdot)(\xi)=\dfrac{\beta_4}{2^k}w(t_k^-,\xi),\qquad t=t_k,\ k=1,2,\dots,\\ \vartheta(\theta,\cdot)=\vartheta_0(\theta,\xi)\in\mathbb{X}=L^2([0,\pi]),\quad \vartheta_0(\cdot,\xi)\in\mathcal{C}\left([-r,0],\mathbb{R}\right),\quad \theta\in[0,\pi],\end{cases}\tag{4.1}$$

where $H\in(\frac{1}{2},1)$, Z_Q^H denotes standard Rosenblatt process defined on a stochastic basis $(\Omega,\mathcal{F},\mathbb{P})$, β_1, β_2, β_3, β_4 are positive constants, $\mathcal{U}=\{\nu\in\mathbb{R}:0<|\nu|\leqslant c,\ c>0\}$, $g:\mathbb{R}^+\to\mathbb{R}^+$ is continuous functions and $\varphi(\theta,\cdot)\in L^2[0,\pi]$ is measurable and satisfies $\mathbb{E}\|\varphi\|^2<\infty$. Let $\mathbb{X}=\mathbb{Y}=\mathbb{U}=L^2([0,\pi])$ with the norm $\|.\|$. Define $A:D(A)\subset\mathbb{X}\to\mathbb{X}$ by $Ax=x''$ with domain

$$D(A)=\{x(.)\in\mathbb{X},\ x,\ x'\ \text{ are absolutely continuous},\ x''\in\mathbb{X},\ x(0)=x(\pi)\}.$$

The spectrum of A consists of the eigenvalues $-n^2$ for $n\in\mathbb{N}$, with associated eigenvectors

$$e_n:=\sqrt{\frac{2}{\pi}}\sin(nx),\quad (n=1,2,3,\dots).$$

Furthermore, the set $\{e_n:n\in\mathbb{N}\}$ is an orthogonal basis in $\mathbb{X}$. Then

$$Ax=\sum_{n=1}^{\infty}n^2\langle x,e_n\rangle e_n,\quad x\in\mathbb{X}.$$

It is well known that A is the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t\geqslant0}$ on $\mathbb{X}$, which is compact and is given by

$$T(t)x=\sum_{n=1}^{\infty}e^{-n^2t}<x,e_n>e_n,\quad x\in\mathbb{X}.$$

Let $\Upsilon:D(A)\subset\mathbb{X}\to\mathbb{X}$ be the operator defined by

$$\Upsilon(t)(\tilde{z})=g(t)A\tilde{z}\ \text{ for }\ t\geqslant0\ \text{ and }\ \tilde{z}\in D(A).$$

Further, define $\mathbb{B}\in L(R,\mathbb{H})$ by $\mathbb{B}u(t)=k(\xi)u, 0\leqslant\xi\leqslant\pi, u\in\mathbb{R}, k(\xi)\in L^2(0,\pi)$. Let $\Gamma u=\int_0^T\mathcal{R}(T-s)\mathbb{B}u(s)ds$, then we claim that Γ is bounded due to Hölder inequality. Take $\mathrm{Ker}(\Gamma)=\{u\in L^2([0,T],\mathbb{R}):\Gamma u=0\}$ as a null space of Γ and let $(\mathrm{Ker}(\Gamma))^{\perp}$ be its complement in $L^2([0,T],\mathbb{R})$. Also take $\Gamma_0:(\mathrm{Ker}(\Gamma))^{\perp}\to\mathrm{Range}(\Gamma)$ is the restriction of Γ to $(\mathrm{Ker}(\Gamma))^{\perp}$. Γ_0 is one-to-one operator. By using the inverse mapping theorem, we get Γ^{-1} is bounded. Now, Γ^{-1} is bounded and have 0 values in $L^2(J,U)\setminus\mathrm{Ker}(W)$, that is, ($\mathbf{H}_9$) is satisfied.

Define the operators $h,f:[0,\infty)\times\mathbb{X}\longrightarrow\mathbb{X}$, $q:[0,\infty)\times\mathbb{H}\mapsto\mathbb{H}$, $q:[0,\infty)\times\mathbb{H}\times\mathbb{H}\mapsto\mathbb{H}$, $I_k:\mathbb{H}\mapsto\mathbb{H}$ by

$$h(t,w(t-\rho(t)))(\xi)=\beta_1 w(t-\rho(t),\xi),\ \xi\in[0,\pi],$$

$$f(t, w(t-\rho(t)))(\xi) = \beta_2 w(t-\rho(t), \xi),\ \xi \in [0, \pi],$$
$$q(t, \phi, y)(\xi) = \beta_3 y w(t-r(t), \xi),\ \xi \in [0, \pi],$$
$$I_k(w(t_k, \cdot)(\xi) = \frac{\beta_4}{2^k} w(t_k^-, \xi), \quad k = 1, 2, \dots,\ \xi \in [0, \pi].$$

In order to rewrite system (4.1) in an abstract form in $\mathbb{X}$, we introduce the following notations

$$\begin{cases} \vartheta(t) = w(t, \xi) \text{ for } t \geqslant 0 \text{ and } \xi \in [0, \pi], \\ \varphi(t)(\tau) = w_0(t, \xi) \text{ for } t \in [-r, 0] \text{ and } \xi \in [0, \pi]. \end{cases}$$

Then equation (4.1) takes the following abstract form

$$\begin{cases} d\left[\vartheta(t) - h(t, \vartheta(t-\delta(t)))\right] = \Big[A\left[\vartheta(t) - h(t, \vartheta(t-\delta(t)))\right] + \int_0^t \Upsilon(t-s)\left[\vartheta(s) - h(s, \vartheta(t-\delta(s)))\right] ds \\ +f\left(t, \vartheta(t-\rho(t))\right) + \mathbb{B}u(t)\Big] dt + \int_{\mathcal{U}} q\left(t, \vartheta(t-\gamma(t)), z\right) \widehat{\eta}(dt, dz) + \sigma(t) dZ_Q^H(t), \quad t \neq t_k \in J := [0, T], \\ \Delta(t_k) = \vartheta(t_k^+) - \vartheta(t_k^-) = I_k(t_k^-), \quad k = 1, 2, \dots, \\ \vartheta_0(t) = \varphi(\cdot) \in \mathcal{C}^0_{\mathcal{F}_0}\left([-r, 0], \mathcal{L}^2(\Omega, \mathbb{H})\right), \quad t \in [-r, 0]. \end{cases}$$

Moreover, if g is bounded and C^1 function such that g' is bounded and uniformly continuous, then $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$ are satisfied, and hence, by Theorem 2.4, Eq. (2.4) has a resolvent operator $(R(t))_{t \geqslant 0}$ on $\mathbb{X}$. Using [14, Lemma 5.2], let $\mu > \delta > 1$ and $g(t) < \exp(-\beta t)$, for all $t \geqslant 0$. Then the above resolvent operator decays exponentially to zero. Specifically, $\|R(t)\| \leqslant \exp(-a t)$ where $a = 1 - 1/\delta$. It is obvious that all the assumptions are satisfied with

$$\mu_0 = 1 - \frac{1}{\delta},\ \tilde{M} = 1,\ C_h = \beta_1\ M_h = \beta_1^2,\ C_f = \beta_2\ M_f = \beta_2^2,$$
$$C_q = M_q = \int_{\mathcal{U}} \beta_3^2 y^2 \lambda(dy),\ C_{I_k} = \frac{\beta_4}{2^k}, \quad d_k = \frac{\beta_4}{2^{2k}}.$$

Thus, by Theorem 3.1, Equation (4.1) is controllable on $J = [0, T]$ provided

$$\beta_1^2 + \left(\sum_{k=1}^{+\infty} C_{I_k}\right)^2 < \frac{1}{5}.$$

5. Conclusion

The controllability of impulsive neutral stochastic delay integrodifferential equations driven by Rosenblatt process and Lévy noise in Hilbert spaces is investigated in this paper. A novel set of adequate criteria is generated by employing a fixed point technique without imposing a strict compactness condition on the resolvent operator. The findings in this paper constitute a generalization and extension of the recent findings on this topic. An example is offered to demonstrate the theoretical conclusion reached. Furthermore, this result could be extended to investigate non-instantaneous impulsive neutral stochastic delay integrodifferential equations driven by Rosenblatt process and Lévy noise in Hilbert space.

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